

# Mathematical Analysis 1

## Lecture #15

Improper integrals. Applications of integration

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- This lecture has four parts:
  - ① Removable and jump discontinuities
  - ② Improper integrals
  - ③ Approximating the definite integral
  - ④ Applications of integration

① Removable and jump discontinuities

- Reminder:

Theorem 5.2.5

Let  $a, b \in \mathbb{R}$  be s.t.  $a < b$ , and let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function. Then  $f$  is Riemann integrable on  $[a, b]$ .

- As we shall see, some discontinuous functions are still Riemann integrable.

### Proposition 5.7.1

Let  $a, b \in \mathbb{R}$  be such that  $a < b$ , and let  $f, g : [a, b] \rightarrow \mathbb{R}$  be bounded functions such that  $f \upharpoonright (a, b) = g \upharpoonright (a, b)$ .<sup>a</sup> Then  $f$  is Riemann integrable on  $[a, b]$  if and only if  $g$  is Riemann integrable on  $[a, b]$ , and in this case, we have  $\int_a^b f(x)dx = \int_a^b g(x)dx$ .

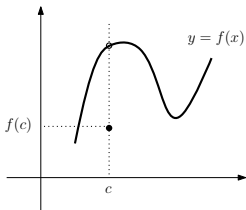
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<sup>a</sup>In other words, we are assuming that  $f$  and  $g$  agree on the open interval  $(a, b)$ , but may possibly differ on  $a$  and on  $b$ , that is, it is possible that  $f(a) \neq g(a)$  and/or  $f(b) \neq g(b)$ .

- Proof: Lecture Notes.

## Definition

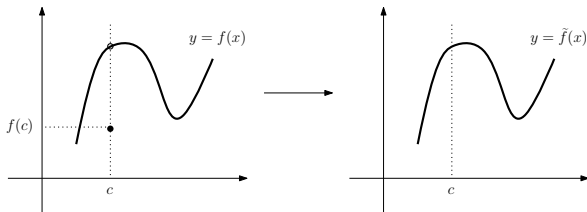
Let  $I \subseteq \mathbb{R}$  be an interval, and let  $f : I \rightarrow \mathbb{R}$  be a function, and let  $c \in I$ . We say that  $f$  has a *removable discontinuity* at  $c \in I$  if  $\lim_{x \rightarrow c} f(x)$  exists (and is a real number), but  $\lim_{x \rightarrow c} f(x) \neq f(c)$ .



## Definition

Let  $I \subseteq \mathbb{R}$  be an interval, and let  $f : I \rightarrow \mathbb{R}$  be a function, and let  $c \in I$ . We say that  $f$  has a *removable discontinuity* at  $c \in I$  if  $\lim_{x \rightarrow c} f(x)$  exists (and is a real number), but  $\lim_{x \rightarrow c} f(x) \neq f(c)$ .

- **Remark:** Removable discontinuities are called “removable” because the function can be made continuous at the point in question simply by changing the value of the function at this point (see the picture below).



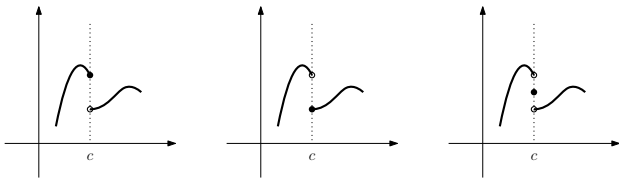
## Definition

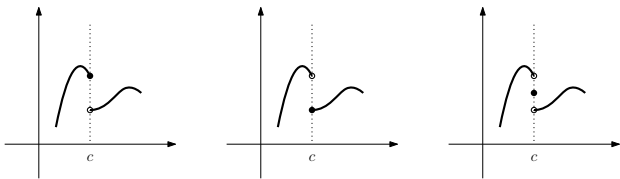
Let  $I \subseteq \mathbb{R}$  be an interval, and let  $f : I \rightarrow \mathbb{R}$  be a function, and let  $c \in I$ . We say that  $f$  has a *jump discontinuity* if the following three conditions are satisfied:

- $c \in I^0$  (i.e.  $c$  belongs to the interior of  $I$ );<sup>a</sup>
- $\lim_{x \rightarrow c^-} f(x)$  exists and is a real number;
- $\lim_{x \rightarrow c^+} f(x)$  exists and is a real number;
- $\lim_{x \rightarrow c^-} f(x) \neq \lim_{x \rightarrow c^+} f(x)$ .

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<sup>a</sup>This means that there exists some  $\delta > 0$  such that  $(c - \delta, c + \delta) \subseteq I$ .





- Note that, for  $I_c^- := I \cap (-\infty, c]$ , the function  $f \upharpoonright I_c^-$  is either continuous at  $c$  (this happens in the first of the three graphs above), or has a removable discontinuity at  $c$  (this happens in the second and third of the three graphs above).
- An analogous remark applies to  $I_c^+ := I \cap [c, +\infty)$ : the function  $f \upharpoonright I_c^+$  is either continuous at  $c$  (this happens in the second of the three pictures above), or has a removable discontinuity at  $c$  (this happens in the first and third of the three graphs above).

