Mathematical Analysis 1

Lecture #13

The indefinite integral

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- This lecture has six parts:
  1. The antiderivative and the indefinite integral: definition, examples, and basic properties
  2. A summary of important indefinite integrals
  3. Addition and scalar multiplication of indefinite integrals
  4. The Substitution Rule
  5. Integration by parts
  6. Convexity and concavity in the context of asymptotes

1. The antiderivative and the indefinite integral: definition, examples, and basic properties

- In the past few lectures, we were interested in finding the derivative of a given function.
  - Here, we will concern ourselves with the opposite problem: determine the functions that have the given derivative.

Definition

For an interval $I \subseteq \mathbb{R}$ and a function $f : A \subseteq \mathbb{R} \to \mathbb{R}$, where $I \subseteq A$, an *antiderivative* of f on I is the function $F : I \to \mathbb{R}$ such that $F'(x) = f(x)$ for all $x \in \mathbb{R}$.

Definition

For an interval $I \subseteq \mathbb{R}$ and a function $f : A \subseteq \mathbb{R} \to \mathbb{R}$, where $I \subseteq A$, an *antiderivative* of f on I is the function $F : I \to \mathbb{R}$ such that $F'(x) = f(x)$ for all $x \in \mathbb{R}$.

Example 5.1.1

a. $F_1(x) = x^2$ is an antiderivative of $f_1(x) = 2x$ on $\mathbb{R} = (-\infty, +\infty)$;
b. $F_2(x) = \sin x$ is an antiderivative of $f_2(x) = \cos x$ on $\mathbb{R} = (-\infty, \infty)$;
c. $F_3(x) = \arcsin x$ is an antiderivative of $f_3(x) = \frac{1}{\sqrt{1-x^2}}$ on $(-1, 1)$.

Theorem 5.1.2

Suppose that F is an antiderivative of a function f on an interval I. Then the general form of the antiderivative of f on the interval I is $F + C$, where C is an arbitrary constant. More precisely, a function G is an antiderivative of f on I if and only if there exists some constant $C \in \mathbb{R}$ such that for all $x \in I$, we have that $G(x) = F(x) + C$.

Proof. Suppose first that there exists some $C \in \mathbb{R}$ such that for all $x \in I$, we have that $G(x) = F(x) + C$. Then for all $x \in I$, we have that $G'(x) = F'(x) = f(x)$, and so G is an antiderivative of f on I.

Suppose conversely that G is an antiderivative of f on I. Then for $x \in I$, we have that

$$\frac{d}{dx}(G(x) - F(x)) = G'(x) - F'(x) = f(x) - f(x) = 0.$$

So, by Theorem 4.13.1(a), the function $G - F$ is constant on the interval I. Thus, there exists a constant $C \in \mathbb{R}$ such that for all $x \in I$, we have that $G(x) - F(x) = C$, that is, $G(x) = F(x) + C$. $\square$

- Suppose that a function f has at least one antiderivative, say F, on an interval I.
  - The *indefinite integral* of f on I, denoted by $\int f(x)dx$, is the set of all antiderivatives of f on I.
  - By Theorem 5.1.2, we have that

$$\int f(x)dx \quad = \quad \{F + C \mid C \in \mathbb{R}\}.$$

  - However, in practice, the set notation is virtually never used in the context of indefinite integrals.
  - Instead, we write

$$\int f(x)dx \quad = \quad F(x) + C.$$

  - Here, C is understood to be an arbitrary constant.
  - Importantly, the above is supposed to be valid for a fixed interval I (which is either specified explicitly or is clear from context).

Theorem 5.1.2

Suppose that F is an antiderivative of a function f on an interval I. Then the general form of the antiderivative of f on the interval I is $F + C$, where C is an arbitrary constant. More precisely, a function G is an antiderivative of f on I if and only if there exists some constant $C \in \mathbb{R}$ such that for all $x \in I$, we have that $G(x) = F(x) + C$.

- **Remark:** If we happen to know that $F'(x) = f(x)$ for all $x \in I$ (where I is some interval), then Theorem 5.1.2 guarantees that $\int f(x)dx = F(x) + C$ on the interval I.

Proposition 5.1.3

Let $F : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, differentiable on some interval $I \subseteq A$. Then $\int F'(x)dx = F(x) + C$ on the interval I.

Proof. This follows immediately from the fact that F is an antiderivative of F' on the interval I. $\square$

Proposition 5.1.3

Let $F : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, differentiable on some interval $I \subseteq A$. Then $\int F'(x)dx = F(x) + C$ on the interval I.

Example 5.1.4

(a) $\int 2xdx = x^2 + C$, valid on the interval $\mathbb{R} = (-\infty, +\infty)$;

(b) $\int \cos xdx = \sin x + C$, valid on the interval $\mathbb{R} = (-\infty, +\infty)$;

(c) $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$, valid on the interval $(-1, +1)$.

- **Remark:** When asked to compute $\int f(x)dx$, we must **include the constant** C!
  - If the constant C is omitted, then our answer is incorrect/incomplete.

Proposition 5.1.3

Let $F : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, differentiable on some interval $I \subseteq A$. Then $\int F'(x)dx = F(x) + C$ on the interval I.

Example 5.1.5

Find the antiderivative of $f(x) = \cos x$ that passes through the point $(\frac{\pi}{2}, 0)$.

Solution. We know that

$$\int \cos x dx = \sin x + C.$$

By solving $\sin(\frac{\pi}{2}) + C = 0$ for C, we get $C = -1$. So, the antiderivative that we need is the function $F(x) = \sin x - 1$. $\square$

- Reminder:

Proposition 4.4.7

$\frac{d}{dx}(\ln x) = \frac{1}{x}$ for $x \in (0, +\infty)$.

