Mathematical Analysis 1: Tutorial #8 - some solutions

Irena Penev
Summer 2026

Exercise 2. *Using derivatives, prove that* $\arcsin x = \arctan\left(\frac{x}{\sqrt{1-x^2}}\right)$ *for all* $x \in (-1,1)$.

Solution. Define $f : (-1,1) \to \mathbb{R}$ by setting

$$f(x) \quad := \quad \arcsin x - \arctan\left(\tfrac{x}{\sqrt{1-x^2}}\right) \qquad \forall x \in (-1,1).$$

Then for $x \in (-1,1)$, we compute:

$$\begin{aligned}
f'(x) &= \tfrac{d}{dx}\Big(\arcsin x - \arctan\left(\tfrac{x}{\sqrt{1-x^2}}\right)\Big) \\
&= \tfrac{1}{\sqrt{1-x^2}} - \tfrac{1}{1+\frac{x^2}{1-x^2}}\tfrac{d}{dx}\Big(\tfrac{x}{\sqrt{1-x^2}}\Big) \\
&= \tfrac{1}{\sqrt{1-x^2}} - \tfrac{1}{\frac{(1-x^2)+x^2}{1-x^2}} \cdot \tfrac{\sqrt{1-x^2}-x\left(\frac{1}{2}\cdot\frac{1}{\sqrt{1-x^2}}\cdot(-2x)\right)}{1-x^2} \\
&= \tfrac{1}{\sqrt{1-x^2}} - (1-x^2)\tfrac{\sqrt{1-x^2}+\frac{x^2}{\sqrt{1-x^2}}}{1-x^2} \\
&= \tfrac{1}{\sqrt{1-x^2}} - \Big(\sqrt{1-x^2}+\tfrac{x^2}{\sqrt{1-x^2}}\Big) \\
&= \tfrac{1}{\sqrt{1-x^2}} - \tfrac{(1-x^2)+x^2}{\sqrt{1-x^2}} \\
&= \tfrac{1}{\sqrt{1-x^2}} - \tfrac{1}{\sqrt{1-x^2}} \\
&= 0.
\end{aligned}$$

It follows that f is a constant function. But clearly, $f(0) = 0$, and it follows that $f(x) = 0$ for all $x \in (-1,1)$. Consequently, $\arcsin x = \arctan\left(\frac{x}{\sqrt{1-x^2}}\right)$ for all $x \in (-1,1)$, which is what we needed to show. □