- Reminder:

### Theorem 5.3.6

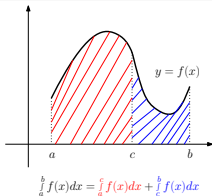
Let  $f : A \subseteq \mathbb{R} \rightarrow \mathbb{R}$ , and let  $a, b, c \in \mathbb{R}$ . Then the following hold:

- (a) if  $\int_a^b f(x)dx$  exists, then so does  $\int_b^a f(x)dx$ , and in this case

$$\int_b^a f(x)dx = -\int_a^b f(x)dx;$$

- (b) if  $\int_a^c f(x)dx$  and  $\int_c^b f(x)dx$  both exist, then so does  $\int_a^b f(x)dx$ , and in this case,

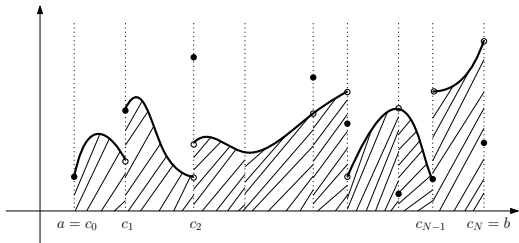
$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

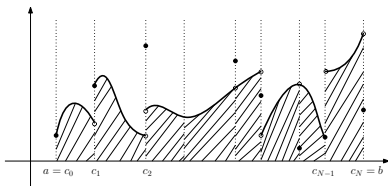


## Theorem 5.7.2

Let  $a, b \in \mathbb{R}$  be such that  $a < b$ , and let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded function. Assume that  $f$  is continuous at all but at most finitely many points in  $[a, b]$ , and assume moreover that all points at which  $f$  is discontinuous are either removable or jump discontinuities. Then  $f$  is Riemann integrable.

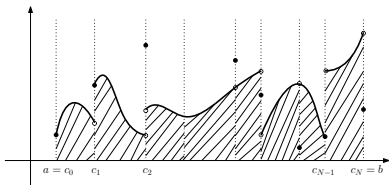
*Proof.* Consider a partition  $P = \{c_0, c_1, \dots, c_N\}$ , with  $a = c_0 < c_1 < \dots < c_N = b$ , such that  $f$  is continuous at all points in  $[a, b] \setminus P$ ; such a  $P$  exists because  $f$  is continuous at all but at most finitely many points in  $f$ .





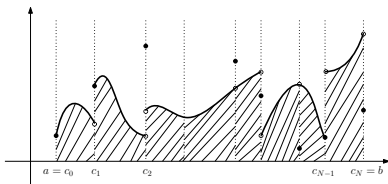
*Proof (continued).* For all  $k \in \{1, \dots, N\}$ , define  $f_k := f \upharpoonright [c_{k-1}, c_k]$ . It now suffices to show that for all  $k \in \{1, \dots, N\}$ , the function  $f_k$  is Riemann integrable on  $[c_{k-1}, c_k]$ , for Theorem 5.3.6(b) will then immediately imply that  $f$  is Riemann integrable on  $[a, b] = [c_0, c_N]$ , and that

$$\int_a^b f(x) dx = \int_{c_0}^{c_N} f(x) dx = \sum_{k=1}^N \int_{c_{k-1}}^{c_k} f_k(x) dx.$$



*Proof (continued).* So, fix  $k \in \{1, \dots, N\}$ ; WTS  $f_k$  is Riemann integrable on  $[c_{k-1}, c_k]$ . Note that  $f_k$  is continuous on  $(c_{k-1}, c_k)$ , and is either continuous or has a removable discontinuity at each of the points  $c_{k-1}, c_k$ . In particular, both  $\lim_{x \rightarrow c_{k-1}^+} f_k(x)$  and

$\lim_{x \rightarrow c_k^-} f_k(x)$  exist and are real numbers.



*Proof (continued).* Now, define  $g_k : [c_{k-1}, c_k] \rightarrow \mathbb{R}$  by setting

$$g_k(x) = \begin{cases} \lim_{x \rightarrow c_{k-1}^+} f_k(x) & \text{if } x = c_{k-1} \\ f_k(x) & \text{if } c_{k-1} < x < c_k \\ \lim_{x \rightarrow c_k^-} f_k(x) & \text{if } x = c_k \end{cases}$$

for all  $x \in [c_{k-1}, c_k]$ . By construction, we have that  $f_k \upharpoonright (a, b) = g_k \upharpoonright (a, b)$ . Moreover,  $g_k$  is continuous on  $[c_{k-1}, c_k]$ , and consequently (by Theorem 5.2.5),  $g_k$  is Riemann integrable on  $[c_{k-1}, c_k]$ . So, by Proposition 5.7.1,  $f_k$  is also Riemann integrable on  $[c_{k-1}, c_k]$ , and we are done.  $\square$

### Theorem 5.7.2

Let  $a, b \in \mathbb{R}$  be such that  $a < b$ , and let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded function. Assume that  $f$  is continuous at all but at most finitely many points in  $[a, b]$ , and assume moreover that all points at which  $f$  is discontinuous are either removable or jump discontinuities. Then  $f$  is Riemann integrable.

### Example 5.7.3

Consider the function  $f : [-3, 5] \rightarrow \mathbb{R}$  given by

$$f(x) = \begin{cases} -7 & \text{if } -3 \leq x < 0 \\ -1 & \text{if } x = 0 \\ e^x & \text{if } 0 < x \leq 5 \end{cases}$$

Determine whether  $f$  is Riemann integrable on  $[-3, 5]$ , and if so, compute  $\int_{-3}^5 f(x) dx$ .

*Solution.* Note that

- $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (-7) = -7$ ;
- $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} e^x = 1$ .

So,  $f$  has a jump discontinuity at  $x = 0$ , and clearly,  $f$  is continuous at all other points in  $[-3, 5]$ . So,  $f$  is Riemann integrable.

*Solution (continued).* We now compute:

$$\int_{-3}^5 f(x) dx = \int_{-3}^0 f(x) dx + \int_0^5 f(x) dx$$

$$= \int_{-3}^0 -7 dx + \int_0^5 e^x dx$$

by Proposition 5.7.1

$$= (-7x) \Big|_{-3}^0 + (e^x) \Big|_0^5$$

$$= (-7 \cdot 0 - 7(-3)) + (e^5 - e^0)$$

$$= 21 + e^5 - 1$$

$$= 20 + e^5.$$



## 2 Improper integrals

### Definition

Suppose that  $a, b \in \mathbb{R} \rightarrow \mathbb{R}$  are such that  $a < b$ .

- For a function  $f : [a, b) \rightarrow \mathbb{R}$ , we define the *improper integral*

$$\int_a^b f(x)dx = \lim_{t \rightarrow b^-} \int_a^t f(x)dx,$$

provided that  $f$  is Riemann integrable on all intervals of the form  $[a, t]$  with  $t \in (a, b)$ , and provided the limit exists.

