Mathematical Analysis 1

Lecture #14

The definite (Riemann) integral and the Fundamental Theorem of Calculus

Irena Penev

May 11, 2026

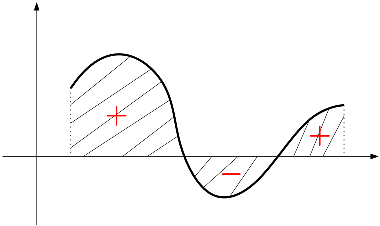
- This lecture has five parts:
  1. The Riemann integral: the definition
  2. Basic properties of the Riemann integral
  3. The Fundamental Theorem of Calculus
  4. The Substitution Rule
  5. Integration by parts

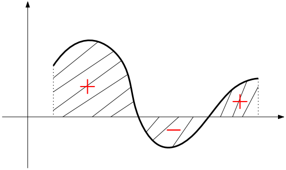
1. The Riemann integral: the definition

- The idea behind the Riemann integral is as follows.
    - We start with a function f that is bounded on some closed interval $[a, b]$.
        - This means that there exist some m, M s.t. for all $x \in [a, b]$, we have that $m \leq f(x) \leq M$.
- We would like to define the Riemann integral of f on $[a, b]$, denoted by

$$\int\limits_a^b f(x)dx$$

to be the area under the curve defined by the equation $y = f(x)$ on the interval $[a, b]$, except that if the curve is below the x-axis, then the area is “negative.”





- Unfortunately, our “definition” (from the previous slide), while intuitive, is not mathematically satisfactory: we never defined the “area” of an arbitrary (bounded) region in the plane, and so we cannot use this concept to define other concepts.
  - As a matter of fact, there is no natural way to define the area of just any bounded region in the plane; this is possible only for sufficiently well-behaved regions.
- So, let us try to formalize our intuition.

Definition

For $a, b \in \mathbb{R}$ s.t. $a < b$, a *partition* of the closed interval $[a, b]$ is a finite (ordered) set $P = \{x_0, x_1, \ldots, x_N\}$, $N \geq 1$, of real numbers s.t. $a = x_0 < x_1 < \cdots < x_N = b$.

Definition

Suppose that $a, b \in \mathbb{R}$ are s.t. $a < b$, that $f : [a, b] \to \mathbb{R}$ is a bounded function, and that $P = \{x_0, x_1, \ldots, x_N\}$ is a partition of $[a, b]$. For each $k \in \{1, \ldots, N\}$, we define

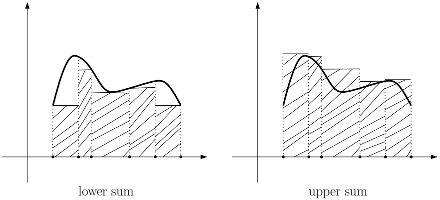
$$m_k := \inf\{f(t) \mid x_{k-1} \le t \le x_k\} \quad \text{and} \quad M_k := \sup\{f(t) \mid x_{k-1} \le t \le x_k\}.$$

Now, we define the *lower sum* of f associated with P to be

$$L(f, P) \quad := \quad \sum_{k=1}^{N} m_k(x_k - x_{k-1}),$$

and we define the *upper sum* of f associated with P to be

$$U(f, P) \quad := \quad \sum_{k=1}^{N} M_k(x_k - x_{k-1}).$$


lower sum

upper sum

Definition

Suppose that $a, b \in \mathbb{R}$ are s.t. $a < b$, that $f : [a, b] \to \mathbb{R}$ is a bounded function, and that $P = \{x_0, x_1, \ldots, x_N\}$ is a partition of $[a, b]$. For each $k \in \{1, \ldots, N\}$, we define

$$m_k := \inf\{f(t) \mid x_{k-1} \leq t \leq x_k\} \quad \text{and} \quad M_k := \sup\{f(t) \mid x_{k-1} \leq t \leq x_k\}.$$

Now, we define the *lower sum* of f associated with P to be

$$L(f, P) \quad := \quad \sum_{k=1}^{N} m_k(x_k - x_{k-1}),$$

and we define the *upper sum* of f associated with P to be

$$U(f, P) \quad := \quad \sum_{k=1}^{N} M_k(x_k - x_{k-1}).$$

- Obviously, $m_k \leq M_k$ for all $k \in \{1, \ldots, N\}$, and so

$$L(f, P) \quad \leq \quad U(f, P).$$

Proposition 5.2.1

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a bounded function. Let $m, M \in \mathbb{R}$ be s.t. for all $x \in [a, b]$, we have that $m \leq f(x) \leq M$. Then for all partitions P of $[a, b]$, we have that

$$m(b-a) \quad \leq L(f, P) \quad \leq \quad U(f, P) \quad \leq \quad M(b-a).$$

Proof. Consider any partition $P = \{x_0, x_1, \dots, x_N\}$ of $[a, b]$, so that $a = x_0 < x_1 < \cdots < x_N = b$. For each $k \in \{1, \dots, N\}$, set

$$m_k := \inf\{f(t) \mid x_{k-1} \leq t \leq x_k\} \quad \text{and} \quad M_k := \sup\{f(t) \mid x_{k-1} \leq t \leq x_k\}.$$

Clearly, for all $k \in \{1, \dots, N\}$, we have that $m \leq m_k \leq M_k \leq M$.

