Mathematical Analysis 1:
Tutorial #8

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Exercise 1. *Compute the limits below, or show that they do not exist. (Note: Some of these limits can be computed L'Hôpital's Rule, whereas others cannot.)*

(a) $\lim_{x\to 0} \frac{1-x\cot x}{x^2}$;

(b) $\lim_{x\to 0} \frac{\sin x}{1+x}$;

(c) $\lim_{x\to a} \frac{a^x-x^a}{x-a}$ *(where $a>0$)*;

(d) $\lim_{x\to a} \frac{x^x-x^a}{x-a}$ *(where $a>0$)*;

(e) $\lim_{x\to 0^+} \frac{\ln x}{\sin x}$;

(f) $\lim_{x\to 0^+} (\cot x)^{\sin x}$;

(g) $\lim_{x\to 0} \frac{e^x}{\cos x}$;

(h) $\lim_{x\to 0} (2-x)^{\tan\left(\frac{\pi x}{2}\right)}$;

(i) $\lim_{x\to (\frac{\pi}{2})^-} (\tan x)^{\sin x}$;

(j) $\lim_{x\to 0} \frac{(1+x)^{1/x}-e}{x}$;

(k) $\lim_{x\to +\infty} \frac{x^{\ln x}}{(\ln x)^x}$;

(l) $\lim_{x\to 0} \frac{\arcsin(3x)-3\arcsin x}{x^3}$.

Exercise 2. *Using derivatives, prove that* $\arcsin x = \arctan\left(\frac{x}{\sqrt{1-x^2}}\right)$ *for all* $x \in (-1,1)$.

Exercise 3. *Let $a>0$ be a fixed constant. Prove that* $\left|\ln\left(\frac{x_2}{x_1}\right)\right| < \frac{|x_2-x_1|}{a}$ *for all distinct* $x_1, x_2 \in (a,+\infty)$.

Definition. *The* interior *of a set $A \subseteq \mathbb{R}$, denoted by A° or int(A), is the set*

$$\big\{a \in A \mid \exists \delta > 0 \text{ s.t. } (a-\delta, a+\delta) \subseteq A\big\}.$$

Exercise 4. *Let $a, b \in \mathbb{R}$ be such that $a<b$. Compute the interiors of the following sets.*

(a) $[a,b]$;

(b) (a,b);

(c) $[a,b)$;

(d) $(a,b]$;

(e) $[a,\infty)$;

(f) (a,∞);

(g) $(-\infty,b)$;

(h) $(-\infty,b]$;

(i) $(-\infty,\infty)$;

(j) $[2,4)\cup[5,6)$;

(k) $(-1,0)\cup(0,1)$;

(l) $\mathbb{Q}$.

Exercise 5. *The goal of this exercise is to generalize Corollary 4.10.3, as follows. Let I be an interval in $\mathbb{R}$, and let $f: I \to \mathbb{R}$ be a function that is continuous on I and differentiable on I°. Prove the following:*

(a) if $f'(x)=0$ for all $x \in I^\circ$, then f constant on I;

(b) if $f'(x)\geq 0$ for all $x \in I^\circ$, then f is non-decreasing on I;

(c) if $f'(x)>0$ for all $x \in I^\circ$, then f is strictly increasing on I;

(d) if $f'(x)\leq 0$ for all $x \in I^\circ$, then f is non-increasing on I;

(e) if $f'(x)<0$ for all $x \in I^\circ$, then f is strictly decreasing on I.

Exercise 6. *Let $a, b \in \mathbb{R}$ be such that $a < b$, and let $f, g, h : [a, b] \to \mathbb{R}$ be functions that are continuous on $[a, b]$ and differentiable on (a, b). Prove that there exists a point $c \in (a, b)$ such that*

$$\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0.$$

Hint: *Construct a suitable function $F : [a, b] \to \mathbb{R}$ given by*

$$F(x) := \begin{vmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{vmatrix} \qquad \forall x \in [a, b].$$