- For a function  $f : [a, +\infty) \rightarrow \mathbb{R}$ , we define the *improper integral*

$$\int_a^{+\infty} f(x)dx = \lim_{t \rightarrow +\infty} \int_a^t f(x)dx,$$

provided that  $f$  is Riemann integrable on all intervals of the form  $[a, t]$  with  $t \in (a, +\infty)$ , and provided the limit exists.

## Definition (continued)

Suppose that  $a, b \in \mathbb{R} \rightarrow \mathbb{R}$  are such that  $a < b$ .

- For a function  $f : (a, b] \rightarrow \mathbb{R}$ , we define the *improper integral*

$$\int_a^b f(x) dx = \lim_{t \rightarrow a^+} \int_t^b f(x) dx,$$

provided that  $f$  is Riemann integrable on all intervals of the form  $[t, b]$  with  $t \in (a, b)$ , and provided the limit exists.

- For a function  $f : (-\infty, b] \rightarrow \mathbb{R}$ , we define the *improper integral*

$$\int_{-\infty}^b f(x) dx = \lim_{t \rightarrow -\infty} \int_t^b f(x) dx,$$

provided that  $f$  is Riemann integrable on all intervals of the form  $[t, b]$  with  $t \in (-\infty, b)$ , and provided the limit exists.

In each of the four cases above, if the limit in question exists and is a real number, then we say that the improper integral in question is *convergent*. Otherwise (i.e. if the limit is equal to  $+\infty$  or  $-\infty$ , or if the limit simply does not exist), we say that the improper integral in question is *divergent*.

### Example 5.8.1

Compute the values of the following improper integrals (if they exist), and determine whether the integrals are convergent or divergent:

(a)  $\int_0^1 \frac{dx}{x};$

(b)  $\int_0^1 \frac{dx}{\sqrt{x}}.$

*Solution.* Note that both integrals are improper: the functions  $\frac{1}{x}$  and  $\frac{1}{\sqrt{x}}$  are both defined (and continuous) on  $(0, 1]$ , but they are undefined at 0.

*Solution (continued).* (a)

$$\begin{aligned}\int_0^1 \frac{dx}{x} &= \lim_{t \rightarrow 0^+} \int_t^1 \frac{dx}{x} \\ &= \lim_{t \rightarrow 0^+} \left( \ln |x| \Big|_t^1 \right) \\ &= \lim_{t \rightarrow 0^+} (\ln 1 - \ln t) \\ &= \lim_{t \rightarrow 0^+} (-\ln t) \\ &= +\infty.\end{aligned}$$

So, the improper integral  $\int_0^1 \frac{dx}{x}$  is divergent.

*Solution (continued).* (b)

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{t \rightarrow 0^+} \int_t^1 \frac{dx}{\sqrt{x}} = \lim_{t \rightarrow 0^+} \left( (2\sqrt{x}) \Big|_t^1 \right) = \lim_{t \rightarrow 0^+} (2 - 2\sqrt{t}) = 2.$$

So, the improper integral is convergent.  $\square$

### Proposition 5.8.2

Let  $p \in \mathbb{R}$ . Then the following hold:

- (a) if  $p > 1$ , then  $\int_1^{+\infty} \frac{dx}{x^p} = \frac{1}{p-1}$ , and in particular, the improper integral  $\int_1^{\infty} \frac{dx}{x^p}$  is convergent;
- (b) if  $p \leq 1$ , then  $\int_1^{+\infty} \frac{dx}{x^p} = +\infty$ , and in particular, the improper integral  $\int_1^{\infty} \frac{dx}{x^p}$  is divergent.

*Proof.* We prove (a) and (b) simultaneously. First, if  $p = 1$ , then

$$\begin{aligned} \int_1^{+\infty} \frac{dx}{x^p} &= \int_1^{+\infty} \frac{dx}{x} = \lim_{t \rightarrow +\infty} \int_1^t \frac{dx}{x} = \lim_{t \rightarrow +\infty} \left( \ln|x| \Big|_1^t \right) \\ &= \lim_{t \rightarrow +\infty} (\ln t - \ln 1) = \lim_{t \rightarrow +\infty} \ln t = +\infty, \end{aligned}$$

as required.

*Proof (continued).* From now on, we assume that  $p \neq 1$ . Then for all  $t \in (1, +\infty)$ , we have the following:

$$\int_1^t \frac{dx}{x^p} = \int_1^t x^{-p} dx = \left( \frac{1}{-p+1} x^{-p+1} \right) \Big|_1^t = \frac{1}{1-p} (t^{1-p} - 1).$$

Suppose first that  $p > 1$ . Then  $1 - p < 0$ , and consequently,  $\lim_{t \rightarrow +\infty} t^{1-p} = 0$ , which in turn implies that

$$\int_1^{+\infty} \frac{dx}{x^p} = \lim_{t \rightarrow +\infty} \int_1^t \frac{dx}{x^p} = \lim_{t \rightarrow +\infty} \frac{1}{1-p} (t^{1-p} - 1) = \frac{1}{p-1}.$$

Suppose now that  $p < 1$ . Then  $1 - p > 0$ , and consequently,  $\lim_{t \rightarrow +\infty} t^{1-p} = +\infty$ , which in turn implies that

$$\int_1^{+\infty} \frac{dx}{x^p} = \lim_{t \rightarrow +\infty} \int_1^t \frac{dx}{x^p} = \lim_{t \rightarrow +\infty} \frac{1}{1-p} (t^{1-p} - 1) = +\infty.$$

This completes the argument.  $\square$

## Proposition 5.8.2

Let  $p \in \mathbb{R}$ . Then the following hold:

- Ⓐ if  $p > 1$ , then  $\int_1^{+\infty} \frac{dx}{x^p} = \frac{1}{p-1}$ , and in particular, the improper integral  $\int_1^{\infty} \frac{dx}{x^p}$  is convergent;
- Ⓑ if  $p \leq 1$ , then  $\int_1^{+\infty} \frac{dx}{x^p} = +\infty$ , and in particular, the improper integral  $\int_1^{\infty} \frac{dx}{x^p}$  is divergent.