Proposition 5.1.6

For $x \in \mathbb{R} \setminus \{0\}$, we have the formula $\frac{d}{dx} \ln |x| = \frac{1}{x}$. Consequently, we have the formula $\int \frac{dx}{x} = \ln |x| + C$, valid on intervals $(-\infty, 0)$ and $(0, +\infty)$.

Proof. For $x \in (0, +\infty)$, we have that

$$\frac{d}{dx} \ln |x| \overset{x \geq 0}{=} \frac{d}{dx} \ln x = \frac{1}{x}.$$

So, the formula $\int \frac{dx}{x} = \ln |x| + C$ is valid on $(0, +\infty)$.

For $x \in (-\infty, 0)$, we have that

$$\frac{d}{dx} \ln |x| \overset{x \leq 0}{=} \frac{d}{dx} \ln(-x) = \frac{1}{-x} \frac{d}{dx}(-x) = \frac{1}{-x}(-1) = \frac{1}{x}.$$

So, the formula $\int \frac{dx}{x} = \ln |x| + C$ is valid on $(-\infty, 0)$. $\square$

Proposition 5.1.6

For $x \in \mathbb{R} \setminus \{0\}$, we have the formula $\frac{d}{dx} \ln |x| = \frac{1}{x}$. Consequently, we have the formula $\int \frac{dx}{x} = \ln |x| + C$, valid on intervals $(-\infty, 0)$ and $(0, +\infty)$.

- **Remark:** Indefinite integrals are computed on intervals, not on arbitrary sets.
  - So, the formula $\int \frac{dx}{x} = \ln |x| + C$ from Proposition 5.1.6 is valid on the interval $(-\infty, 0)$, and also on the interval $(0, +\infty)$, as well as for any subinterval of those two intervals.
  - However, the formula is **not** valid for the set $(-\infty, 0) \cup (0, +\infty)$, since this set is not an interval.
  - Let us explain the reasoning (next slide).

Proposition 5.1.6

For $x \in \mathbb{R} \setminus \{0\}$, we have the formula $\frac{d}{dx} \ln |x| = \frac{1}{x}$. Consequently, we have the formula $\int \frac{dx}{x} = \ln |x| + C$, valid on intervals $(-\infty, 0)$ and $(0, +\infty)$.

- Define the function $F : (-\infty, 0) \cup (0, +\infty) \to \mathbb{R}$ by setting

$$F(x) = \begin{cases} \ln |x| - 1 & \text{if} \quad x < 0 \\ \ln |x| + 1 & \text{if} \quad x > 0 \end{cases}$$

for all $x \in (-\infty, 0) \cup (0, +\infty)$.

  - Then for all $x \in (-\infty, 0) \cup (0, +\infty)$, we do indeed have that $F'(x) = \frac{1}{x}$.
  - However, there is no fixed constant C such that for all $x \in (-\infty, 0) \cup (0, +\infty)$, we have that $F(x) = \ln |C| + C$.
  - The reason for this "pathology" is the fact that $(-\infty, 0) \cup (0, +\infty)$ is not an interval.

2. A summary of important indefinite integrals

- Reminder:

function $f(x)$	derivative $f'(x)$	differentiable for
$c =$ const.	0	$x \in \mathbb{R}$
x^α (for a fixed constant $\alpha \in \mathbb{R}$)	$\alpha x^{\alpha-1}$	$x > 0$
e^x	e^x	$x \in \mathbb{R}$
$\ln x$	$\frac{1}{x}$	$x > 0$
$\sin x$	$\cos x$	$x \in \mathbb{R}$
$\cos x$	$-\sin x$	$x \in \mathbb{R}$
$\tan x$	$\frac{1}{\cos^2 x}$	$x \neq \frac{2k+1}{2}\pi,\ k \in \mathbb{Z}$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$x \in (-1, 1)$
$\arctan x$	$\frac{1}{1+x^2}$	$x \in \mathbb{R}$

- Using the table above, as well as Proposition 5.1.6, we get the following table of indefinite integrals (next slide).

antiderivative $\int f(x)dx$	constraints on the parameter α	constraints on the variable x
$\int 0dx = C$		
$\int x^\alpha dx = \frac{1}{\alpha+1}x^{\alpha+1} + C$	$\alpha \in \mathbb{R} \setminus \{-1\}$	$x > 0$
$\int \frac{dx}{x} = \ln\lvert x\rvert + C$		$x \neq 0$
$\int e^x dx = e^x + C$		
$\int \sin x dx = -\cos x + C$		
$\int \cos x dx = \sin x + C$		
$\int \frac{dx}{\cos^2 x} = \tan x + C$		$x \neq \frac{2k+1}{2}\pi,\ k \in \mathbb{Z}$
$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$		$x \in (-1, 1)$
$\int \frac{dx}{1+x^2} = \arctan x + C$		

- **Remark:** When our table gives no constraints on the variable x, it means that the formula is valid on the entire interval $\mathbb{R} = (-\infty, +\infty)$.

antiderivative $\int f(x)dx$	constraints on the parameter α	constraints on the variable x
$\int 0dx = C$		
$\int x^\alpha dx = \frac{1}{\alpha+1}x^{\alpha+1} + C$	$\alpha \in \mathbb{R} \setminus \{-1\}$	$x > 0$
$\int \frac{dx}{x} = \ln\|x\| + C$		$x \neq 0$
$\int e^x dx = e^x + C$		
$\int \sin x dx = -\cos x + C$		
$\int \cos x dx = \sin x + C$		
$\int \frac{dx}{\cos^2 x} = \tan x + C$		$x \neq \frac{2k+1}{2}\pi,\ k \in \mathbb{Z}$
$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$		$x \in (-1, 1)$
$\int \frac{dx}{1+x^2} = \arctan x + C$		

- **Remark:** The formula $\int \frac{dx}{x} = \ln|x| + C$ is valid on the intervals $(-\infty, 0)$ and $(0, +\infty)$.
  - However, the formula is **not** valid on the set $(-\infty, 0) \cup (0, +\infty)$, since that set is not an interval.