Proof (continued). Reminder: $m \leq m_k \leq M_k \leq M$ $\forall k \in \{1, \ldots, N\}$.

We now compute:

$$\begin{aligned} L(f, P) &= \sum_{k=1}^{N} m_k(x_k - x_{k-1}) \\ &\geq \sum_{k=1}^{N} m(x_k - x_{k-1}) \\ &= m \sum_{k=1}^{N} (x_k - x_{k-1}) \\ &= m(x_N - x_0) \\ &= m(b - a), \end{aligned}$$

Proof (continued). Reminder: $m \leq m_k \leq M_k \leq M$ $\forall k \in \{1, \ldots, N\}$. whereas

$$\begin{aligned} U(f, P) &= \sum_{k=1}^{N} M_k (x_k - x_{k-1}) \\ &\leq \sum_{k=1}^{N} M (x_k - x_{k-1}) \\ &\leq M \sum_{k=1}^{N} (x_k - x_{k-1}) \\ &= M(x_N - x_0) \\ &= M(b - a). \end{aligned}$$

Proposition 5.2.1

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a bounded function. Let $m, M \in \mathbb{R}$ be s.t. for all $x \in [a, b]$, we have that $m \leq f(x) \leq M$. Then for all partitions P of $[a, b]$, we have that

$$m(b-a) \leq L(f, P) \leq U(f, P) \leq M(b-a).$$

Proof (continued). We have now shown that $m(b-a) \leq L(f, P)$ and $U(f, P) \leq M(b-a)$. We already know that $L(f, P) \leq U(f, P)$, and the result follows. □

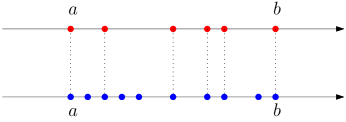
Definition

For $a, b \in \mathbb{R}$ s.t. $a < b$, a *partition* of the closed interval $[a, b]$ is a finite (ordered) set $P = \{x_0, x_1, \ldots, x_N\}$, $N \geq 1$, of real numbers s.t. $a = x_0 < x_1 < \cdots < x_N = b$.

Definition

Let $a, b \in \mathbb{R}$ be s.t. $a < b$. A *refinement* of a partition P of an interval $[a, b]$ is any partition Q of $[a, b]$ s.t. $P \subseteq Q$ (when P and Q are considered as unordered sets).

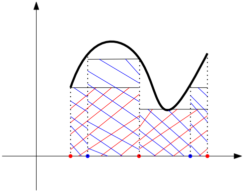
- This is illustrated in the picture below: a partition of $[a, b]$ is in red, and a refinement of the partition is in blue.



Proposition 5.2.2

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, let $f : [a, b] \to \mathbb{R}$ be a bounded function, and let P and Q be partitions of $[a, b]$ s.t. Q is a refinement of P. Then $L(f, P) \leq L(f, Q)$ and $U(f, Q) \leq U(f, P)$.

- Proof: Obvious!
- For lower sums, this is illustrated below. The partition P is in red, whereas its refinement Q consists both of the red and blue points. The lower sum $L(f, P)$ is represented by red shading, whereas the lower sum $L(f, Q)$ is represented by blue shading. Clearly, $L(f, P) \leq L(f, Q)$.



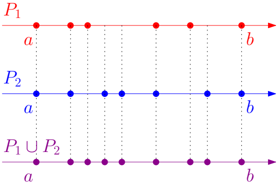
Proposition 5.2.2

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, let $f : [a, b] \to \mathbb{R}$ be a bounded function, and let P and Q be partitions of $[a, b]$ s.t. Q is a refinement of P. Then $L(f, P) \leq L(f, Q)$ and $U(f, Q) \leq U(f, P)$.

- **Remark:** By the definition of upper and lower sums, we have that $L(f, Q) \leq U(f, Q)$, and so under the hypotheses of the proposition, we in fact get that

$$L(f, P) \quad \leq \quad L(f, Q) \quad \leq U(f, Q) \quad \leq \quad U(f, P).$$

- Suppose that $a, b \in \mathbb{R}$ are s.t. $a < b$, and that P_1 and P_2 are partitions of $[a, b]$.
  - Then $P_1 \cup P_2$ is the partition of $[a, b]$ obtained by first taking the union of P_1 and P_2 as unordered sets, and then ordering the elements of $P_1 \cup P_2$ from smallest to largest.
  - This is illustrated in the picture below. Clearly, $P_1 \cup P_2$ is a refinement of both P_1 and P_2.



- Reminder:

Proposition 5.2.2

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, let $f : [a, b] \to \mathbb{R}$ be a bounded function, and let P and Q be partitions of $[a, b]$ s.t. Q is a refinement of P. Then $L(f, P) \leq L(f, Q)$ and $U(f, Q) \leq U(f, P)$.