## Definition

Let  $a, b \in \mathbb{R} \cup \{\pm\infty\}$  be such that  $a < b$ , and let  $f : (a, b) \rightarrow \mathbb{R}$  be a function that is Riemann integrable on all closed intervals included in  $(a, b)$ .<sup>a</sup> We then define the *improper integral*

$$\int_a^b f(x) dx := \int_a^c f(x) dx + \int_c^b f(x) dx,$$

where  $c \in (a, b)$  is chosen arbitrarily, provided that both  $\int_a^c f(x) dx$  and  $\int_c^b f(x) dx$  exist (as real numbers or as  $\pm\infty$ ), as does their sum.

If this sum is a real number, then we say that  $\int_a^b f(x) dx$  is convergent. Otherwise, we say that  $\int_a^b f(x) dx$  is divergent.

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<sup>a</sup>Thus, we are assuming that for all  $a', b' \in (a, b)$  such that  $a' < b'$ , the function  $f$  is bounded and Riemann integrable on  $[a', b']$ . However, it is possible that  $f$  is **not** bounded on the entire open interval  $(a, b)$ .

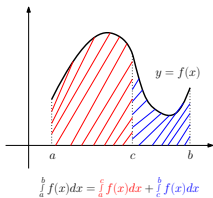
• **Remarks:** Regarding the definition from the previous slide:

① We consider “ $\infty - \infty$ ” to be undefined.

- In particular, if  $\int_a^c f(x)dx = +\infty$  and  $\int_c^b f(x)dx = -\infty$  (or vice versa), then  $\int_a^b f(x)dx$  is undefined, and we say that  $\int_a^b f(x)dx$  is divergent.

② It can be shown that the existence and the value of  $\int_a^b f(x)dx$  do **not** depend on the choice of  $c$ .

- In other words, we will get the same answer concerning the existence and value of  $\int_a^b f(x)dx$  for every single  $c \in (a, b)$ .
- This essentially follows from Theorem 5.3.6, but we omit the details.



### Example 5.8.3

Compute the value of  $\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$  (if it exists). Is  $\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$  convergent or divergent?

*Solution.* We compute:

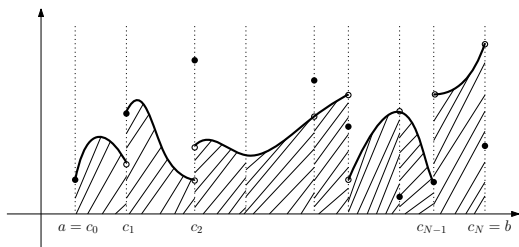
$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} &= \int_{-\infty}^0 \frac{dx}{1+x^2} + \int_0^{+\infty} \frac{dx}{1+x^2} \\ &= \left( \lim_{t \rightarrow -\infty} \int_t^0 \frac{dx}{1+x^2} \right) + \left( \lim_{t \rightarrow +\infty} \int_0^t \frac{dx}{1+x^2} \right) \\ &= \lim_{t \rightarrow -\infty} \left( \arctan x \Big|_t^0 \right) + \lim_{t \rightarrow +\infty} \left( \arctan x \Big|_0^t \right) \\ &= \lim_{t \rightarrow -\infty} \left( \arctan 0 - \arctan t \right) + \lim_{t \rightarrow +\infty} \left( \arctan t - \arctan 0 \right) \\ &= \frac{\pi}{2} + \frac{\pi}{2} = \pi.\end{aligned}$$

Thus, the improper integral  $\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$  is convergent.  $\square$

- Reminder:

### Theorem 5.7.2

Let  $a, b \in \mathbb{R}$  be such that  $a < b$ , and let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded function. Assume that  $f$  is continuous at all but at most finitely many points in  $[a, b]$ , and assume moreover that all points at which  $f$  is discontinuous are either removable or jump discontinuities. Then  $f$  is Riemann integrable.



- We now turn to the case when a function  $f$  is defined at all but possibly finitely many points in an interval, and may possibly be defined but discontinuous at a finite number of points in this interval.
  - Of course, Theorem 5.7.2 deals with removable and jump discontinuities, but we would like to be able to handle discontinuities that are not necessarily of these two types.
  - To this end, we give the definition on the next slide.
  - This definition generalizes both the Riemann integral, and the improper integrals that we have defined thus far.

## Definition

Let  $c_0, c_1, \dots, c_N \in \mathbb{R} \cup \{\pm\infty\}$  be such that  $c_0 < c_1 < \dots < c_N$ , and let  $f : A \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be such that

$(c_0, c_1) \cup (c_1, c_2) \cup \dots \cup (c_{N-1}, c_N) \subseteq A$ . Assume that for all  $k \in \{1, \dots, N\}$ , the integral  $\int_{c_{k-1}}^{c_k} f(x) dx$  exists (either as a real

number or as  $\pm\infty$ ; here, it is possible that  $\int_{c_{k-1}}^{c_k} f(x) dx$  is an ordinary Riemann integral or is an improper integral). We define

$$\int_{c_0}^{c_N} f(x) dx := \sum_{k=1}^N \int_{c_{k-1}}^{c_k} f(x) dx,$$

provided that the sum exists (either as a real number or as  $\pm\infty$ ).

If the sum exists and is a real number, then we say that  $\int_{c_0}^{c_N} f(x) dx$

is *convergent*. Otherwise, we say that  $\int_{c_0}^{c_N} f(x) dx$  is *divergent*.

## Definition (continued)

Moreover, we define

$$\int_{c_N}^{c_0} f(x)dx := - \int_{c_0}^{c_N} f(x)dx,$$

provided that  $\int_{c_0}^{c_N} f(x)dx$  exists.

