Mathematical Analysis 1:
Tutorial #6

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Summer 2026

The Intermediate Value Theorem. *Let a and b be real numbers such that $a < b$, let $f : [a,b] \to \mathbb{R}$ be a continuous function, and let $M \in \mathbb{R}$ be such that* $\min\{f(a), f(b)\} < M < \max\{f(a), f(b)\}$.[1] *Then there exists some $c \in (a,b)$ such that $f(c) = M$.*

Exercise 1. *Prove that each of the following equations has a solution in the specified interval.*

(a) equation: $x^5 - 4x^4 - 1 = 0$*, interval:* $(0,5)$*;*

(b) equation: $\cos x = x$*, interval:* $(0,1)$*.*

Exercise 2. *Prove that each of the following equations has a (real) solution:*

(a) $\arctan x = 5 - x$*;* *(b)* $\cos x = -x^7$*;* *(c)* $\ln x = 3 - 2x$*.*

Exercise 3. *Let $a, b \in \mathbb{R}$ be such that $a < b$, and let $f : (a,b) \to \mathbb{R}$ be a continuous function. Assume that $L_a := \lim\limits_{x \to a^+} f(x)$ and $L_b := \lim\limits_{x \to b^-} f(x)$ both exist (as real numbers or as $\pm\infty$), and assume furthermore that $L_a \neq L_b$. Let $M \in \mathbb{R}$ be such that* $\min\{L_a, L_b\} < M < \max\{L_a, L_b\}$. *Prove that there exists some $c \in (a,b)$ such that $f(c) = M$.*

Exercise 4. *Prove that the equation* $\tan x = e^x$ *has a solution in the interval* $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Theorem 3.7.2. $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$.

Exercise 5. *Compute* $\lim\limits_{x \to 0} \frac{1 - \cos x}{x}$.

Remark. *We proved in lecture that the sequence $\{(1 + \frac{1}{n})^n\}_{n=1}^{\infty}$ converges, and we defined $e := \lim\limits_{n \to \infty} (1 + \frac{1}{n})^n$. With some work, it can be proven that we have the following **function** limit:*

$$\lim_{x \to \infty} (1 + \tfrac{1}{x})^x \quad = \quad e.$$

*Note that in the limit above, x takes **real** (not just integer) values. You may use this fact below.*

Exercise 6. *Prove the following:*

(a) $\lim\limits_{x \to 0} (1 + x)^{1/x} = e$ *(use the substitution $y = \frac{1}{x}$);*

(b) $\lim\limits_{x \to +\infty} \left(\frac{x+1}{x-1}\right)^x = e^2$*;*

(c) $\lim\limits_{x \to 0} \frac{a^x - 1}{x} = \ln a$*, where $a > 0$ is a fixed constant (use the substitution $y = a^x - 1$);*

[1] So, we are assuming that $f(a) \neq f(b)$, and that either $f(a) < M < f(b)$ or $f(b) < M < f(a)$.

Definition. *Let* $f : A \subseteq \mathbb{R} \to \mathbb{R}$ *be a function and let* $a \in \mathbb{R}$. *We say that* $x = a$ *is a* vertical asymptote *of* f *if at least one of* $\lim_{x\to a^-} f(x)$ *and* $\lim_{x\to a^+} f(x)$ *exists and is equal to* $+\infty$ *or* $-\infty$.

Definition. *For a function* $f : A \subseteq \mathbb{R} \to \mathbb{R}$ *and a real number* $b \in \mathbb{R}$, *we say that:*

- $y = b$ *is a* horizontal asymptote *of* f *as* $x \to +\infty$ *if* $\lim_{x\to+\infty} f(x) = b$;
- $y = b$ *is a* horizontal asymptote *of* f *as* $x \to -\infty$ *if* $\lim_{x\to-\infty} f(x) = b$;
- y *is a* horizontal asymptote *of* f *if at least one of* $\lim_{x\to+\infty} f(x)$ *and* $\lim_{x\to-\infty} f(x)$ *exists and is equal to* b.

Definition. *For a function* $f : A \subseteq \mathbb{R} \to \mathbb{R}$ *and real numbers* $a \neq 0$ *and* b, *we say that:*

- $y = ax + b$ *is a* slant asymptote *of* f *as* $x \to +\infty$ *if* $\lim_{x\to+\infty} \big(f(x) - (ax + b)\big) = 0$;[2]
- $y = ax + b$ *is a* slant asymptote *of* f *as* $x \to -\infty$ *if* $\lim_{x\to-\infty} \big(f(x) - (ax + b)\big) = 0$;[3]
- $y = ax + b$ *is a* slant asymptote *of* f *as* $x \to -\infty$ *if either* $\lim_{x\to+\infty} \big(f(x) - (ax + b)\big) = 0$ *or* $\lim_{x\to-\infty} \big(f(x) - (ax + b)\big) = 0$.

Proposition 3.6.15 1. *Let* $f : A \subseteq \mathbb{R} \to \mathbb{R}$ *be a function, and let* $a \neq 0$ *and* b *be real numbers. Then:*

(a) $y = ax + b$ *is a slant asymptote of* f *as* $x \to +\infty$ *if and only if the following hold:*

$$\lim_{x\to+\infty} \frac{f(x)}{x} = a \qquad \text{and} \qquad \lim_{x\to+\infty} \big(f(x) - ax\big) = b;$$

(b) $y = ax + b$ *is a slant asymptote of* f *as* $x \to -\infty$ *if and only if the following hold:*

$$\lim_{x\to-\infty} \frac{f(x)}{x} = a \qquad \text{and} \qquad \lim_{x\to-\infty} \big(f(x) - ax\big) = b.$$

Exercise 7. *For each of the following functions, compute (i) the domain of the function, (ii) all the vertical asymptotes, (iii) all the horizontal asymptotes, and (iv) all the slant asymptotes. In parts (iii) and (iv), distinguish between asymptotes at* $x \to +\infty$ *and* $x \to -\infty$.

(a) $f_1(x) = \frac{2x^3+6x^2+4x}{x^2+5x+6}$;

(b) $f_2(x) = \frac{3x-7}{x^2+2x-8}$;

(c) $f_3(x) = \frac{2x-5}{3x+7}$;

(d) $f_4(x) = \frac{x^7}{x^4+1}$.

Exercise 8. *For each of the following functions, compute (i) the domain of the function, (ii) all the vertical asymptotes, (iii) all the horizontal asymptotes, and (iv) all the slant asymptotes. In parts (iii) and (iv), distinguish between asymptotes at* $x \to +\infty$ *and* $x \to -\infty$.

(a) $f = \sqrt{x^2 - 3x + 2}$; *(b)* $g(x) = \frac{\arcsin x}{\sqrt{1-x^2}}$; *(c)* $h(x) = \sqrt[3]{x^2 - x^3}$.

[2]So, for very large x, we have that $f(x) \approx ax + b$.

[3]So, for a negative real number x of very large absolute value, we have that $f(x) \approx ax + b$.