Lemma 5.2.3

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, let P_1 and P_2 be partitions of $[a, b]$, and let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then $L(f, P_1) \leq U(f, P_2)$.

Proof. Set $P := P_1 \cup P_2$, so that P is a refinement both of P_1 and of P_2. Then

$$L(f, P_1) \overset{(*)}{\leq} L(f, P) \overset{(**)}{\leq} U(f, P) \overset{(*)}{\leq} U(f, P_2),$$

where (*) follows from Proposition 5.2.2, and (**) follows from the definition of upper and lower sums. ☐

- Let us now define the Riemann integral!
- Let $a, b \in \mathbb{R}$ be s.t. $a < b$, let $f : [a, b] \to \mathbb{R}$ be a bounded function, and set
  - $L := \sup \big\{ L(f, P) \mid P \text{ is a partition of } [a, b] \big\}$;
  - $U := \inf \big\{ U(f, P) \mid P \text{ is a partition of } [a, b] \big\}$.

  Remark: By Proposition 5.2.1, the two sets above are bounded, and so by the completeness of the ordered field $\mathbb{R}$, L and U are both defined. Moreover, by Lemma 5.2.3, $L \leq U$.
- If $L = U$, then we say that f is *Riemann-integrable* (or simply *integrable*) on $[a, b]$, and the *Riemann integral* (or the *definite integral*) of f from a to b is defined to be

$$\int\limits_a^b f(x)dx \quad := \quad L = U,$$

  whereas the *Riemann integral* (or the *definite integral*) of f from b to a is defined to be

$$\int\limits_b^a f(x)dx \quad := \quad -\int\limits_a^b f(x)dx.$$

- If $L \neq U$ (equivalently: $L < U$), then f is not Riemann-integrable on $[a, b]$.

- For any real-valued function f, and any $a \in \mathbb{R}$ that belongs to the domain of f, we define

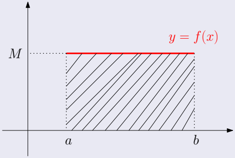
$$\int\limits_a^a f(x)dx \quad := \quad 0.$$

- **Terminology:** In the expression $\int\limits_a^b f(x)dx$, numbers a and b are referred to as the *limits of integration*.

Proposition 5.2.4

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a constant function. Let $M \in \mathbb{R}$ be s.t. for all $x \in \mathbb{R}$, we have that $f(x) = M$. Then f is Riemann integrable on $[a, b]$, and moreover,

$$\int_a^b f(x)dx \quad = \quad M(b-a).$$



Proof. Let $P_0 = \{x_0, x_1, \dots, x_N\}$ be any partition of $[a, b]$, so that $a = x_0 < x_1 < \dots < x_N \leq b$. For each $k \in \{1, \dots, N\}$, set

$$m_k := \inf\{f(t) \mid x_{k-1} \leq t \leq x_k\} \quad \text{and} \quad M_k := \sup\{f(t) \mid x_{k-1} \leq t \leq x_k\}.$$

But since f is a constant function that satisfies $f(x) = M$ for all $x \in [a, b]$, we see that for all $k \in \{1, \dots, N\}$, we have that $m_k = M_k = M$.

Proof (continued). Consequently,

$$\begin{aligned} L(f, P_0) &= \sum_{k=1}^{N} m_k(x_k - x_{k-1}) \\ &= \sum_{k=1}^{N} M(x_k - x_{k-1}) \\ &= M \sum_{k=1}^{N} (x_k - x_{k-1}) \\ &= M(x_N - x_0) \\ &= M(b - a), \end{aligned}$$

and analogously,

$$U(f, P_0) = M(b - a).$$

Proof (continued). Reminder: $L(f, P_0) = M(b - a)$ and $U(f, P_0) = M(b - a)$.

Since the partition P_0 of $[a, b]$ was chosen arbitrarily, it follows that

- $L := \sup \{L(f, P) \mid P \text{ is a partition of } [a, b]\} = M(b - a)$;
- $U := \inf \{U(f, P) \mid P \text{ is a partition of } [a, b]\} = M(b - a)$.

Thus, $L = U = M(b - a)$.

So, by the definition of the Riemann integral, we have that f is Riemann integrable on $[a, b]$, and that

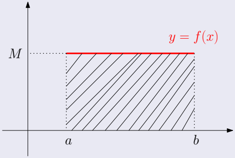
$$\int_a^b f(x)dx = M(b - a),$$

which is what we needed to show. $\square$

Proposition 5.2.4

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a constant function. Let $M \in \mathbb{R}$ be s.t. for all $x \in \mathbb{R}$, we have that $f(x) = M$. Then f is Riemann integrable on $[a, b]$, and moreover,

$$\int_a^b f(x)dx \quad = \quad M(b-a).$$



Theorem 5.2.5

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f is Riemann integrable on $[a, b]$.