- **Remark:** As before, we consider “ $\infty - \infty$ ” to be undefined. So, if one of the integrals  $\int_{c_0}^{c_1} f(x)dx, \dots, \int_{c_{N-1}}^{c_N} f(x)dx$  is equal to  $+\infty$ , while another one is equal to  $-\infty$ , then  $\int_{c_0}^{c_N} f(x)dx$  is simply undefined, and we say that  $\int_{c_0}^{c_N} f(x)dx$  is divergent.

- **Remark:** Admittedly, our definition raises some questions.
  - For instance, might the convergence/divergence or the value of  $\int_{c_0}^{c_N} f(x)dx$  change if we choose different intermediate points  $c_1, \dots, c_{N-1}$ ?
  - Also, if  $f$  is defined and bounded on some  $[c_{k-1}, c_k]$ , might we get different values for the integral  $\int_{c_{k-1}}^{c_k} f(x)dx$  by interpreting it as an ordinary Riemann integral on the one hand, and as one of the improper integrals defined above on the other?
  - In fact, it can be shown that none of these pathologies can occur; in view of Theorem 5.7.2, this should not be too surprising, but we omit the full details.

### Theorem 5.7.2

Let  $a, b \in \mathbb{R}$  be such that  $a < b$ , and let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded function. Assume that  $f$  is continuous at all but at most finitely many points in  $[a, b]$ , and assume moreover that all points at which  $f$  is discontinuous are either removable or jump discontinuities. Then  $f$  is Riemann integrable.

- **Remark:** Suppose that for some  $a, b \in \mathbb{R} \cup \{\pm\infty\}$  such that  $a < b$ , and for some function  $f$ , the integral  $\int_a^b f(x)dx$  is defined.
  - It can then be shown that according to the same definition, the integral  $\int_{a'}^{b'} f(x)dx$  is defined for all  $a', b' \in \mathbb{R} \cup \{\pm\infty\}$  such that  $a \leq a' \leq b' \leq b$ .
  - Once again, we omit the technical details.

### Example 5.8.4

Compute the values of the integrals below (if they exist).  
Determine whether the integrals are convergent or divergent.

(a)  $\int_{-1}^1 \frac{dx}{x};$

(b)  $\int_{-2}^3 \frac{dx}{\sqrt[3]{x}}.$

*Solution.* (a) The function  $\frac{1}{x}$  is undefined at 0. So, we have that

$$\int_{-1}^1 \frac{dx}{x} = \int_{-1}^0 \frac{dx}{x} + \int_0^1 \frac{dx}{x},$$

provided that both improper integrals on the right-hand side are defined, as is their sum. We know that  $\int \frac{dx}{x} = \ln|x| + C$ .

Therefore,

$$\int_{-1}^0 \frac{dx}{x} = \lim_{t \rightarrow 0^-} \int_{-1}^t \frac{dx}{x} = \lim_{t \rightarrow 0^-} \left( \ln|x| \Big|_{-1}^t \right) = \lim_{t \rightarrow 0^-} (\ln|t| - \ln 1) = -\infty.$$

Similarly,

$$\int_0^1 \frac{dx}{x} = \lim_{t \rightarrow 0^+} \int_t^1 \frac{dx}{x} = \lim_{t \rightarrow 0^+} \left( \ln|x| \Big|_t^1 \right) = \lim_{t \rightarrow 0^+} (\ln 1 - \ln|t|) = +\infty.$$

### Example 5.8.4

Compute the values of the integrals below (if they exist). Determine whether the integrals are convergent or divergent.

(a)  $\int_{-1}^1 \frac{dx}{x};$

(b)  $\int_{-2}^3 \frac{dx}{\sqrt[3]{x}}.$

*Solution (continued).* (a) Reminder:

•  $\int_{-1}^0 \frac{d}{dx} = -\infty;$

•  $\int_0^1 \frac{d}{dx} = +\infty.$

Since “ $\infty - \infty$ ” is undefined,  $\int_{-1}^1 \frac{dx}{x}$  does not exist. In particular,

the integral  $\int_{-1}^1 \frac{dx}{x}$  is divergent.

*Solution (continued).* (b) The function  $\frac{1}{\sqrt[3]{x}}$  is undefined at  $x = 0$ . So, we have that

$$\int_{-2}^3 \frac{dx}{\sqrt[3]{x}} = \int_{-2}^0 \frac{dx}{\sqrt[3]{x}} + \int_0^3 \frac{dx}{\sqrt[3]{x}},$$

provided that both improper integrals on the right-hand side are defined, as is their sum. First, note that

$$\int \frac{dx}{\sqrt[3]{x}} = \int x^{-1/3} dx = \frac{3}{2}x^{2/3} + C.$$

Thus, we have that

$$\begin{aligned} \int_{-2}^0 \frac{dx}{\sqrt[3]{x}} &= \lim_{t \rightarrow 0^-} \int_{-2}^t \frac{dx}{\sqrt[3]{x}} = \lim_{t \rightarrow 0^-} \left( \frac{3}{2}x^{2/3} \Big|_{-2}^t \right) \\ &= \lim_{t \rightarrow 0^-} \left( \frac{3}{2}t^{2/3} - \frac{3}{2}(-2)^{2/3} \right) = -\frac{3}{2}\sqrt[3]{4}, \end{aligned}$$

and that

$$\begin{aligned} \int_0^3 \frac{dx}{\sqrt[3]{x}} &= \lim_{t \rightarrow 0^+} \int_t^3 \frac{dx}{\sqrt[3]{x}} = \lim_{t \rightarrow 0^+} \left( \frac{3}{2}x^{2/3} \Big|_t^3 \right) \\ &= \lim_{t \rightarrow 0^+} \left( \frac{3}{2}3^{2/3} - \frac{3}{2}t^{2/3} \right) = \frac{3}{2}\sqrt[3]{9}. \end{aligned}$$

### Example 5.8.4

Compute the values of the integrals below (if they exist). Determine whether the integrals are convergent or divergent.