antiderivative $\int f(x)dx$	constraints on the parameter α	constraints on the variable x
$\int 0dx = C$		
$\int x^\alpha dx = \frac{1}{\alpha+1}x^{\alpha+1} + C$	$\alpha \in \mathbb{R} \setminus \{-1\}$	$x > 0$
$\int \frac{dx}{x} = \ln\lvert x\rvert + C$		$x \neq 0$
$\int e^x dx = e^x + C$		
$\int \sin x dx = -\cos x + C$		
$\int \cos x dx = \sin x + C$		
$\int \frac{dx}{\cos^2 x} = \tan x + C$		$x \neq \frac{2k+1}{2}\pi,\ k \in \mathbb{Z}$
$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$		$x \in (-1, 1)$
$\int \frac{dx}{1+x^2} = \arctan x + C$		

- **Remark:** The formula $\int \frac{dx}{\cos^2 x} = \tan x + C$ is valid for any interval of the form $\left(\frac{2k-1}{2}\pi, \frac{2k+1}{2}\pi\right)$, where $k \in \mathbb{Z}$.
  - However, the formula is **not** valid on the set $\mathbb{R} \setminus \{\frac{2k+1}{2}\pi \mid k \in \mathbb{Z}\}$, since this set is **not** an interval.

antiderivative $\int f(x)dx$	constraints on the parameter α	constraints on the variable x
$\int x^\alpha dx = \frac{1}{\alpha+1}x^{\alpha+1} + C$	$\alpha \in \mathbb{R} \setminus \{-1\}$	$x > 0$

- **Remark:** The formula above is guaranteed to work on the interval $(0, +\infty)$.
  - However, assuming everything is defined, we do not have to assume that $x > 0$.
  - For instance, formulas $\int x^3 dx = \frac{1}{4}x^4 + C$ and $\int x^{1/3} dx = \frac{3}{4}x^{4/3} + C$ are valid on the interval $\mathbb{R} = (-\infty, +\infty)$, because everything is defined.
  - The formula $\int x^{-1/5} dx = \frac{5}{4}x^{4/5} + C$ is valid on the intervals $(-\infty, 0)$ and $(0, +\infty)$, because everything is defined on those intervals.
    - Note that $x^{-1/5}$ is undefined for $x = 0$, which is why the formula $\int x^{-1/5} dx = \frac{5}{4}x^{4/5} + C$ is **not** valid on the entire interval $\mathbb{R} = (-\infty, +\infty)$.
    - Once again, we do **not** say that $\int x^{-1/5} dx = \frac{5}{4}x^{4/5} + C$ is valid on the set $(-\infty, 0) \cup (0, +\infty)$, because that set is not an interval.

antiderivative $\int f(x)dx$	constraints on the parameter α	constraints on the variable x
$\int x^\alpha dx = \frac{1}{\alpha+1}x^{\alpha+1} + C$	$\alpha \in \mathbb{R} \setminus \{-1\}$	$x > 0$
$\int \frac{dx}{x} = \ln\|x\| + C$		$x \neq 0$

- **Remark:** For $\alpha = 0$, the formula from the table becomes

$$\int 1dx = x + C \qquad \text{or} \qquad \int dx = x + C,$$

which is valid on the interval $\mathbb{R} = (-\infty, +\infty)$.

- **Remark:** Note that for $\alpha = -1$, we have a special formula, namely

$$\int x^{-1}dx \;=\; \ln|x| + C,$$

which is valid on the intervals $(-\infty, 0)$ and $(0, +\infty)$.

antiderivative $\int f(x)dx$	constraints on the parameter α	constraints on the variable x
$\int 0dx = C$		
$\int x^\alpha dx = \frac{1}{\alpha+1}x^{\alpha+1} + C$	$\alpha \in \mathbb{R} \setminus \{-1\}$	$x > 0$
$\int \frac{dx}{x} = \ln \lvert x \rvert + C$		$x \neq 0$
$\int e^x dx = e^x + C$		
$\int \sin x dx = -\cos x + C$		
$\int \cos x dx = \sin x + C$		
$\int \frac{dx}{\cos^2 x} = \tan x + C$		$x \neq \frac{2k+1}{2}\pi,\ k \in \mathbb{Z}$
$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$		$x \in (-1, 1)$
$\int \frac{dx}{1+x^2} = \arctan x + C$		

3. Addition and scalar multiplication of indefinite integrals

- Suppose that functions f_1 and f_2 have antiderivatives F_1 and F_2, respectively, on an interval I, so that

$$\int f_1(x)dx = F_1(x) + C \qquad \text{and} \qquad \int f_2(x)dx = F_2(x) + C.$$

  - Then we have that

$$\begin{aligned} \int f_1(x)dx + \int f_2(x)dx &= \big(F_1(x) + C_1\big) + \big(F_2(x) + C_2\big) \\ &= F_1(x) + F_2(x) + C_1 + C_2 \\ &= F_1(x) + F_2(x) + C. \end{aligned}$$

  - Here, C_1 and C_2 are arbitrary real constants, and $C_1 + C_2$ yields an arbitrary real constant, which we simply denote by C.
  - Similarly, $\int f_1(x)dx - \int f_2(x)dx = F_1(x) - F_2(x) + C$.

Proposition 5.1.7

Suppose that functions f_1 and f_2 have antiderivatives on an interval I. Then $f_1 + f_2$ and $f_1 - f_2$ also have antiderivatives on I, and moreover, we have the formula

$$\int f_1(x) \pm f_2(x) dx \quad = \quad \int f_1(x) dx \pm \int f_2(x) dx,$$

valid on the interval I.

- **Notation:** It is customary to omit parentheses as we did above, i.e. for a (potentially complicated) expression $\Box$, we normally write $\int \Box dx$ instead of $\int (\Box) dx$.