- Proof: Lecture Notes (optional reading).
- The proof relies on theory that we have not studied in this course, and in particular, on “compactness’ and “uniform continuity.”
  - Details: Lecture Notes (section 5.4).
- The converse of Theorem 5.2.5 is false: there are functions that are Riemann integrable on a closed interval, but are not continuous on that interval.
- However, it is by no means the case that all functions defined on a closed interval are Riemann integrable on it.
  - Example: next slide!

Example 5.2.6

Let $f : [0,1] \to \mathbb{R}$ be given by

$$f(x) \quad := \quad \begin{cases} 1 & \text{if} \quad x \in \mathbb{Q} \\ 0 & \text{if} \quad x \notin \mathbb{Q} \end{cases} \qquad \forall x \in [0,1].$$

Show that f is **not** Riemann integrable from 0 to 1.

Solution. Let $P_0 = \{x_0, \ldots, x_N\}$ be a partition of $[0,1]$, with $0 = x_0 < \cdots < x_N = 1$. As usual, for all $k \in \{1, \ldots, N\}$, we set

$$m_k = \inf\{f(t) \mid x_{k-1} \leq t \leq x_k\} \quad \text{and} \quad M_k = \sup\{f(t) \mid x_{k-1} \leq t \leq x_k\}.$$

But since each interval $[x_{k-1}, x_k]$ contains both a rational and an irrational number, we see that $m_k = 0$ and $M_k = 1$ for all $k \in \{1, \ldots, N\}$. Consequently,

$$L(f, P_0) \quad = \quad \sum_{k=1}^{N} \underbrace{m_k}_{=0}(x_k - x_{k-1}) \quad = \quad 0$$

and

$$U(f, P_0) \quad = \quad \sum_{k=1}^{N} \underbrace{M_k}_{=1}(x_k - x_{k-1}) \quad = \quad \sum_{k=1}^{N}(x_k - x_{k-1}) \quad = \quad x_N - x_0 \quad = \quad 1.$$

Example 5.2.6

Let $f : [0,1] \to \mathbb{R}$ be given by

$$f(x) \quad := \quad \begin{cases} 1 & \text{if} \quad x \in \mathbb{Q} \\ 0 & \text{if} \quad x \notin \mathbb{Q} \end{cases} \qquad \forall x \in [0,1].$$

Show that f is **not** Riemann integrable from 0 to 1.

Solution (continued). Reminder: $L(f, P_0) = 0$ and $U(f, P_0) = 1$.

Since the partition P_0 of $[0,1]$ was chosen arbitrarily, it follows that

- $L := \sup \{ L(f, P) \mid P \text{ is a partition of } [a, b] \} = 0$;
- $U := \inf \{ U(f, P) \mid P \text{ is a partition of } [a, b] \} = 1$.

Since $L \neq U$, we see that f is not Riemann integrable on $[0,1]$. □

2. Basic properties of the Riemann integral

- Here, we state a number of important properties of the Riemann integral.
    - Most of the proofs are omitted (details: Lecture Notes).

Proposition 5.3.3

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f, g : [a, b] \to \mathbb{R}$ be bounded functions, Riemann integrable on $[a, b]$. Then $f + g$ is also Riemann integrable, and we have that

$$\int_a^b f(x) + g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

Proposition 5.3.4

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and $f : [a, b] \to \mathbb{R}$ be a bounded function, Riemann integrable on $[a, b]$, and let $c \in \mathbb{R}$. Then cf is also Riemann integrable on $[a, b]$, and we have that

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx.$$

Theorem 5.3.6

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$, and let $a, b, c \in \mathbb{R}$. Then the following hold:

- (a) if $\int\limits_a^b f(x)dx$ exists, then so does $\int\limits_b^a f(x)dx$, and in this case
$$\int\limits_b^a f(x)dx = -\int\limits_a^b f(x)dx;$$
- (b) if $\int\limits_a^c f(x)dx$ and $\int\limits_c^b f(x)dx$ both exist, then so does $\int\limits_a^b f(x)dx$, and in this case,
$$\int\limits_a^b f(x)dx = \int\limits_a^c f(x)dx + \int\limits_c^b f(x)dx.$$

- **Remark:** Note that we make no assumptions about the relationship between a, b, and c. In particular, we are **not** assuming that $a < c < b$.
- Part (a) follows from the definition.
- Part (b) is proven in the Lecture Notes.

Theorem 5.3.6

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$, and let $a, b, c \in \mathbb{R}$. Then the following hold:

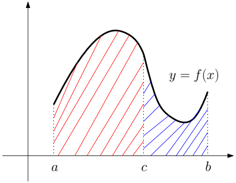
ⓐ if $\int_a^b f(x)dx$ exists, then so does $\int_b^a f(x)dx$, and in this case

$$\int_b^a f(x)dx = -\int_a^b f(x)dx;$$

ⓑ if $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ both exist, then so does $\int_a^b f(x)dx$, and in this case,

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

- For the special case when $a < c < b$, part (b) can be illustrated as follows:



Theorem 5.3.7

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a bounded function, Riemann integrable on $[a, b]$. If $f(x) \geq 0$ for all $x \in [a, b]$, then $\int\limits_a^b f(x)dx \geq 0$.