(a)  $\int_{-1}^1 \frac{dx}{x};$

(b)  $\int_{-2}^3 \frac{dx}{\sqrt[3]{x}}.$

*Solution (continued).* (b) It follows that

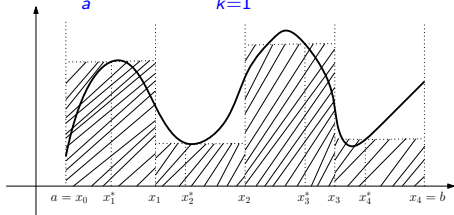
$$\int_{-2}^3 \frac{dx}{\sqrt[3]{x}} = \int_{-2}^0 \frac{dx}{\sqrt[3]{x}} + \int_0^3 \frac{dx}{\sqrt[3]{x}} = -\frac{3}{2}\sqrt[3]{4} + \frac{3}{2}\sqrt[3]{9} = \frac{3}{2}(\sqrt[3]{9} - \sqrt[3]{4}).$$

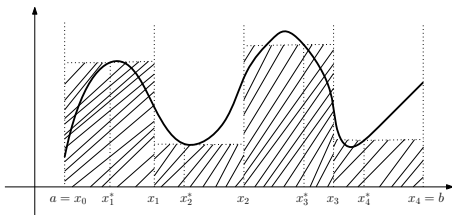
In particular, the integral  $\int_{-2}^3 \frac{dx}{\sqrt[3]{x}}$  is convergent.  $\square$

### 3 Approximating the definite integral

- We defined the definite (or Riemann) integral in terms of “upper sums” and “lower sums.”

- Now, suppose  $a, b \in \mathbb{R}$  are such that  $a < b$  and that  $f : [a, b] \rightarrow \mathbb{R}$  is an integrable function.
- Let  $n \in \mathbb{N}$ . For all  $k \in \{0, \dots, n\}$ , set  $x_k = a + \frac{k}{n}(b - a)$ . (In particular,  $a = x_0 < x_1 < \dots < x_n = b$ .)
- If we divide  $[a, b]$  into  $n$  equal-length subintervals, then the  $k$ -th interval is of the form  $[x_{k-1}, x_k]$ .
- Each of these  $n$  intervals is of length  $\Delta x = \frac{b-a}{n}$ .
- Further, for each  $k \in \{1, \dots, n\}$ , let  $x_k^*$  be an arbitrary point in the  $k$ -th interval  $[x_{k-1}, x_k]$ .
- If  $n$  is large, then  $\int_a^b f(x) dx \approx \sum_{k=1}^n f(x_k^*) \Delta x$ .





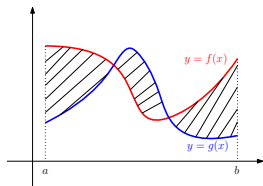
- Remark:** Sometimes, it is convenient to give a rule for choosing the  $x_k^*$ 's. For instance, we may choose:
  - to always take  $x_k^* := x_{k-1}$  (the left endpoint of the  $k$ -th subinterval), or
  - to always take  $x_k^* = x_k$  (the right endpoint of the  $k$ -th subinterval), or
  - to always take  $x_k^* = \frac{1}{2}(x_{k-1} + x_k)$  (the midpoint of the  $k$ -th subinterval).

#### 4 Applications of integration

- Recall that, intuitively,  $\int_a^b f(x)dx$  (where  $a < b$ ) is simply the area under the curve  $y = f(x)$ , from  $x = a$  to  $x = b$ , with parts below the  $x$ -axis counting as “negative area.”
  - Now, suppose we are given two integrable continuous functions,  $f$  and  $g$ , and real numbers  $a < b$ .
  - Then the area between  $y = f(x)$  and  $y = g(x)$ , and between  $x = a$  to  $x = b$ , is

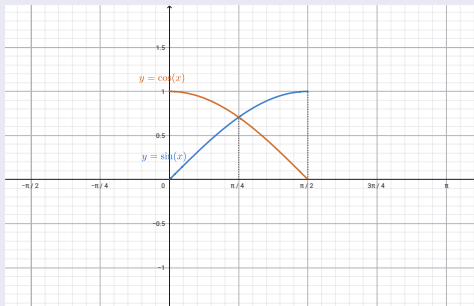
$$A = \int_a^b |f(x) - g(x)| dx.$$

- Remark:** We have absolute value because we never want to get “negative area.”



### Example 5.10.1

Compute the area of the region bounded by the curves  $y = \sin x$ ,  $y = \cos x$ ,  $x = 0$ , and  $x = \frac{\pi}{2}$ .



*Solution.* Note that for  $x \in [0, \frac{\pi}{4}]$ , we have that  $\sin x \leq \cos x$ , and so  $|\sin x - \cos x| = \cos x - \sin x$ ; on the other hand, for  $x \in [\frac{\pi}{4}, \frac{\pi}{2}]$ , we have that  $\sin x \geq \cos x$ , and so  $|\sin x - \cos x| = \sin x - \cos x$ .

So, the area we need is the following (next slide):

*Solution (continued).*

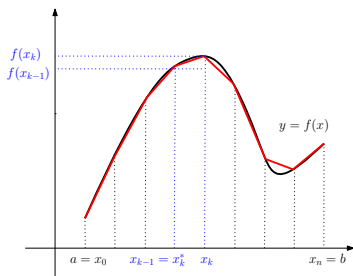
$$\begin{aligned} A &= \int_0^{\pi/2} |\sin x - \cos x| dx \\ &= \int_0^{\pi/4} |\sin x - \cos x| dx + \int_{\pi/4}^{\pi/2} |\sin x - \cos x| dx \\ &= \int_0^{\pi/4} (\cos x - \sin x) dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x) dx \\ &= \left( \sin x + \cos x \right) \Big|_0^{\pi/4} + \left( -\cos x - \sin x \right) \Big|_{\pi/4}^{\pi/2} \\ &= 2\sqrt{2} - 2. \end{aligned}$$