Proposition 5.1.7

Suppose that functions f_1 and f_2 have antiderivatives on an interval I. Then $f_1 + f_2$ and $f_1 - f_2$ also have antiderivatives on I, and moreover, we have the formula

$$\int f_1(x) \pm f_2(x) dx \quad = \quad \int f_1(x) dx \pm \int f_2(x) dx,$$

valid on the interval I.

Proof. In what follows, all formulas, and all antiderivatives, are assumed to be on the interval I. Let F_1 and F_2 be antiderivatives of f_1 and f_2, respectively, so that

$$\int f_1(x) dx = F_1(x) + C \qquad \text{and} \qquad \int f_2(x) dx = F_2(x) + C.$$

Note that

$$\tfrac{d}{dx}\left(F_1(x) + F_2(x)\right) \stackrel{(*)}{=} F_1'(x) + F_2'(x) \stackrel{(**)}{=} f_1(x) + f_2(x),$$

where (*) follows from the properties of the derivative, whereas (**) follows from the fact that F_1 and F_2 are antiderivatives of f_1 and f_2, respectively.

Proposition 5.1.7

Suppose that functions f_1 and f_2 have antiderivatives on an interval I. Then $f_1 + f_2$ and $f_1 - f_2$ also have antiderivatives on I, and moreover, we have the formula

$$\int f_1(x) \pm f_2(x)dx \quad = \quad \int f_1(x)dx \pm \int f_2(x)dx,$$

valid on the interval I.

Proof. Reminder:

- $\int f_1(x)dx = F_1(x) + C$ and $\int f_2(x)dx = F_2(x) + C$;
- $\frac{d}{dx}(F_1(x) + F_2(x)) = f_1(x) + f_2(x)$.

Thus, $F_1 + F_2$ is an antiderivative of $f_1 + f_2$, and it follows that

$$\int f_1(x) + f_2(x)dx \quad = \quad F_1(x) + F_2(x) + C \quad = \quad \int f_1(x)dx + \int f_2(x)dx.$$

The formula

$$\int f_1(x) - f_2(x)dx \quad = \quad \int f_1(x)dx - \int f_2(x)dx,$$

is obtained analogously. □

- Suppose that a function f has an antiderivative F on an interval I, so that $\int f(x)dx = F(x) + C$, valid on I.
  - For a constant $\alpha \in \mathbb{R} \setminus \{0\}$, we have that

$$\begin{aligned} \alpha \int f(x)dx &= \alpha\big(F(x) + C_1\big) \\ &= \alpha F(x) + \alpha C_1 \\ &= \alpha F(c) + C \end{aligned}$$

    is valid on the interval I.
  - Here C_1 is an arbitrary constant, which yields an arbitrary constant αC_1 (because $\alpha \neq 0$), which we simply denote by C.

Proposition 5.1.8

Suppose that a function f has a antiderivative on an interval I, and let $\alpha \in \mathbb{R} \setminus \{0\}$ be a constant. Then αf also has an antiderivative on I, and moreover, we have the formula

$$\int \alpha f(x)dx \quad = \quad \alpha \int f(x)dx,$$

valid on the interval I.

Proof. In what follows, all formulas, and all antiderivatives, are assumed to be on the interval I. Let F be an antiderivative of f, so that $\int f(x)dx = F(x) + C$. Note that

$$\frac{d}{dx}(\alpha F(x)) \quad \stackrel{(*)}{=} \quad \alpha F'(x) \quad \stackrel{(**)}{=} \quad \alpha f(x),$$

where (*) follows from the properties of the derivative, whereas (**) follows from the fact that F is an antiderivative of f. Therefore, αF is an antiderivative of αf, and it follows that

$$\int \alpha f(x)dx \quad = \quad \alpha F(x) + C \quad \stackrel{\alpha \neq 0}{=} \quad \alpha \int f(x)dx. \qquad \square$$

Proposition 5.1.8

Suppose that a function f has a antiderivative on an interval I, and let $\alpha \in \mathbb{R} \setminus \{0\}$ be a constant. Then αf also has an antiderivative on I, and moreover, we have the formula

$$\int \alpha f(x)dx \ = \ \alpha \int f(x)dx,$$

valid on the interval I.

- **Remark:** Suppose that a function f has an antiderivative F on an interval I, so that $\int f(x)dx = F(x) + C$, valid on I.
  - The formula from Proposition 5.1.8 fails for $\alpha = 0$, since we have the following:
    - $\int 0f(x)dx = \int 0dx = C$;
    - $0 \int f(x)dx = 0(F(x) + C) = 0$.

Proposition 5.1.7

Suppose that functions f_1 and f_2 have antiderivatives on an interval I. Then $f_1 + f_2$ and $f_1 - f_2$ also have antiderivatives on I, and moreover, we have the formula

$$\int f_1(x) \pm f_2(x)dx \;\;=\;\; \int f_1(x)dx \pm \int f_2(x)dx,$$

valid on the interval I.

Proposition 5.1.8

Suppose that a function f has a antiderivative on an interval I, and let $\alpha \in \mathbb{R} \setminus \{0\}$ be a constant. Then αf also has an antiderivative on I, and moreover, we have the formula

$$\int \alpha f(x)dx \;\;=\;\; \alpha \int f(x)dx,$$

valid on the interval I.

Example 5.1.9

Compute $\int x + \frac{7}{x} dx$.

Solution. We compute:

$$\int x + \frac{7}{x} dx \quad = \quad \int x dx + 7 \int \frac{dx}{x} \quad = \quad \frac{1}{2}x^2 + 7 \ln|x| + C.$$

□

4. The Substitution Rule

The Substitution Rule

Let $I, J \subseteq \mathbb{R}$ be intervals, let f be a function that has an antiderivative F on the interval I, and let $\varphi : J \to I$ is a continuously differentiable function.[a] Then

$$\int f(\varphi(x))\varphi'(x)dx = F(\varphi(x)) + C$$

on the interval J.