Proof. Assume that $f(x) \geq 0$ for all $x \in [a, b]$. Define

- $L := \sup\big\{L(f, P) \mid P \text{ is a partition of } [a, b]\big\}$;
- $U := \inf\big\{U(f, P) \mid P \text{ is a partition of } [a, b]\big\}$.

Since f is Riemann integrable on $[a, b]$, we have that $\int\limits_a^b f(x)dx = L = U$. So, it is enough to show that $L \geq 0$.

By the definition of L, it suffices to show that for all partitions P of $[a, b]$, we have that $L(f, P) \geq 0$.

Theorem 5.3.7

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a bounded function, Riemann integrable on $[a, b]$. If $f(x) \geq 0$ for all $x \in [a, b]$, then $\int\limits_a^b f(x)dx \geq 0$.

Proof. Fix any partition $P = \{x_0, x_1, \dots, x_N\}$ of $[a, b]$, where $a = x_0 < x_1 < \dots < x_N = b$. WTS $L(f, P) \geq 0$.

For each $k \in \{1, \dots, N\}$, set $m_k := \inf\{f(t) \mid x_{k-1} \leq t \leq x_k\}$. Since $f(x) \geq 0$ for all $x \in [a, b]$, we see that $m_k \geq 0$ for all $k \in \{1, \dots, N\}$. Meanwhile, it is clear that for all $k \in \{1, \dots, N\}$, we have that $x_k - x_{k-1} > 0$. Therefore,

$$L(f, P) \;\geq\; \sum_{k=1}^{N} m_k(x_k - x_{k-1}) \;\geq\; 0,$$

which is what we needed to show. □

Theorem 5.3.8

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f, g : [a, b] \to \mathbb{R}$ be bounded functions, Riemann integrable on $[a, b]$. Assume that $f(x) \leq g(x)$ for all $x \in [a, b]$. Then $\int\limits_a^b f(x)dx \leq \int\limits_a^b g(x)dx$.

Proof. Define $h : [a, b] \to \mathbb{R}$ by setting $h(x) := g(x) - f(x)$. By Propositions 5.3.3 and 5.2.4, h is Riemann integrable on $[a, b]$, and moreover,

$$\int\limits_a^b h(x)dx = \int\limits_a^b g(x)dx - \int\limits_a^b f(x)dx.$$

Moreover, note that $h(x) \geq 0$ for all $x \in [a, b]$, and so by Theorem 5.3.7, we have that $\int\limits_a^b h(x)dx \geq 0$. It follows that

$$0 \leq \int\limits_a^b h(x)dx = \int\limits_a^b g(x)dx - \int\limits_a^b f(x)dx,$$

and consequently, $\int\limits_a^b f(x)dx \leq \int\limits_a^b g(x)dx$. □

Corollary 5.3.9

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a bounded function, Riemann integrable on $[a, b]$. Let $m, M \in \mathbb{R}$ be s.t. for all $x \in [a, b]$, we have that $m \leq f(x) \leq M$. Then

$$m(b-a) \quad \leq \quad \int\limits_a^b f(x)dx \quad \leq \quad M(b-a).$$

Proof. Define $f_1, f_2 : [a, b] \to \mathbb{R}$ by setting $f_1(x) := m$ and $f_2(x) := M$ for all $x \in [a, b]$. Clearly,

$$f_1(x) \quad \leq \quad f(x) \quad \leq f_2(x) \qquad\qquad \forall x \in [a, b].$$

By Proposition 5.2.4, both f_1 and f_2 are Riemann integrable, and moreover, $\int\limits_a^b f_1(x)dx = m(b-a)$ and $\int\limits_a^b f_2(x)dx = M(b-a)$. We now have that

$$m(b-a) \quad = \quad \int\limits_a^b f_1(x)dx \quad \overset{(*)}{\leq} \quad \int\limits_a^b f(x)dx \quad \overset{(*)}{\leq} \quad \int\limits_a^b f_1(x)dx \quad = \quad M(b-a),$$

where (*) follows from Theorem 5.3.8. □

3. The Fundamental Theorem of Calculus

- The Fundamental Theorem of Calculus has two parts.
  - The first part essentially states that any function that is continuous on a closed interval also has an antiderivative on that closed interval.
  - The second part is useful for actually computing the Riemann integral in many cases.

The Fundamental Theorem of Calculus, Part I

Let $I \subseteq \mathbb{R}$ be an interval, let $a \in I$, and let $f : I \to \mathbb{R}$ be a continuous function. Then the function $g : I \to \mathbb{R}$ given by

$$g(x) \quad := \quad \int\limits_a^x f(t)dt \qquad\qquad \forall x \in I$$

is well-defined and differentiable (and therefore continuous) on I. Moreover, $g'(x) = f(x)$ for all $x \in I$, and so g is an antiderivative of f on I.

Proof (slightly informal). Let us first explain why g is well-defined.