□

- Let  $a, b \in \mathbb{R}$  be such that  $a < b$ , and suppose  $f$  is continuously differentiable on  $[a, b]$ . Then the arc length of the curve  $y = f(x)$  from  $x = a$  to  $x = b$  is

$$s := \int_a^b \sqrt{1 + (f'(x))^2} dx.$$

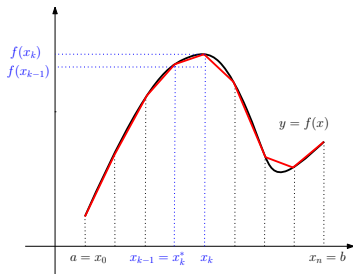
- Let us try to justify this (not a formal proof).
- Let  $n$  be a very large positive integer.
- For all  $k \in \{0, \dots, n\}$ , we set  $x_k = a + \frac{k}{n}(b - a)$ . (In particular,  $a = x_0 < x_1 < \dots < x_n = b$ .)
- Now, if we divide  $[a, b]$  into  $n$  equal-length subintervals, then the  $k$ -th interval is of the form  $[x_{k-1}, x_k]$ .
- Each of these  $n$  intervals is of length  $\Delta x = \frac{b-a}{n}$ .
- Further, for each  $k \in \{1, \dots, n\}$ , we set  $x_k^* = x_{k-1}$ .
- We approximate the curve  $y = f(x)$  from  $x = a$  to  $x = b$  by a **piecewise linear curve** connecting consecutive points of the form  $(x_k, f(x_k))$ .



- Now, by the Pythagorean theorem, the line segment connecting points  $(x_{k-1}, f(x_{k-1}))$  and  $(x_k, f(x_k))$  has length

$$\begin{aligned} \sqrt{(x_k - x_{k-1})^2 + (f(x_k) - f(x_{k-1}))^2} &= (x_k - x_{k-1}) \sqrt{1 + \left(\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}\right)^2} \\ &\stackrel{(*)}{\approx} \Delta x \sqrt{1 + (f'(x_{k-1}))^2} \\ &= \Delta x \sqrt{1 + (f'(x_k^*))^2}, \end{aligned}$$

where  $(*)$  follows from the fact that  $f$  is differentiable and  $x_k \approx x_{k-1}$  (because  $n$  is large).



- So, the length of our entire **piecewise linear curve** is

$$\sum_{k=1}^n \Delta x \sqrt{1 + (f'(x_k^*))^2} \approx \int_a^b \sqrt{1 + (f'(x))^2} dx.$$

### Example 5.10.2

Find the length of the arc given by  $y = x^2 - \frac{1}{8} \ln x$  from  $x = 1$  to  $x = 3$ .

*Solution.* Set  $f(x) = x^2 - \frac{1}{8} \ln x$ . So, we need to find the length of the arc  $y = f(x)$  from  $x = 1$  to  $x = 3$ . Now, we have that  $f'(x) = 2x - \frac{1}{8x}$  and

$$1 + (f'(x))^2 = 1 + (2x - \frac{1}{8x})^2 = 4x^2 + \frac{1}{2} + \frac{1}{64x^2} = (2x + \frac{1}{8x})^2.$$

For  $x > 0$ , we have that  $2x + \frac{1}{8x} > 0$ , and so

$$\sqrt{1 + (f'(x))^2} = \sqrt{(2x + \frac{1}{8x})^2} = 2x + \frac{1}{8x}.$$

So, the length of our arc is

$$\begin{aligned} s &= \int_1^3 \sqrt{1 + (f'(x))^2} dx = \int_1^3 (2x + \frac{1}{8x}) dx \\ &= \left( x^2 + \frac{1}{8} \ln |x| \right) \Big|_1^3 = 8 + \frac{\ln 3}{8}. \end{aligned}$$

□

- In the 3-dimensional space,  $P_x$  is the plane perpendicular to the  $x$ -axis (and parallel to the  $yz$ -plane), intersecting the  $x$ -axis in the point  $x$ .
- Now, suppose  $S$  is a solid that lies between  $x = a$  and  $x = b$  (with  $a < b$ ).
  - Note that  $x = a$  and  $x = b$  define planes perpendicular to the  $x$ -axis and parallel to the  $yz$ -plane.
- If the cross-sectional area of  $S$  in the plane  $P_x$  is  $A(x)$ , where  $A$  is a continuous function, then the *volume* of  $S$  is defined to be

$$V := \int_a^b A(x) dx.$$

- Let us try to explain why this makes sense.
  - This is not a formal proof of anything!
  - We **defined** the volume of  $S$  by the formula  $V = \int_a^b A(x) dx$ .
  - Here, we simply try to provide the intuition behind the definition.

- Let  $n$  be a very large positive integer.
- For all  $k \in \{0, \dots, n\}$ , we set  $x_k = a + \frac{k}{n}(b - a)$ .
- We divide the interval  $[a, b]$  into  $n$  intervals, each of length  $\Delta x = \frac{b-a}{n}$ .
- The  $k$ -th interval is of the form  $[x_{k-1}, x_k]$ . For each  $k \in \{1, \dots, n\}$ , let  $x_k^*$  be an arbitrary point in the  $k$ -th interval  $[x_{k-1}, x_k]$ .
- Now, the part of  $S$  that is between the planes  $P_{x_{k-1}}$  and  $P_{x_k}$  should have volume approximately  $A(x_k^*)\Delta x$ .
- So, by summing up, we see that the volume of  $S$  should be approximately

$$\sum_{i=1}^n A(x_k^*)\Delta x \approx \int_a^b A(x)dx.$$

### Example 5.10.3

Find the volume of the sphere of radius  $r$ .