[a] So, we are assuming φ is differentiable at all points $x \in J$, and that $\varphi' : J \to \mathbb{R}$ is continuous.

- Proof: Lecture Notes (relies on the Chain Rule).

The Substitution Rule

Let $I, J \subseteq \mathbb{R}$ be intervals, let f be a function that has an antiderivative F on the interval I, and let $\varphi : J \to I$ is a continuously differentiable function.[a] Then

$$\int f(\varphi(x))\varphi'(x)dx \quad = \quad F(\varphi(x)) + C$$

on the interval J.

[a] So, we are assuming φ is differentiable at all points $x \in J$, and that $\varphi' : J \to \mathbb{R}$ is continuous.

- **Notation:** For $u = \varphi(x)$, where φ is a differentiable function, we write $du(x) = \varphi'(x)dx$, or simply $du = \varphi'(x)dx$. So, under the assumptions of the Substitution Rule, we get the following formula:

$$\int f(\varphi(x))\varphi'(x)dx \quad = \quad \int f(u)du, \qquad \begin{array}{l} u = \varphi(x) \\ du = \varphi'(x)dx. \end{array}$$

- The Substitution Rule:

$$\int f(\varphi(x))\varphi'(x)dx \quad = \quad \int f(u)du, \qquad \begin{array}{l} u = \varphi(x) \\ du = \varphi'(x)dx. \end{array}$$

- **Remark:** In applying substitution, the most difficult part tends to be finding the right choice of u.
    - While there are some tricks, there is no algorithm for choosing the right kind of substitution, and so this is largely a matter of trial and error.
- Let's take a look at some examples!

$$\int f(\varphi(x))\varphi'(x)dx = \int f(u)du, \qquad u = \varphi(x), \quad du = \varphi'(x)dx.$$

Example 5.1.10

Compute $\int \frac{x}{1+x^2}dx$.

Solution. We introduce the substitution $u = 1 + x^2$ (so, $du = 2xdx$, and therefore $xdx = \frac{1}{2}du$), and we compute:

$$\begin{aligned}
\int \frac{x}{1+x^2}dx &= \frac{1}{2}\int \frac{du}{u} && u = 1 + x^2, \quad du = 2xdx \\
&= \frac{1}{2}\ln|u| + C \\
&= \frac{1}{2}\ln|1 + x^2| + C \\
&= \frac{1}{2}\ln(1 + x^2) + C && \text{because } 1 + x^2 > 0.
\end{aligned}$$

Example 5.1.10

Compute $\int \frac{x}{1+x^2} dx$.

Solution (continued). Reminder: $\int \frac{x}{1+x^2} dx = \frac{1}{2}\ln(1+x^2) + C$.

Optional: We can check that our answer is correct by differentiating the answer:

$$\frac{d}{dx}\left(\frac{1}{2}\ln(1+x^2)\right) = \frac{1}{2}\frac{1}{1+x^2}\frac{d}{dx}(1+x^2) = \frac{1}{2}\frac{1}{1+x^2}\cdot 2x = \frac{x}{1+x^2}.$$

Since we got the function that we originally integrated, our answer is correct.

Remark: Note that we omitted the constant C when we differentiated to check our answer. This is because the derivative of a constant function is zero, and so the constant C would immediately vanish from our calculation. $\square$

$$\int f(\varphi(x))\varphi'(x)dx = \int f(u)du, \qquad \begin{array}{l} u = \varphi(x) \\ du = \varphi'(x)dx. \end{array}$$

Example 5.1.11

Compute $\int x^2 e^{-x^3} dx$.

Solution. We compute:

$$\begin{aligned} \int x^2 e^{-x^3} dx &= -\tfrac{1}{3}\int e^u du \qquad \begin{array}{l} u = -x^3 \\ du = -3x^2 dx \end{array} \\ &= -\tfrac{1}{3}e^u + C \\ &= -\tfrac{1}{3}e^{-x^3} + C. \end{aligned}$$

Optional: We check our answer as follows:

$$\tfrac{d}{dx}\left(-\tfrac{1}{3}e^{-x^3}\right) = -\tfrac{1}{3}e^{-x^3}(-3x^2) = x^2 e^{-x^3}.$$

So, our answer is correct. □

$$\int f(\varphi(x))\varphi'(x)dx = \int f(u)du, \qquad \begin{array}{l} u = \varphi(x) \\ du = \varphi'(x)dx. \end{array}$$

Example 5.1.12

Let $a \in (0,1) \cup (1,+\infty)$ be a constant. Compute $\int a^x dx$.

Solution. We compute:

$$\begin{aligned} \int a^x dx &= \int e^{x \ln a} dx \\ &= \frac{1}{\ln a} \int e^u du \\ &= \frac{1}{\ln a} e^u + C \\ &= \frac{1}{\ln a} e^{x \ln a} + C \\ &= \frac{a^x}{\ln a} + C. \end{aligned} \qquad \begin{array}{l} u = x \ln a \\ du = \ln a dx \end{array}$$

Example 5.1.12

Let $a \in (0,1) \cup (1,+\infty)$ be a constant. Compute $\int a^x dx$.

Solution (continued). Reminder: $\int a^x dx = \frac{a^x}{\ln a} + C$.

Optional: We check our answer as follows:

$$\frac{d}{dx}\left(\frac{a^x}{\ln a}\right) = \frac{1}{\ln a}\left(\frac{d}{dx}a^x\right) = \frac{1}{\ln a}\left(\frac{d}{dx}e^{x \ln a}\right) = \frac{1}{\ln a} \cdot \underbrace{e^{x \ln a}}_{=a^x} \ln a = a^x.$$

So, our answer is correct. □

$$\int f(\varphi(x))\varphi'(x)dx = \int f(u)du, \qquad \begin{aligned} u &= \varphi(x) \\ du &= \varphi'(x)dx. \end{aligned}$$

Example 5.1.13

Compute $\int \tan x dx$.