For $x = a$, we have that $\int\limits_a^x f(t)dt = \int\limits_a^a f(t)dt = 0$.

Next, by hypothesis, f is continuous on I. Therefore, for all $x \in I \setminus \{a\}$, f is continuous on $[a, x]$ or $[x, a]$ (depending on whether $a < x$ or $x < a$), and so by Theorem 5.2.5, f is integrable on $[a, x]$ or $[x, a]$, i.e. $\int\limits_a^x f(t)dt$ exists. This proves that g is indeed well-defined.

The Fundamental Theorem of Calculus, Part I

Let $I \subseteq \mathbb{R}$ be an interval, let $a \in I$, and let $f : I \to \mathbb{R}$ be a continuous function. Then the function $g : I \to \mathbb{R}$ given by

$$g(x) \quad := \quad \int_a^x f(t)dt \qquad \forall x \in I$$

is well-defined and differentiable (and therefore continuous) on I. Moreover, $g'(x) = f(x)$ for all $x \in I$, and so g is an antiderivative of f on I.

Proof (continued). Now, note that for any $x, y \in I$ s.t. $x < y$, the function f is continuous on $[x, y]$, and is therefore (by Theorem 5.2.5) integrable on $[x, y]$, and consequently, both $\int_x^y f(t)dt$ and $\int_y^x f(t)dt$ exist.

Recall that every differentiable function is continuous (by Theorem 4.3.1). Thus, it remains to show that for all $x \in I$, g is differentiable at x and satisfies $g'(x) = f(x)$. Fix an arbitrary $x \in I$.

Proof (continued). For $h \in \mathbb{R}$ s.t. $x + h \in I$, we have that

$$\begin{aligned} g(x+h) - g(x) &= \int_a^{x+h} f(t)dt - \int_a^x f(t)dt \\ &= \Big(\int_a^x f(t)dt + \int_x^{x+h} f(t)dt \Big) - \int_a^x f(t)dt \quad \text{by Theorem 5.3.6(b)} \\ &= \int_x^{x+h} f(t)dt, \end{aligned}$$

and so if $h \neq 0$, then

$$\frac{g(x+h)-g(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t)dt.$$

Proof (continued). Reminder: For $h \in \mathbb{R} \setminus \{0\}$ s.t. $x + h \in I$:
$\frac{g(x+h)-g(x)}{h} = \frac{1}{h} \int\limits_{x}^{x+h} f(t)dt$.

For the sake of simplicity, let us suppose that $h > 0$. (The proof is similar for $h < 0$.) By the Extreme Value Theorem, f reaches both a maximum value M_h and a minimum value m_h on the interval $[x, x+h]$. Fix $u_h, v_h \in [x, x+h]$ s.t. $f(u_h) = m_h$ and $f(v_h) = M_h$. So, M_h, m_h, u, v are essentially functions of h; the point x is fixed throughout. By Corollary 5.3.9, we have that

$$m_h h \quad \leq \quad \int\limits_{x}^{x+h} f(t)dt \quad \leq \quad M_h h,$$

and since $h > 0$, we have that

$$m_h \quad \leq \quad \frac{1}{h} \int\limits_{x}^{x+h} f(t)dt \quad \leq \quad M_h.$$

It follows that

$$f(u_h) \quad \leq \quad \frac{g(x+h)-g(x)}{h} \quad \leq \quad f(v_h).$$

We proved the above inequality for the case when $h > 0$, but it is not hard to show that it also holds if $h < 0$.

Proof (continued). Reminder: For $h \in \mathbb{R} \setminus \{0\}$ s.t. $x + h \in I$: $f(u_h) \leq \frac{g(x+h)-g(x)}{h} \leq f(v_h)$.

Now, for $h \to 0$, we get that $u_h, v_h \to x$ (because u_h, v_h are between x and $x + h$). By the continuity of f, we have that

$$\lim_{h \to 0} f(u_h) \;=\; f(\lim_{h \to 0} u_h) \;=\; f(x),$$

and similarly,

$$\lim_{h \to 0} f(v_h) \;=\; f(\lim_{h \to 0} v_h) \;=\; f(x),$$

The Squeeze Theorem now implies that

$$\lim_{h \to 0} \frac{g(x+h)-g(x)}{h} \;=\; f(x).$$

By definition, the left-hand-side of this last equality is precisely $g'(x)$. It follows that $g'(x) = f(x)$, and we are done. $\square$

The Fundamental Theorem of Calculus, Part I

Let $I \subseteq \mathbb{R}$ be an interval, let $a \in I$, and let $f : I \to \mathbb{R}$ be a continuous function. Then the function $g : I \to \mathbb{R}$ given by

$$g(x) \quad := \quad \int\limits_a^x f(t)dt \qquad \forall x \in I$$

is well-defined and differentiable (and therefore continuous) on I. Moreover, $g'(x) = f(x)$ for all $x \in I$, and so g is an antiderivative of f on I.

Example 5.5.1

Let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = \int\limits_{17/3}^{x} \sqrt{1+t^2}dt$. Compute $g'(x)$.

