Mathematical Analysis 1:
Tutorial #4

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Exercise 2 of Tutorial 3. *Prove the following variant of the Comparison Test.*

> ***The Comparison Test (stronger version).*** *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers, and assume that there exists some $N \in \mathbb{N}$, such that for all $n \in \mathbb{N}$, if $n \geq N$, then $0 \leq a_n \leq b_n$. Then:*
>
> *(a) if the series $\sum\limits_{n=1}^{\infty} b_n$ converges, then so does the series $\sum\limits_{n=1}^{\infty} a_n$;*
>
> *(b) if the series $\sum\limits_{n=1}^{\infty} a_n$ diverges, then so does the series $\sum\limits_{n=1}^{\infty} b_n$.*

Warning: *Note that part (a) gives us no information about the relationship between the sums of the series $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$. This is because of the first N terms of the two sequences: the first N terms have no effect on convergence/divergence of a series, but if the series does converge, they do affect the value of the sum.*

Exercise 3 of Tutorial 3. *Prove the following theorem (which we can think of as a variant of the Comparison Test, or alternatively, as a kind of "Squeeze Theorem" for series).*

> ***Theorem.*** *Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, and $\{c_n\}_{n=1}^{\infty}$ be sequences of real numbers such that*
>
> $$a_n \quad \leq \quad b_n \quad \leq \quad c_n \qquad \forall n \in \mathbb{N}.$$
>
> *Then the following hold:*
>
> *(a) if the series $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} c_n$ converge, then so does $\sum\limits_{n=1}^{\infty} b_n$, and in this case,*
>
> $$\sum_{n=1}^{\infty} a_n \quad \leq \quad \sum_{n=1}^{\infty} b_n \quad \leq \quad \sum_{n=1}^{\infty} c_n;$$
>
> *(b) if the series $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} c_n$ converge absolutely, then so does $\sum\limits_{n=1}^{\infty} b_n$.*

*(**Hint:** For both parts, you will need to use the Comparison Test in a convenient way. For (b), start by explaining why $|b_n| \leq \max\{|a_n|, |c_n|\}$ $\forall n \in \mathbb{N}$.)*

Definition. *An* accumulation point *of a set $A \subseteq \mathbb{R}$ is a point $a \in \mathbb{R}$ (note that a may or may not belong to A) such that for all real numbers $\varepsilon > 0$, there exists some $a' \in A$ such that $0 < |a' - a| < \varepsilon$.*

Definition. *Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, let $a \in \mathbb{R}$ be an accumulation point of A, and let $L \in \mathbb{R}$. We say that L* is the limit of $f(x)$ as x approaches a, *or that $f(x)$* tends to L as x approaches a, *provided that the following holds:*

> *for every $\varepsilon > 0$, there exists some $\delta > 0$, such that for all $x \in A$, if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.*

Under such circumstances, we write $L = \lim\limits_{x \to a} f(x)$.

Exercise 1. *Compute the following limits. (You do **not** need to give a formal ε-δ proof.)*

(a) $\lim\limits_{x\to 2} \frac{x-2}{x+3}$; *(b)* $\lim\limits_{x\to 2} \frac{x^2-3x+2}{x^2-5x+6}$; *(c)* $\lim\limits_{x\to 1} \frac{3x^4-4x^3+1}{(x-1)^2}$; *(d)* $\lim\limits_{x\to 1} \frac{x^4-1}{x-1}$.

Exercise 2. *Compute the limit $\lim\limits_{x\to -2} \frac{x^2+7x+10}{x+2}$, and give an ε-δ proof of the correctness of you answer.*

Exercise 3. *Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by:*

$$f(x) \;:=\; \begin{cases} (x-3)^3 & \text{if} \;\; x > 3 \\ 0 & \text{if} \;\; x = 3 \\ |x-3| & \text{if} \;\; x < 3 \end{cases} \qquad \forall x \in \mathbb{R}.$$

Compute the limit $\lim\limits_{x\to 3} f(x)$, and give an ε-δ proof of the correctness of you answer. Is the function f continuous at $a = 3$?

Exercise 4. *Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by:*

$$f(x) \;:=\; \begin{cases} (x+1)^2 & \text{if} \;\; x < 0 \\ 0 & \text{if} \;\; x = 0 \\ |x-1| & \text{if} \;\; x > 0 \end{cases} \qquad \forall x \in \mathbb{R}.$$

Compute the limit $\lim\limits_{x\to 0} f(x)$, and give an ε-δ proof of the correctness of you answer. Is the function f continuous at $a = 0$?

Exercise 5. *Consider the constant function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 1$ for all $x \in \mathbb{R}$. Find the error in the following "proof":*

> *We claim that $\lim\limits_{x\to 3} f(x) = 0$. Fix a real number $\delta > 0$. Set $\varepsilon := 2$. Fix $x \in \mathbb{R}$ such that $0 < |x-3| < \delta$. Then $|f(x)-0| = |1-0| = 1 < \varepsilon$. This proves that $\lim\limits_{x\to 3} f(x) = 0$, as we had claimed.*

Theorem 3.3.1. *Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$, let $a \in \mathbb{R}$ be an accumulation point of the set A, and let $L \in \mathbb{R}$. Then the following are equivalent:*

- *(i)* $\lim\limits_{x\to a} f(x) = L$;
- *(ii) for all sequences $\{a_n\}_{n=1}^{\infty}$ of real numbers that all belong to the set $A \setminus \{a\}$, if $\lim\limits_{n\to\infty} a_n = a$, then $\lim\limits_{n\to\infty} f(a_n) = L$.*

Exercise 6. *Let $f : \mathbb{R} \to \mathbb{R}$ given by*

$$f(x) \;:=\; \begin{cases} 1 & \text{if} \;\; x \text{ is rational} \\ 0 & \text{if} \;\; x \text{ is irrational} \end{cases} \qquad \forall x \in \mathbb{R}.$$

Find the error in the following "proof":

> *Consider the sequences $\{\frac{1}{n}\}_{n=1}^{\infty}$. Clearly, $\lim\limits_{n\to\infty} \frac{1}{n} = 0$, and $\lim\limits_{n\to\infty} f\left(\frac{1}{n}\right) \overset{(*)}{=} \lim\limits_{n\to\infty} 1 = 1$, where (*) follows from the fact that that $\frac{1}{n}$ is rational for all $n \in \mathbb{N}$. Therefore, by Theorem 3.3.1, $\lim\limits_{x\to 0} f(x) = 1$.*