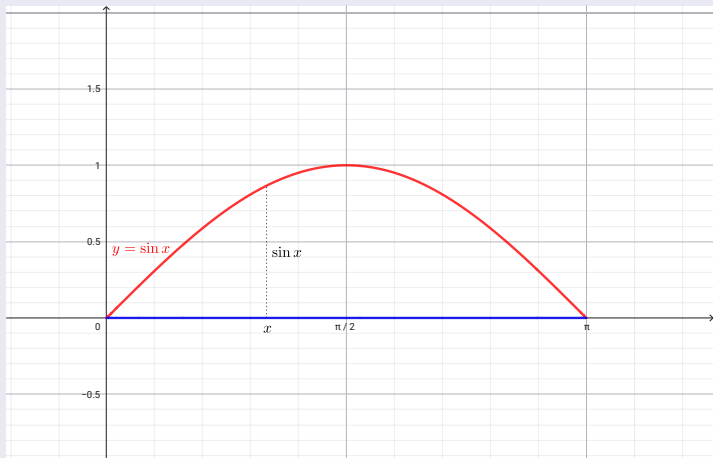
*Solution.* We place the sphere  $S$  of radius  $r$  in the 3-dimensional space, with the center of  $S$  placed at the origin. The intersection of  $S$  and the plane  $P_x$  (with  $-r \leq x \leq r$ ) is a disk of radius  $\sqrt{r^2 - x^2}$ , and the area of this disk is  $A(x) = \pi(r^2 - x^2)$ . So, the volume of  $S$  is

$$\begin{aligned} A &= \int_{-r}^r A(x) dx \\ &= \int_{-r}^r \pi(r^2 - x^2) dx \\ &= \pi \left( r^2 x - \frac{x^3}{3} \right) \Big|_{-r}^r \\ &= \frac{4}{3} r^3 \pi. \end{aligned}$$



### Example 5.10.4

Find the area of the solid  $S$  obtained by rotating about the  $x$ -axis the region between  $y = \sin x$  and  $y = 0$ , and between  $x = 0$  and  $x = \pi$ .



*Solution.* For  $x \in [0, \pi]$ , the cross-sectional area of  $S$  in the plane  $P_x$  is  $A(x) = \pi \sin^2 x$ . So, the volume of  $S$  is

$$\begin{aligned} V &= \int_0^{\pi} A(x) dx \\ &= \pi \int_0^{\pi} \sin^2 x dx \\ &= \pi \int_0^{\pi} \frac{1 - \cos 2x}{2} dx \\ &= \pi \left( \frac{x}{2} - \frac{\sin 2x}{4} \right) \Big|_0^{\pi} \\ &= \frac{\pi^2}{2}. \end{aligned}$$



- There are a couple of similar examples in the Lecture Notes.
- The formula  $V = \int_a^b A(x)dx$  is usually reasonably convenient for computing the volume of a solid obtained by rotating a region enclosed by a couple of curves about the  $x$ -axis (or some other horizontal line).
  - The formula is, however, more difficult to use if we rotate about the  $y$ -axis (or some other vertical line).
  - Let us find a better formula.

- Suppose  $0 \leq a < b$ , and suppose  $y = f(x)$  is non-negative and continuous on  $[a, b]$ .
- Let  $R$  be the region between  $y = 0$ ,  $y = f(x)$ ,  $x = a$  and  $x = b$ .
- Let  $S$  be the solid obtained by rotating  $R$  about the  $y$ -axis.
- Then the volume of  $S$  is

$$V = \int_a^b 2\pi x f(x) dx.$$

- Let us justify this formula (not a formal proof).

- Let  $n$  be a very large positive integer.
- For all  $k \in \{0, \dots, n\}$ , we set  $x_k = a + \frac{k}{n}(b - a)$ .
- We divide the interval  $[a, b]$  into  $n$  intervals, each of length  $\Delta x = \frac{b-a}{n}$ .
- The  $k$ -th interval is of the form  $[x_{k-1}, x_k]$ .
- For each  $k \in \{1, \dots, n\}$ , let  $x_k^* = \frac{x_{k-1} + x_k}{2}$  (so,  $x_k^*$  is the midpoint of the interval  $[x_{k-1}, x_k]$ ).
- Now, consider the solid obtained by rotating about the  $y$ -axis the region between  $y = f(x)$  and  $y = 0$ , and between  $x = x_{k-1}$  and  $x = x_k$ .
- It is, approximately, a “cylindrical shell” of volume:

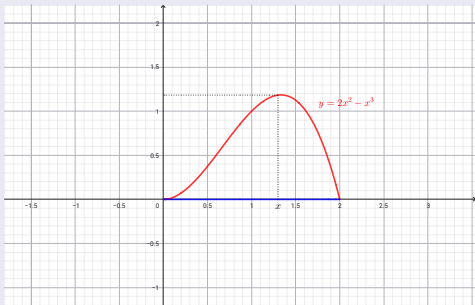
$$\begin{aligned}
 (\pi x_k^2 f(x_k^*)) - (\pi x_{k-1}^2 f(x_k^*)) &= \pi(x_k^2 - x_{k-1}^2) f(x_k^*) \\
 &= \pi(x_k - x_{k-1})(x_k + x_{k-1}) f(x_k^*) \\
 &= \pi \Delta x (2x_k^*) f(x_k^*) \\
 &= 2\pi x_k^* f(x_k^*) \Delta x.
 \end{aligned}$$

- So, the total volume of  $S$  should be approximately

$$\sum_{k=1}^n (2\pi x_k^*) f(x_k^*) \Delta x \approx \int_a^b 2\pi x f(x) dx.$$

### Example 5.10.7

Find the volume of the solid  $S$  obtained by rotating about the  $y$ -axis the region bounded by  $y = 2x^2 - x^3$  and  $y = 0$ .



*Solution.* The curves  $y = 2x^2 - x^3$  and  $y = 0$  intersect in points  $(0, 0)$  and  $(2, 0)$ , and so we integrate from 0 to 2. For  $x \in [0, 2]$ , we have that  $2x^2 - x^3 \geq 0$ , and so the volume of  $S$  is

$$V = \int_0^2 2\pi x(2x^2 - x^3) dx = 2\pi \int_0^2 (2x^3 - x^4) dx = 2\pi \left( \frac{x^4}{2} - \frac{x^5}{5} \right) \Big|_0^2 = \frac{16\pi}{5}.$$