Solution. Since the function $f(t) = \sqrt{1+t^2}$ is continuous, we see that

$$g'(x) \quad = \quad f(x) \quad = \quad \sqrt{1+x^2}$$

for all $x \in \mathbb{R}$. ☐

Example 5.5.2

Compute $\frac{d}{dx}\int_{2x}^{3x}\frac{t^2-1}{t^2+1}\,dt$.

Solution. First, we note that the function $f(t)=\frac{t^2-1}{t^2+1}$ is continuous on $\mathbb{R}$. We now compute:

$$\begin{aligned}
\frac{d}{dx}\int_{2x}^{3x}\frac{t^2-1}{t^2+1}\,dt &= \frac{d}{dx}\left(\int_{2x}^{0}\frac{t^2-1}{t^2+1}\,dt+\int_{0}^{3x}\frac{t^2-1}{t^2+1}\,dt\right)\\
&= \frac{d}{dx}\left(-\int_{0}^{2x}\frac{t^2-1}{t^2+1}\,dt+\int_{0}^{3x}\frac{t^2-1}{t^2+1}\,dt\right)\\
&= -\frac{(2x)^2-1}{(2x)^2+1}\cdot 2+\frac{(3x)^2-1}{(3x)^2+1}\cdot 3 && \text{by the Fundamental Theorem of Calculus, Part I, and by the Chain Rule}\\
&= \frac{3(9x^2-1)}{9x^2+1}-\frac{2(4x^2-1)}{4x^2+1}.
\end{aligned}$$

☐

The Fundamental Theorem of Calculus, Part II

Let $a, b \in \mathbb{R}$ be s.t. $a < b$. If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Proof. Let $g : [a, b] \to \mathbb{R}$ be given by $g(x) = \int_a^x f(t)dt$; by the Fundamental Theorem of Calculus, Part I, the function g is well-defined and differentiable on $[a, b]$, and moreover, we have that $g'(x) = f(x)$ for all $x \in [a, b]$, i.e. g is an antiderivative of f on $[a, b]$. So, by Theorem 5.1.2, there is a constant $C \in \mathbb{R}$ s.t. for all $x \in [a, b]$, we have that $F(x) = g(x) + C$. Further, note that $g(a) = \int_a^a f(t)dt = 0$. We now have that

$$F(b) - F(a) = \big(g(b) + C\big) - \big(\underbrace{g(a)}_{=0} + C\big) = g(b) = \int_a^b f(t)dt.$$

□

The Fundamental Theorem of Calculus, Part II

Let $a, b \in \mathbb{R}$ be s.t. $a < b$. If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, and F is any antiderivative of f on $[a, b]$, then

$$\int\limits_a^b f(x)dx \quad = \quad F(b) - F(a).$$

- **Remark:** Note that the Fundamental Theorem of Calculus, Part II, implies that if a function F is continuously differentiable on a closed interval $[a, b]$, then

$$\int\limits_a^b F'(x)dx = F(b) - F(a).$$

The Fundamental Theorem of Calculus, Part II

Let $a, b \in \mathbb{R}$ be s.t. $a < b$. If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

- **Remark:** To apply the formula from the Fundamental Theorem of Calculus, Part II, we do not actually need to have $a < b$. We just need f to be continuous on some interval I that contains both a and b, and we need F to be an antiderivative of f on that interval. Indeed, note that under the hypotheses of the Fundamental Theorem of Calculus, Part II, we have that

$$\int_b^a f(x)dx = -\int_a^b f(x)dx = -\Big(F(b) - F(a)\Big) = F(a) - F(b),$$

even though $a < b$. On the other hand, we obviously have that

$$\int_a^a f(x)dx = 0 = F(a) - F(a).$$

The Fundamental Theorem of Calculus, Part II

Let $a, b \in \mathbb{R}$ be s.t. $a < b$. If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, and F is any antiderivative of f on $[a, b]$, then

$$\int\limits_a^b f(x)dx = F(b) - F(a).$$

- **Notation:** $F(x)\Big|_{x=a}^{b} := F(b) - F(a)$ and $F(x)\Big|_{a}^{b} := F(b) - F(a)$. Thus, the equality from the Fundamental Theorem of Calculus, Part II, can be rewritten as

$$\int\limits_a^b f(x)dx = F(x)\Big|_{x=a}^{b} \qquad \text{or} \qquad \int\limits_a^b f(x)dx = F(x)\Big|_{a}^{b}.$$

The Fundamental Theorem of Calculus, Part II

Let $a, b \in \mathbb{R}$ be s.t. $a < b$. If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)dx \quad = \quad F(b) - F(a).$$

Example 5.3.5

Evaluate $\int_1^3 e^x dx$.

