Mathematical Analysis 1:
Tutorial #3

Irena Penev
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Exercise 5 of Tutorial 2. *(a) Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers that coincide except possibly for the first few (finitely many) terms. More precisely, assume that there exists some $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N$, then $a_n = b_n$. Explain why $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.*

(b) Let $k \in \mathbb{N}$. Explain why $\sum_{n=k}^{\infty} \frac{1}{n}$ diverges.

Exercise 6 of Tutorial 2. *Determine which of the following series converge:*

(a) $\sum_{n=2}^{\infty} \frac{1}{n-\sqrt{n}}$;

(b) $\sum_{n=1}^{\infty} \frac{n+1}{n^2}$;

(c) $\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^2+1}$;

(d) $\sum_{n=1}^{\infty} \frac{n-1}{n2^n}$;

(e) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$;

(f) $\sum_{n=1}^{\infty} \frac{1+\sin n}{3^n}$;

(g) $\sum_{n=1}^{\infty} \frac{\sin n}{3^n}$;

(h) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$;

(i) $\sum_{n=1}^{\infty} \frac{1}{n^{1+(1/n)}}$;

(j) $\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{n!}$;

(k) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3n-1}$;

(l) $\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)}{n!}$;

(m) $\sum_{n=1}^{\infty} \left(-\frac{n}{5}\right)^n$;

(n) $\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2+1}$;

(o) $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\ln n}$;

(p) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[100]{n}}$.

Exercise 1. *Determine which of the following series converge. For those series that converge, determine if they converge absolutely or conditionally.*

(a) $\sum_{n=1}^{\infty} n^2 e^{-n^2}$;

(b) $\sum_{n=1}^{\infty} \frac{n-1}{n^2+1}$;

(c) $\sum_{n=1}^{\infty} \frac{1}{n^2+n+1}$;

(d) $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$;

(e) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$;

(f) $\sum_{n=0}^{\infty} \frac{(-5)^n}{(2n)!}$;

(g) $\sum_{n=1}^{\infty} (-1)^n \frac{5^n}{n^5}$;

(h) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$;

(i) $\sum_{n=1}^{\infty} \frac{\sin(4n)}{4^n}$;

(j) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{6-n}$;

(k) $\sum_{n=1}^{\infty} \frac{n(-5)^n}{7^n}$;

(l) $\sum_{n=1}^{\infty} \frac{7^n}{n \cdot 5^n}$;

(m) $\sum_{n=1}^{\infty} \frac{n^n}{2^{1+2n}}$;

(n) $\sum_{n=1}^{\infty} \left(\frac{1-n^2}{1+2n^2}\right)^n$;

(o) $\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n!}$;

(p) $\sum_{n=1}^{\infty} \frac{n^5+1}{n^6+1}$;

(q) $\sum_{n=1}^{\infty} \frac{(2n)^n}{n^{2n}}$;

(r) $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$;

(s) $\sum_{n=1}^{\infty} \left(\sqrt[n]{2}-1\right)^n$;

(t) $\sum_{n=1}^{\infty} \frac{1}{n+n\sin^2 n}$.

Exercise 2. *Prove the following variant of the Comparison Test.*

> ***The Comparison Test (stronger version).*** *Let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be sequences of real numbers, and assume that there exists some* $N \in \mathbb{N}$*, such that for all* $n \in \mathbb{N}$*, if* $n \geq N$*, then* $0 \leq a_n \leq b_n$*. Then:*
>
> *(a) if the series* $\sum_{n=1}^{\infty} b_n$ *converges, then so does the series* $\sum_{n=1}^{\infty} a_n$*;*
>
> *(b) if the series* $\sum_{n=1}^{\infty} a_n$ *diverges, then so does the series* $\sum_{n=1}^{\infty} b_n$*.*

Warning: *Note that part (a) gives us no information about the relationship between the sums of the series* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$*. This is because of the first* N *terms of the two sequences: the first* N *terms have no effect on convergence/divergence of a series, but if the series does converge, they do affect the value of the sum.*

Exercise 3. *Prove the following theorem (which we can think of as a variant of the Comparison Test, or alternatively, as a kind of "Squeeze Theorem" for series).*

> ***Theorem.*** *Let* $\{a_n\}_{n=1}^{\infty}$*,* $\{b_n\}_{n=1}^{\infty}$*, and* $\{c_n\}_{n=1}^{\infty}$ *be sequences of real numbers such that*
>
> $$a_n \quad \leq \quad b_n \quad \leq \quad c_n \qquad \forall n \in \mathbb{N}.$$
>
> *Then the following hold:*
>
> *(a) if the series* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} c_n$ *converge, then so does* $\sum_{n=1}^{\infty} b_n$*, and in this case,*
>
> $$\sum_{n=1}^{\infty} a_n \quad \leq \quad \sum_{n=1}^{\infty} b_n \quad \leq \quad \sum_{n=1}^{\infty} c_n;$$
>
> *(b) if the series* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} c_n$ *converge absolutely, then so does* $\sum_{n=1}^{\infty} b_n$*.*

*(**Hint:** For both parts, you will need to use the Comparison Test in a convenient way. For (b), start by explaining why* $|b_n| \leq \max\{|a_n|, |c_n|\} \;\; \forall n \in \mathbb{N}$*.)*