Mathematical Analysis 1:
Tutorial #2

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Exercise 1. *For each of the following sequences $\{a_n\}_{n=1}^{\infty}$ of real numbers, compute:*

(i) *the first few terms of the sequence* $\{a_n\}_{n=1}^{\infty}$;

(ii) *the first few terms of the sequence* $\{\sup\limits_{n \geq k} a_n\}_{k=1}^{\infty}$;

(iii) *the first few terms of the sequence* $\{\inf\limits_{n \geq k} a_n\}_{k=1}^{\infty}$;

(iv) $\limsup\limits_{n \to \infty} a_n$;

(v) $\liminf\limits_{n \to \infty} a_n$.

Using your answer to (iv) and (v) above, either compute $\lim\limits_{n \to \infty} a_n$, *or explain why it doesn't exist.*

(a) $a_n = \frac{1}{n}$ *for all* $n \in \mathbb{N}$;

(b) $a_n = -\frac{1}{n}$ *for all* $n \in \mathbb{N}$;

(c) $a_n = \frac{(-1)^n}{\sqrt{n}}$ *for all* $n \in \mathbb{N}$;

(d) $a_n = n$ *for all* $n \in \mathbb{N}$;

(e) $a_n = -n$ *for all* $n \in \mathbb{N}$;

(f) $a_n = (-1)^n n$ *for all* $n \in \mathbb{N}$;

(g) $a_n = (-1)^n + n$ *for all* $n \in \mathbb{N}$;

(h) $a_n = (-1)^n + \frac{1}{n}$.

Exercise 2. *Let* $p \in (0,1)$. *Prove that the series* $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ *diverges.*

Exercise 3. *(a) Show that the series* $\sum\limits_{n=1}^{\infty} \frac{1}{n(n+1)}$ *converges and find its sum.*

> ***Hint:*** *Express* $\frac{1}{n(n+1)}$ *as a difference of two fractions, and then find a nice formula for the k-th partial sum of the series.*

(b) Using part (a) and the Comparison Test, prove that the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ *converges. (You don't need to find the sum of this series. Just prove convergence.)*

> ***Hint:*** *Note that* $0 < \frac{1}{n(n+1)} < \frac{1}{n^2}$ *for all* $n \in \mathbb{N}$, *and so you cannot apply the Comparison Test to the series* $\sum\limits_{n=1}^{\infty} \frac{1}{n(n+1)}$ *and* $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ *directly. Slightly "edit" one of the two series first, and then apply the test.*

(c) *Using part (b) and the Comparison Test, prove that for all real numbers* $p \geq 2$*, the series* $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ *converges. (You don't need to find the sum of this series. Just prove convergence.)*

> ***Remark:*** *In fact, the series* $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ *converges for all real numbers* $p > 1$*. However, you are not asked to prove this.*

Exercise 4. *Show that the series* $\sum\limits_{n=1}^{\infty} \left(\frac{1}{3^n} - \frac{2}{n(n+1)} \right)$ *converges, and compute its sum.*

Exercise 5. *(a) Let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be sequences of real numbers that coincide except possibly for the first few (finitely many) terms. More precisely, assume that there exists some* $N \in \mathbb{N}$ *such that for all* $n \in \mathbb{N}$*, if* $n \geq N$*, then* $a_n = b_n$*. Explain why* $\sum\limits_{n=1}^{\infty} a_n$ *and* $\sum\limits_{n=1}^{\infty} b_n$ *either both converge or both diverge.*

(b) *Let* $k \in \mathbb{N}$*. Explain why* $\sum\limits_{n=k}^{\infty} \frac{1}{n}$ *diverges.*

Exercise 6. *Determine which of the following series converge:*

(a) $\sum\limits_{n=2}^{\infty} \frac{1}{n-\sqrt{n}}$;

(b) $\sum\limits_{n=1}^{\infty} \frac{n+1}{n^2}$;

(c) $\sum\limits_{n=1}^{\infty} \frac{\cos^2(n)}{n^2+1}$;

(d) $\sum\limits_{n=1}^{\infty} \frac{n-1}{n2^n}$;

(e) $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$;

(f) $\sum\limits_{n=1}^{\infty} \frac{1+\sin n}{3^n}$;

(g) $\sum\limits_{n=1}^{\infty} \frac{\sin n}{3^n}$;

(h) $\sum\limits_{n=1}^{\infty} \frac{n!}{n^n}$;

(i) $\sum\limits_{n=1}^{\infty} \frac{1}{n^{1+(1/n)}}$;

(j) $\sum\limits_{n=1}^{\infty} (-1)^n \frac{n^n}{n!}$;

(k) $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{3n-1}$;

(l) $\sum\limits_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)}{n!}$;

(m) $\sum\limits_{n=1}^{\infty} \left(-\frac{n}{5}\right)^n$;

(n) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{4n^2+1}$;

(o) $\sum\limits_{n=2}^{\infty} \frac{(-1)^{n+1}}{\ln n}$;

(p) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[100]{n}}$.