Mathematical Analysis 1: Tutorial #5

Irena Penev
Summer 2026

Definition. *An* accumulation point *of a set* $A \subseteq \mathbb{R}$ *is a point* $a \in \mathbb{R}$ *(note that* a *may or may not belong to* A*) such that for all real numbers* $\varepsilon > 0$*, there exists some* $a' \in A$ *such that* $0 < |a' - a| < \varepsilon$*.*

Definition. *Let* $f : A \subseteq \mathbb{R} \to \mathbb{R}$ *be a function, let* $a \in \mathbb{R}$ *be an accumulation point of* A*, and let* $L \in \mathbb{R}$*. We say that* L is the limit of $f(x)$ as x approaches a*, or that* $f(x)$ tends to L as x approaches a*, provided that the following holds:*

> *for every* $\varepsilon > 0$*, there exists some* $\delta > 0$*, such that for all* $x \in A$*, if* $0 < |x - a| < \delta$*, then* $|f(x) - L| < \varepsilon$*.*

Under such circumstances, we write $L = \lim_{x \to a} f(x)$*.*

Definition. *A function* $f : A \subseteq \mathbb{R} \to \mathbb{R}$ *is* continuous *at a point* $a \in A$ *if the following holds:*

> *for all* $\varepsilon > 0$*, there exists some* $\delta > 0$*, such that for all* $x \in A$*, if* $|x - a| < \delta$*, then* $|f(x) - f(a)| < \varepsilon$*.*

If f *is continuous at all points* $a \in A$*, then we simply say that* f *is* continuous*. If* $I \subseteq A$ *is an interval, then we say that* f *is* continuous *on* I *provided that* $f \restriction I$ *is continuous at all points in* I*.*[1]

Theorem 3.3.1. *Let* $f : A \subseteq \mathbb{R} \to \mathbb{R}$*, let* $a \in \mathbb{R}$ *be an accumulation point of the set* A*, and let* $L \in \mathbb{R}$*. Then the following are equivalent:*

(i) $\lim_{x \to a} f(x) = L$*;*

(ii) for all sequences $\{a_n\}_{n=1}^{\infty}$ *of real numbers that all belong to the set* $A \setminus \{a\}$*, if* $\lim_{n \to \infty} a_n = a$*, then* $\lim_{n \to \infty} f(a_n) = L$*.*

The Intermediate Value Theorem. *Let* a *and* b *be real numbers such that* $a < b$*, let* $f : [a, b] \to \mathbb{R}$ *be a continuous function, and let* $M \in \mathbb{R}$ *be such that* $\min\{f(a), f(b)\} < M < \max\{f(a), f(b)\}$*.*[2] *Then there exists some* $c \in (a, b)$ *such that* $f(c) = M$*.*

The Extreme Value Theorem. *Let* $a, b \in \mathbb{R}$ *be such that* $a < b$*, and let* $f : [a, b] \to \mathbb{R}$ *be a continuous function. Then* f *reaches both a maximum and a minimum on* $[a, b]$*.*

[1] Here, $f \restriction I$ is the restriction of f to I. In other words, $(f \restriction I) : I \to \mathbb{R}$ is the function given by $(f \restriction I)(x) = f(x)$ for all $x \in I$.

[2] So, we are assuming that $f(a) \neq f(b)$, and that either $f(a) < M < f(b)$ or $f(b) < M < f(a)$.

Exercise 6 of Tutorial 4. *Let $f : \mathbb{R} \to \mathbb{R}$ given by*

$$f(x) \quad := \quad \begin{cases} 1 & \text{if} \quad x \text{ is rational} \\ 0 & \text{if} \quad x \text{ is irrational} \end{cases} \qquad \forall x \in \mathbb{R}.$$

Find the error in the following "proof":

> *Consider the sequences $\{\frac{1}{n}\}_{n=1}^{\infty}$. Clearly, $\lim_{n\to\infty} \frac{1}{n} = 0$, and $\lim_{n\to\infty} f\left(\frac{1}{n}\right) \overset{(*)}{=} \lim_{n\to\infty} 1 = 1$, where (*) follows from the fact that that $\frac{1}{n}$ is rational for all $n \in \mathbb{N}$. Therefore, by Theorem 3.3.1, $\lim_{x\to 0} f(x) = 1$.*

Exercise 1. *Let $a, b \in \mathbb{R}$ be such that $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Prove that $Im(f) := \{f(x) \mid x \in [a, b]\}$ is a (possibly degenerate) closed interval.*

> ***Terminology:*** *A "degenerate" closed interval is a set of the form $[x, x] = \{x\}$, where x is a real number.*
>
> ***Hint:*** *First use the Extreme Value Theorem to come up with a suitable "candidate" for the closed interval that you need, and then use the Mean Value Theorem to prove that your answer is correct.*

Exercise 2. *Let $a, b \in \mathbb{R}$ be such that $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a one-to-one continuous function.*

(a) Prove that if $f(a) < f(b)$, then f is strictly increasing, and moreover, $Im(f) = \big[f(a), f(b)\big]$.

(b) Prove that if $f(a) > f(b)$, then f is strictly decreasing, and moreover, $Im(f) = \big[f(b), f(a)\big]$.

> ***Remark:*** *Since f is one-to-one and $a \neq b$, we know that $f(a) \neq f(b)$.*

Exercise 3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous bijective function. Prove that $f^{-1} : \mathbb{R} \to \mathbb{R}$ is continuous.*

> ***Hint:*** *Fix $y_0 \in \mathbb{R}$ and $\varepsilon > 0$. You need to choose a suitable $\delta > 0$ such that for all $y \in \mathbb{R}$, if $|y - y_0| < \delta$, then $|f^{-1}(y) - f^{-1}(y_0)| < \varepsilon$. Take any $a, b \in \mathbb{R}$ such that $f^{-1}(y_0) - \varepsilon < a < f^{-1}(y_0) < b < f^{-1}(y_0) + \varepsilon$. Now apply Exercise 2 to f and the interval $[a, b]$, and then choose a suitable $\delta > 0$.*

Exercise 4. *Prove that any open or half open interval in $\mathbb{R}$ is the union of a nested family of closed intervals. More precisely, prove that for any open or half-open interval $I \subseteq \mathbb{R}$, there exists a sequence $\{I_n\}_{n=1}^{\infty}$ of closed intervals such that $I_1 \subseteq I_2 \subseteq I_3 \subseteq I_4 \subseteq \dots$ and $I = \bigcup_{n=1}^{\infty} I_n$.*

Exercise 5. *Let $\{I_n\}_{n=1}^{\infty}$ be a nested sequence of closed intervals in $\mathbb{R}$, i.e. one that satisfies $I_1 \subseteq I_2 \subseteq I_3 \subseteq I_4 \subseteq \dots$. Prove that $I = \bigcup_{n=1}^{\infty} I_n$ is an interval (open, closed, or half-open).*

Exercise 6. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous function, where I is an interval (open, closed, or half-open). Prove that $Im(f)$ is an interval (possibly a degenerate closed interval, i.e. one that contains only one point).*

Exercise 7. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous one-to-one function, where I is an interval (open, closed, or half-open). Prove that f is either strictly increasing or strictly decreasing, and that $Im(f)$ is an interval of the same type (open, closed, or half-open) as I.*