Solution. We compute:

$$\begin{aligned} \int \tan x dx &= \int \frac{\sin x}{\cos x} dx \\ &= \int \frac{-du}{u} \\ &= -\ln|u| + C \\ &= -\ln|\cos x| + C. \end{aligned} \qquad \begin{aligned} u &= \cos x \\ du &= -\sin x dx \end{aligned}$$

Example 5.1.13

Compute $\int \tan x dx$.

Solution (continued). Reminder: $\int \tan x dx = -\ln|\cos x| + C$.

Optional: We check our answer as follows:

$$\begin{aligned}
\frac{d}{dx}\left(-\ln|\cos x|\right) &= -\frac{d}{dx}\ln|\cos x| \\
&= -\frac{1}{\cos x}\left(\frac{d}{dx}\cos x\right) && \text{by Proposition 5.1.6 and the Chain Rule} \\
&= -\frac{1}{\cos x}(-\sin x) \\
&= \tan x.
\end{aligned}$$

So, our answer is correct. □

5. Integration by parts

Integration by parts

Let function u and v be continuously differentiable on an interval I. Then $\int u(x)dv(x) = u(x)v(x) - \int v(x)du(x)$ on the interval I.

- Proof: Later!
- **Notation:** To simplify notation, we may write the formula above simply as

$$\int udv \quad = \quad uv - \int vdu$$

- **Remark:** As with substitution, the hardest part of using integration by parts is figuring out how to choose u and v.
    - The rule of thumb is that we want v to be easy to integrate, and we want du to be simpler than u.
    - Apart from that, this is largely a matter of trial and error.

- Integration by parts: $\int u dv = uv - \int v du$.
- **Remark:** When integrating by parts, we choose suitable u and dv, and we compute du and v.
  - In our examples, we will color code this as follows:

$$u = ??, \quad v = ??$$
$$du = ??, \quad dv = ??$$

    where the question marks are replaced by the appropriate formulas.
  - Here, we choose the red terms, and based on those, we compute the blue terms.
- Let's take a look at some examples!

- Integration by parts: $\int u dv = uv - \int v du$.

Example 4.1.14

Compute $\int xe^x dx$.

Solution. We compute:

$$\begin{aligned} \int xe^x dx &= xe^x - \int e^x dx \\ &= xe^x - e^x + C. \end{aligned}$$

$$\begin{aligned} u &= x, & v &= e^x \\ du &= dx, & dv &= e^x dx \end{aligned}$$

Optional: We check our answer as follows:

$$\frac{d}{dx}\left(xe^x - e^x\right) = \left(e^x + xe^x\right) - e^x = xe^x.$$

So, our answer is correct. □

- Integration by parts: $\int udv = uv - \int vdu$.

Example 4.1.14

Compute $\int x^2 \sin x dx$.

Solution. In this example, we perform integration by parts twice, as follows:

$$\begin{aligned}
& \int x^2 \sin x dx \\
= \quad & -x^2 \cos x + 2 \int x \cos x dx \\
= \quad & -x^2 \cos x + 2(x \sin x - \int \sin x dx) \\
= \quad & -x^2 \cos x + 2x \sin x + 2 \cos x + C.
\end{aligned}$$

$$u_1 = x^2, \quad v_1 = -\cos x$$
$$du_1 = 2xdx, \quad dv_1 = \sin x dx$$

$$u_2 = x, \quad v_2 = \sin x$$
$$du_2 = dx, \quad dv_2 = \cos x dx$$

Example 4.1.14

Compute $\int x^2 \sin x dx$.

Solution (continued). Reminder:
$\int x^2 \sin x dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C$.

Optional: We check our answer as follows:

$$
\begin{aligned}
& \frac{d}{dx}\big(- x^2 \cos x + 2x \sin x + 2 \cos x \big) \\
= \; & \big(- 2x \cos x - x^2(- \sin x)\big) + \big(2 \sin x + 2x \cos x\big) + 2(- \sin x) \\
= \; & -2x \cos x + x^2 \sin x + 2 \sin x + 2x \cos x - 2 \sin x \\
= \; & 2x \sin x.
\end{aligned}
$$

So, our answer is correct. □

- Integration by parts: $\int udv = uv - \int vdu$.
- We now take a look at a couple of more complicated examples, which rely on recursive formulas.

Example 5.1.16

Compute $\int e^x \cos x dx$.

Solution. We compute:

$$\begin{aligned} I &:= \int e^x \cos x dx \\ &= e^x \sin x - \int e^x \sin x dx. \end{aligned}$$

$$u = e^x, \quad v = \sin x$$
$$du = e^x dx, \quad dv = \cos x dx$$

Now, let us try to compute $\int e^x \sin x dx$:

$$\begin{aligned} \int e^x \sin x dx &= -e^x \cos x + \int e^x \cos x dx \\ &= -e^x \cos x + I. \end{aligned}$$

$$u = e^x, \quad v = -\cos x$$
$$du = e^x dx, \quad dv = \sin x dx$$

So now we have that

$$\begin{aligned} I = e^x \sin x - \int e^x \sin x dx &= e^x \sin x - \left(-e^x \cos x + I\right) \\ &= e^x(\sin x + \cos x) - I. \end{aligned}$$

Example 5.1.16

Compute $\int e^x \cos x dx$.

Solution (continued). Reminder: $I = e^x(\sin x + \cos x) - I$.

Here, we must be careful: I is an indefinite integral, and so it has a "built in" arbitrary additive constant C, and this constant C need not be the same on both sides of the equation! So, when we solve for I, we account for the constant as follows:

$$2I = e^x(\sin x + \cos x) + C.$$

This yields

$$I = \tfrac{1}{2}e^x(\sin x + \cos x) + C,$$

and we are done.

Remark: In the last line of the computation above, we write C, and not $\frac{1}{2}C$.

