Mathematical Analysis 1: Tutorial #7

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function $f(x)$	derivative $f'(x)$	differentiable for
$c = \text{const.}$	0	$x \in \mathbb{R}$
x^α (for a fixed constant $\alpha \in \mathbb{R}$)	$\alpha x^{\alpha-1}$	$x > 0$
e^x	e^x	$x \in \mathbb{R}$
$\ln x$	$\frac{1}{x}$	$x > 0$
$\sin x$	$\cos x$	$x \in \mathbb{R}$
$\cos x$	$-\sin x$	$x \in \mathbb{R}$
$\tan x$	$\frac{1}{\cos^2 x}$	$x \neq \frac{2k+1}{2}\pi,\ k \in \mathbb{Z}$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$x \in (-1, 1)$
$\arctan x$	$\frac{1}{1+x^2}$	$x \in \mathbb{R}$

Differentiation Rules. *Let $f, g : A \subseteq \mathbb{R} \to \mathbb{R}$ be functions, and let $a_0 \in A$ be an accumulation point of A. Assume that f and g are both differentiable at a_0. Then the following hold:*

1. *$f + g$ is differentiable at a_0, and $(f + g)'(a_0) = f'(a_0) + g'(a_0)$;*
2. *for all constants $c \in \mathbb{R}$, cf is differentiable at a_0, and $(cf)'(a_0) = c\big(f'(a_0)\big)$;*
3. *[Product Rule] fg is differentiable at a_0, and $(fg)'(a_0) = f'(a_0)g(a_0) + f(a_0)g'(a_0)$;*
4. *[Quotient Rule] if $g(a_0) \neq 0$, then $\frac{f}{g}$ is differentiable at a_0, and*

$$\left(\frac{f}{g}\right)'(a_0) \quad = \quad \frac{f'(a_0)g(a_0) - f(a_0)g'(a_0)}{(g(a_0))^2}.$$

The Chain Rule. *Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and $g : B \subseteq \mathbb{R} \to \mathbb{R}$ be functions such that $Im(f) \subseteq B$ (so that $g \circ f : A \to \mathbb{R}$ is defined). Assume that f is differentiable at a point $a_0 \in A$, and that g is differentiable at the point $b_0 := f(a_0)$. (In particular, a_0 belongs to and is an accumulation point of A, whereas b_0 belongs to and is an accumulation point of B.) Then $g \circ f$ is differentiable at a_0, and moreover, we have that $(g \circ f)'(a_0) = g'(b_0)f'(a_0)$.*

Exercise 1. *Compute the derivatives of the following:*

(a) $5x^7 - 4x^5 + 3x^3 - 7$;

(b) $\sqrt[7]{x^4}$;

(c) $\frac{1+x\sqrt{x}}{1-x\sqrt{x}}$;

(d) $\frac{\sin x + \cos x}{\sin x - \cos x}$;

(e) $3^x(x^3 + 1)$;

(f) $x \arcsin x$;

(g) $\ln\left(x - \sqrt{x^2 - 1}\right)$;

(h) $\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$;

(i) $x^{\sin x}$;

(j) $(\sin x)^{\cos x}$;

(k) $\ln(\ln(\ln x))$;

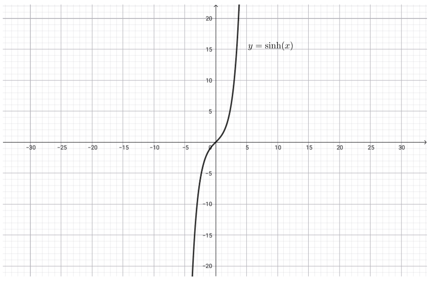
(l) $\arcsin \frac{2x}{1+x^2}$;

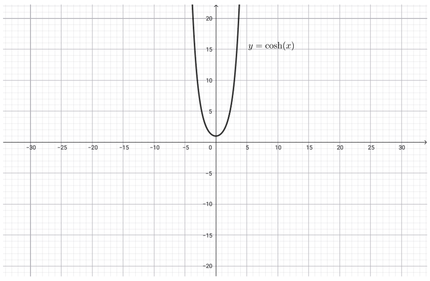
(m) $\frac{1}{2}\arctan \frac{2x}{1-x^2}$;

(n) $\log_x^e$;

(o) $\log_{\cos x}(\sin x)$;

Definition. *The* hyperbolic sine *and the* hyperbolic cosine *are, respectively, the functions* $\sinh, \cosh : \mathbb{R} \to \mathbb{R}$ *given by* $\sinh x = \frac{e^x - e^{-x}}{2}$ *and* $\cosh x = \frac{e^x + e^{-x}}{2}$ *for all* $x \in \mathbb{R}$*.*





Exercise 2. *Prove the following:*

(a) $\sinh x$ *is odd and* $\cosh x$ *is even;*

(b) $\cosh^2(x) - \sinh^2(x) = 1$*;*

(c) $\sinh'(x) = \cosh(x)$ *and* $\cosh'(x) = \sinh'(x)$*.*

Theorem 3.3.1. *Let* $f : A \subseteq \mathbb{R} \to \mathbb{R}$*, let* $a \in \mathbb{R}$ *be an accumulation point of the set* A*, and let* $L \in \mathbb{R}$*. Then the following are equivalent:*

(i) $\lim_{x \to a} f(x) = L$*;*

(ii) for all sequences $\{a_n\}_{n=1}^{\infty}$ *of real numbers that all belong to the set* $A \setminus \{a\}$*, if* $\lim_{n \to \infty} a_n = a$*, then* $\lim_{n \to \infty} f(a_n) = L$*.*

Exercise 3. *Consider the function* $f : \mathbb{R} \to \mathbb{R}$ *given by*

$$f(x) \quad := \quad \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

for all $x \in \mathbb{R}$*. Prove that* f *continuous at* 0*, but is* ***not*** *differentiable at* 0*.*

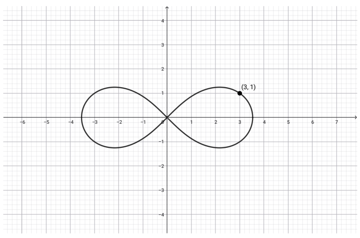
Exercise 4. *Consider the function* $g : \mathbb{R} \to \mathbb{R}$ *given by*

$$g(x) \quad := \quad \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

for all $x \in \mathbb{R}$*. Prove that* g *is differentiable at* 0*.*

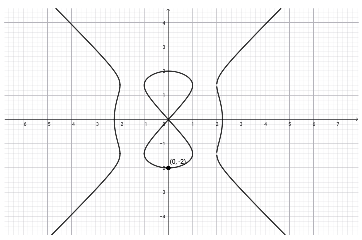
Exercise 5. *Use implicit differentiation to compute the equation of the tangent to the given curve through the specified point.*

(a) $2(x^2 + y^2)^2 = 25(x^2 - y^2)$*,* $(3, 1)$*;*



(lemniscate)

(b) $y^2(y^2 - 4) = x^2(x^2 - 5)$*,* $(0, -2)$*.*



(devil's curve)