

Mathematical Analysis 1:

Tutorial #6 - some solutions

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Exercise 3. Let $a, b \in \mathbb{R}$ be such that $a < b$, and let $f : (a, b) \rightarrow \mathbb{R}$ be a continuous function. Assume that $L_a := \lim_{x \rightarrow a^+} f(x)$ and $L_b := \lim_{x \rightarrow b^-} f(x)$ both exist (as real numbers or as $\pm\infty$), and assume furthermore that $L_a \neq L_b$. Let $M \in \mathbb{R}$ be such that $\min\{L_a, L_b\} < M < \max\{L_a, L_b\}$. Prove that there exists some $c \in (a, b)$ such that $f(c) = M$.

Remark: We will give a solution for the special case when L_a and L_b are real numbers. We give two solutions. The first one is easier, but the second one can be generalized to the case when L_a or L_b is $\pm\infty$.

Solution#1 (only for the case when $L_a, L_b \in \mathbb{R}$). Define the function $g : [a, b] \rightarrow \mathbb{R}$ by setting

$$g(x) := \begin{cases} L_a & \text{if } x = a \\ f(x) & \text{if } a < x < b \\ L_b & \text{if } x = b \end{cases}$$

for all $x \in [a, b]$. Then g is continuous on $[a, b]$, and we have that $\min\{g(a), g(b)\} < M < \max\{g(a), g(b)\}$. So, by the Intermediate Value Theorem, there exists some $c \in (a, b)$ such that $g(c) = M$. □

Solution#2 (only for the case when $L_a, L_b \in \mathbb{R}$). Set $m := \frac{a+b}{2}$, so that m is the midpoint of the interval (a, b) . Our goal is to choose (i.e. prove the existence of) suitable points $a' \in (a, m)$ and $b' \in (m, b)$, so that $a' < b'$ and $[a', b'] \subsetneq (a, b)$, that additionally satisfy $\min\{f(a'), f(b')\} < M < \max\{f(a'), f(b')\}$. The result will then follow from the Intermediate Value Theorem applied to the function f , the interval $[a', b']$, and the “intermediate value” M . We consider two cases: when $L_a < L_b$ and when $L_b < L_a$.

Case 1: $L_a < L_b$.

We now have that $L_a < M < L_b$, and we set $\varepsilon_a := M - L_a$ and $\varepsilon_b := L_b - M$. (Clearly, $\varepsilon_a, \varepsilon_b > 0$.)

Using the fact that $L_a = \lim_{x \rightarrow a^+} f(x)$, we fix $\delta_a > 0$ such that for all $x \in (a, b)$, if $a < x < a + \delta_a$, then $|f(x) - L_a| < \varepsilon_a$, that is, $L_a - \varepsilon_a < f(x) < L_a + \varepsilon_a$. Fix any $a' \in (a, \min\{m, a + \delta_a\})$. Then $a' \in (a, m) \subsetneq (a, b)$ and $a < a' < a + \delta_a$, and it follows that $f(a') < L_a + \varepsilon_a = M$.

Similarly, using the fact that $L_b = \lim_{x \rightarrow b^-} f(x)$, we fix $\delta_b > 0$ such that for all $x \in (a, b)$, if $b - \delta_b < x < b$, then $|f(x) - L_b| < \varepsilon_b$, that is, $L_b - \varepsilon_b < f(x) < L_b + \varepsilon_b$. Fix $b' \in (\max\{m, b - \delta_b\}, b)$. Then $b' \in (m, b) \subsetneq (a, b)$ and $b - \delta_b < b' < b$, and it follows that $M = L_b - \varepsilon_b < f(b')$.

We now have that $a < a' < m < b' < b$, and in particular, $a' < b'$ and $[a', b'] \subsetneq (a, b)$. So, since f is continuous on (a, b) , it is also continuous on $[a', b']$. Moreover, we have that $f(a') < M < f(b')$. So, by the Intermediate Value Theorem, there exists some $c \in (a, b)$ such that $f(c) = M$.

Case 2: $L_b < L_a$.

We now have that $L_b < M < L_a$. Here, we could proceed similarly as in the previous case. However, let us instead reduce the problem to Case 1, as follows. Consider the function $-f$. Since f is continuous on (a, b) , so is $-f$. Moreover, $-L_a = \lim_{x \rightarrow a^+} (-f)(x)$ and $-L_b = \lim_{x \rightarrow b^-} (-f)(x)$, and we have that $-L_a < -M < -L_b$. We are now back in Case 1, only for the function $-f$, the interval (a, b) , and the “intermediate value” $-M$. So, by Case 1, there exists some $c \in (a, b)$ such that $(-f)(c) = -M$. So, $f(c) = M$. \square

Remark. We proved in lecture that the sequence $\{(1 + \frac{1}{n})^n\}_{n=1}^\infty$ converges, and we defined $e := \lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n$. With some work, it can be proven that we have the following **function** limit:

$$\lim_{x \rightarrow \infty} (1 + \frac{1}{x})^x = e.$$

Note that in the limit above, x takes **real** (not just integer) values. You may use this fact below.

Exercise 6. Prove the following:

(c) $\lim_{x \rightarrow 0} (1 + x)^{1/x} = e$ (use the substitution $y = \frac{1}{x}$);

(d) $\lim_{x \rightarrow +\infty} (\frac{x+1}{x-1})^x = e^2$;

(e) $\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \ln a$, where $a > 0$ is a fixed constant (use the substitution $y = a^x - 1$);

Solution of (e).

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{a^x - 1}{x} &= \lim_{y \rightarrow 0} \frac{y}{\log_a(y+1)} && \text{using the substitution} \\ &= \lim_{y \rightarrow 0} \frac{y}{\frac{\ln(y+1)}{\ln a}} && y = a^x - 1, \text{ and so} \\ &= \lim_{y \rightarrow 0} \frac{\ln a}{\frac{1}{y} \ln(y+1)} && x = \log_a(y+1) \\ &= (\ln a) \left(\lim_{y \rightarrow 0} \frac{1}{\ln((y+1)^{1/y})} \right) \\ &= (\ln a) \frac{1}{\ln \left(\lim_{y \rightarrow 0} (y+1)^{1/y} \right)} \\ &= (\ln a) \frac{1}{\ln e} && \text{by part (a)} \\ &= \ln a. \end{aligned}$$

\square