- This is because C is an arbitrary constant, which means that $\frac{1}{2}C$ is also an arbitrary constant. So, we may just as well write C, and not the more complicated $\frac{1}{2}C$.

Example 5.1.16

Compute $\int e^x \cos x dx$.

Solution (continued). Reminder:
$\int e^x \cos x dx = \frac{1}{2}e^x(\sin x + \cos x) + C$.

Optional: We check our answer as follows:

$$\begin{aligned} \frac{d}{dx}\left(\frac{1}{2}e^x(\sin x + \cos x)\right) &= \frac{1}{2}e^x(\sin x + \cos x) + \frac{1}{2}e^x(\cos x - \sin x) \\ &= e^x \cos x. \end{aligned}$$

So, our answer is correct. □

- Integration by parts: $\int udv = uv - \int vdu$.

Example 5.1.17

Show that for all positive integers n, we have that

$$\int \ln^n x dx = x \ln^n x - n \int \ln^{n-1} x dx.$$

Then, using the recursive formula above, compute $\int \ln^3 x dx$.

- **Notation:** For $n \in \mathbb{N}_0$ and $x \in (0, +\infty)$, we write $\ln^n x := (\ln x)^n$.

Solution. First, for a positive integer n, we have:

$$\begin{aligned} \int \ln^n x dx &= x \ln^n x - \int x \frac{n \ln^{n-1} x}{x} dx \\ &= x \ln^n x - n \int \ln^{n-1} x dx, \end{aligned}$$

$$\begin{aligned} u &= \ln^n x, & v &= x \\ du &= \frac{n \ln^{n-1} x}{x} dx, & dv &= dx \end{aligned}$$

which proves our recursive formula.

Example 5.1.17

Show that for all positive integers n, we have that

$$\int \ln^n x dx \quad = \quad x \ln^n x - n \int \ln^{n-1} x dx.$$

Then, using the recursive formula above, compute $\int \ln^3 x dx$.

Solution (continued). Using the recursive formula that we just proved, we compute:

$$\begin{aligned}
\int \ln^3 x dx &= x \ln^3 x - 3 \int \ln^2 x dx \\
&= x \ln^3 x - 3\Big(x \ln^2 x - 2 \int \ln x dx\Big) \\
&= x \ln^3 x - 3x \ln^2 x + 6 \int \ln x dx \\
&= x \ln^3 x - 3x \ln^2 x + 6\Big(x \ln x - \int dx\Big) \\
&= x \ln^3 x - 3x \ln^2 x + 6x \ln x - 6x + C.
\end{aligned}$$

Example 5.1.17

Show that for all positive integers n, we have that

$$\int \ln^n x dx \quad = \quad x \ln^n x - n \int \ln^{n-1} x dx.$$

Then, using the recursive formula above, compute $\int \ln^3 x dx$.

Solution (continued). Reminder:
$\int \ln^n x dx = x \ln^3 x - 3x \ln^2 x + 6x \ln x - 6x + C$.

Optional: We check our answer as follows:

$$\begin{aligned} & \tfrac{d}{dx}\left(x \ln^3 x - 3x \ln^2 x + 6x \ln x - 6x\right) \\ = \quad & \ln^3 x + x \cdot 3 \ln^2 x \cdot \tfrac{1}{x} - 3 \ln^2 x - 3x(2 \ln x) \cdot \tfrac{1}{x} + 6(\ln x) + 6x \cdot \tfrac{1}{x} - 6 \\ = \quad & \ln^3 x + 3 \ln^2 x - 3 \ln^2 x - 6 \ln x + 6 \ln x + 6 - 6 \quad = \quad \ln^3 x. \end{aligned}$$

So, our answer is correct. □

Integration by parts

Let function u and v be continuously differentiable on an interval I. Then $\int u(x)dv(x) = u(x)v(x) - \int v(x)du(x)$ on the interval I.

Proof. By the Product Rule, we have that

$$(u(x)v(x))' = u'(x)v(x) + u(x)v'(x).$$

When we integrate, we get that

$$\int (u(x)v(x))'dx = \int u'(x)v(x)dx + \int u(x)v'(x)dx,$$

or equivalently,

$$\int u(x)v'(x)dx = \int (u(x)v(x))'dx - \int u'(x)v(x)dx.$$

Integration by parts

Let function u and v be continuously differentiable on an interval I. Then $\int u(x)dv(x) = u(x)v(x) - \int v(x)du(x)$ on the interval I.

Proof. Reminder:

$$\int u(x)v'(x)dx = \int (u(x)v(x))'dx - \int u'(x)v(x)dx.$$

Now, by Proposition 5.1.3, we have that

$$\int (u(x)v(x))'dx = u(x)v(x) + C.$$

However, both $\int u(x)v'(x)dx$ and $\int u'(x)v(x)dx$ have arbitrary additive constants built into them, and so we can simply write

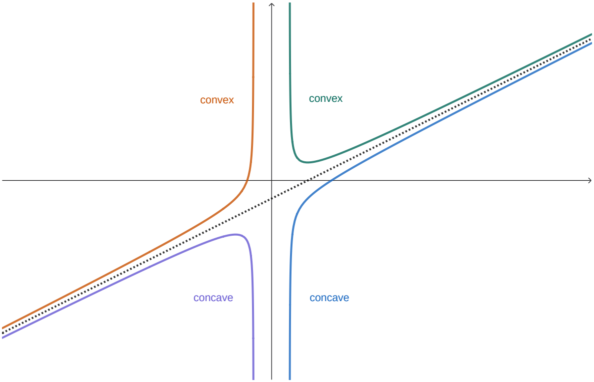
$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx,$$

that is,

$$\int u(x)dv(x) = u(x)v(x) - \int v(x)du(x).$$

This completes the argument. □

6. Convexity and concavity in the context of asymptotes



- Convex functions lie above their horizontal/slant asymptotes, whereas concave functions lie below them.
  - This fact is sometimes helpful for sketching graphs of functions.
- Proof: Lecture Notes (subsection 4.14.6).