Solution. Since e^x is an antiderivative of e^x (everywhere on $\mathbb{R}$), we have that

$$\int_1^3 e^x dx \quad = \quad e^x \Big|_1^3 \quad = \quad e^3 - e^1 \quad = \quad e^3 - e.$$

□

The Fundamental Theorem of Calculus, Part II

Let $a, b \in \mathbb{R}$ be s.t. $a < b$. If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, and F is any antiderivative of f on $[a, b]$, then

$$\int\limits_a^b f(x)dx \quad = \quad F(b) - F(a).$$

- **Warning:** To apply the formula $\int\limits_a^b f(x)dx = F(b) - F(a)$, we need to make sure that f is defined and continuous on $[a, b]$, and that F is an antiderivative of f on that interval. Otherwise, we must use other methods to compute the integral (assuming the integral exists).

Example 5.5.5

The following calculation is **WRONG**:

$$\int\limits_{-1}^{3} \frac{dx}{x^2} \;=\; \left(-\frac{1}{x}\right)\Bigg|_{-1}^{3} \;=\; -\frac{1}{3}-\left(-\frac{1}{-1}\right) \;=\; -\frac{4}{3}.$$

It is wrong because $f(0)$ is undefined, and so f is **not** defined on the entire interval $[-1,3]$. As a matter of fact, $\int\limits_{-1}^{3} \frac{dx}{x^2}$ does not exist.

4. The Substitution Rule

The Substitution Rule

Let $I, J \subseteq \mathbb{R}$ be intervals, let $f : I \to \mathbb{R}$ be a continuous function, and let $\varphi : J \to I$ be a continuously differentiable function. Then for all $a, b \in J$, we have that

$$\int_a^b f\big(\varphi(x)\big)\varphi'(x)dx \quad = \quad \int_{\varphi(a)}^{\varphi(b)} f(u)du.$$

- Proof: Lecture Notes.
- **Notation:** Under the assumptions of the Substitution Rule, we get the following formula:

$$\int_a^b f\big(\varphi(x)\big)\varphi'(x)dx \quad = \quad \int_{\varphi(a)}^{\varphi(b)} f(u)du, \qquad \begin{array}{ll} u = \varphi(x), & du = \varphi'(x)dx \\ x = a, & x = b \\ u = \varphi(a), & u = \varphi(b). \end{array}$$

Example 5.6.1

Compute $\int\limits_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^2 x} dx$.

Solution. Here, we will use the function $\varphi(x) = \sin x$, and so $u = \sin x$. We compute:

$$\begin{aligned} \int\limits_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^2 x} dx &= \int\limits_{1/\sqrt{2}}^{1} \frac{du}{u^2} \\ &= \left(-\frac{1}{u}\right)\Bigg|_{1/\sqrt{2}}^{1} \\ &= \sqrt{2} - 1. \end{aligned}$$

$$\begin{array}{ll} u = \sin x, & du = \cos x dx \\ x = \frac{\pi}{4}, & x = \frac{\pi}{2} \\ u = \frac{1}{\sqrt{2}}, & u = 1 \end{array}$$

□

Example 5.6.2

Compute $\int\limits_0^4 \frac{du}{1+\sqrt{u}}$.

Solution. Here, we use "reverse substitution" $\varphi(x) = x^2$, with $u = x^2$. We compute

$$\begin{aligned} \int\limits_0^4 \frac{du}{1+\sqrt{u}} &= \int\limits_0^2 \frac{2xdx}{1+x} \\ &= 2\Big(\int\limits_0^2 dx - \int\limits_0^2 \frac{dx}{x+1} \Big) \\ &= 2\Big(x\Big|_0^2 - \ln|x+1|\Big|_0^2 \Big) \\ &= 2(2 - \ln 3) \end{aligned}$$

$$\begin{aligned} &u = x^2, \quad && du = 2xdx \\ &u = 0, && u = 4 \\ &x = 0, && x = 2 \end{aligned}$$

$\square$

5. Integration by Parts

Integration by Parts

If $u(x)$ and $v(x)$ are continuously differentiable on an interval I, then for all $a, b \in I$, we have that

$$\int_a^b u(x)v'(x)dx \quad = \quad \left(u(x)v(x)\right)\Big|_a^b - \int_a^b v(x)u'(x)dx$$

- Proof: Lecture Notes.
- **Remark:** Note that when computing definite integrals using substitution, we need to adjust the limits of integration. When integrating by parts, we do not.

Integration by Parts

If $u(x)$ and $v(x)$ are continuously differentiable on an interval I, then for all $a, b \in I$, we have that

$$\int_a^b u(x)v'(x)dx = (u(x)v(x))\Big|_a^b - \int_a^b v(x)u'(x)dx$$

Example 5.6.4

Compute $\int_0^1 xe^{-x}dx$.

Solution.

$$\begin{aligned}
\int_0^1 xe^{-x}dx &= (-xe^{-x})\Big|_0^1 + \int_0^1 e^{-x}dx \\
&= (-1 \cdot e^{-1} + 0 \cdot e^0) + (-e^{-x})\Big|_0^1 \\
&= 1 - \tfrac{2}{e}
\end{aligned}$$

$$\begin{array}{ll} u = x, & v = -e^{-x} \\ du = dx, & dv = e^{-x}dx \end{array}$$

□