Theorem 4.14.7

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and let $c \in \mathbb{R}$ be such that $(c, +\infty) \subseteq A$. Let $a, b \in \mathbb{R}$, and assume that $y = ax + b$ is a horizontal/slant asymptote of f as $x \to +\infty$.[a] Then all the following hold:

(a) if f is convex on $(c, +\infty)$, then $f(x) \geq ax + b$ for all $x \in (c, +\infty)$;

(b) if f is strictly convex on $(c, +\infty)$, then $f(x) > ax + b$ for all $x \in (c, +\infty)$;

(c) if f is concave on $(c, +\infty)$, then $f(x) \leq ax + b$ for all $x \in (c, +\infty)$;

(d) if f is strictly concave on $(c, +\infty)$, then $f(x) < ax + b$ for all $x \in (c, +\infty)$.

---

[a] If $a = 0$, then $y = ax + b$ (i.e. $y = b$) is a horizontal asymptote of f as $x \to +\infty$. On the other hand, if $a \neq 0$, then $y = ax + b$ is a slant asymptote of f as $x \to +\infty$.

Theorem 4.14.8

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and let $c \in \mathbb{R}$ be such that $(-\infty, c) \subseteq A$. Let $a, b \in \mathbb{R}$, and assume that $y = ax + b$ is a horizontal/slant asymptote of f as $x \to -\infty$.[a] Then all the following hold:

(a) if f is convex on $(-\infty, c)$, then $f(x) \geq ax + b$ for all $x \in (-\infty, c)$;

(b) if f is strictly convex on $(-\infty, c)$, then $f(x) > ax + b$ for all $x \in (-\infty, c)$;

(c) if f is concave on $(-\infty, c)$, then $f(x) \leq ax + b$ for all $x \in (-\infty, c)$;

(d) if f is strictly concave on $(-\infty, c)$, then $f(x) < ax + b$ for all $x \in (-\infty, c)$.

---

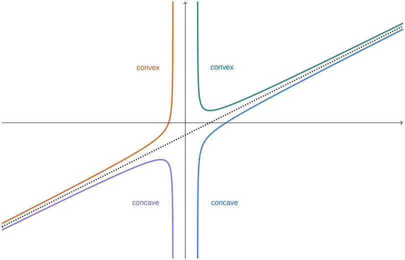
[a] If $a = 0$, then $y = ax + b$ (i.e. $y = b$) is a horizontal asymptote of f as $x \to -\infty$. On the other hand, if $a \neq 0$, then $y = ax + b$ is a slant asymptote of f as $x \to -\infty$.

- The key ingredient is the following lemma, which deals with the special case when $y = 0$ (i.e. the x-axis) is a horizontal asymptote of our function f as $x \to +\infty$.

Lemma 4.14.6

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and let $c \in \mathbb{R}$ be such that $(c, +\infty) \subseteq A$. Assume that $y = 0$ (i.e. the x-axis) is a horizontal asymptote of f as $x \to +\infty$. Then all the following hold:

a. if f is convex on $(c, +\infty)$, then $f(x) \geq 0$ for all $x \in (c, +\infty)$;
b. if f is strictly convex on $(c, +\infty)$, then $f(x) > 0$ for all $x \in (c, +\infty)$;
c. if f is concave on $(c, +\infty)$, then $f(x) \leq 0$ for all $x \in (c, +\infty)$;
d. if f is strictly concave on $(c, +\infty)$, then $f(x) < 0$ for all $x \in (c, +\infty)$.



- To prove Theorem 4.14.7 (where f has a horizontal/slant asymptote $y = ax + b$ as $x \to +\infty$), we define an auxiliary function $g(x) := f(x) - (ax - b)$, which has the x-axis as a horizontal asymptote as $x \to +\infty$, and also has the same convexity/concavity as f.
  - We apply Lemma 4.14.6 to g, and the result for f follows.
- Theorem 4.14.8 ("horizontal/slant asymptotote as $x \to -\infty$") can be reduced to Theorem 4.14.7 ("horizontal/slant asymptotote as $x \to +\infty$").
  - Indeed, if f has a horizontal/slant asymptote $y = ax + b$ as $x \to -\infty$, then we reflect f about the y-axis, and we apply Theorem 4.14.7 to the resulting function.

Example 4.15.3

Sketch the graph of the function $f(x) = x \arctan x$.

Solution. The domain of the function is $\mathbb{R}$, and clearly, f is continuous. Therefore, f has no vertical asymptotes. Next, note that for all $x \in \mathbb{R}$, we have that

$$f(-x) \;=\; (-x)\arctan(-x) \;\stackrel{(*)}{=}\; (-x)(-\arctan x) \;=\; x\arctan x \;=\; f(x),$$

where (*) follows from the fact that the arctangent function is odd. So, f is even, which means that its graph is symmetric about the y-axis.

Example 4.15.3

Sketch the graph of the function $f(x) = x \arctan x$.

Solution (continued). Now, recall that

$$\lim_{x \to +\infty} \arctan x = \tfrac{\pi}{2} \qquad \text{and} \qquad \lim_{x \to -\infty} \arctan x = -\tfrac{\pi}{2}.$$

It then readily follows that

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} (x \arctan x) = +\infty$$

and

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} (x \arctan x) = +\infty$$

Thus, f has no horizontal asymptotes.

Example 4.15.3

Sketch the graph of the function $f(x) = x \arctan x$.

Solution (continued). On the other hand, we have that

$$\lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \arctan x = \frac{\pi}{2} =: a,$$

and that

$$\begin{aligned}
\lim_{x\to+\infty} \left(f(x) - ax\right) &= \lim_{x\to+\infty} \left(x \arctan x - \tfrac{\pi}{2}x\right) \\
&= \lim_{x\to+\infty} x(\arctan x - \tfrac{\pi}{2}) \\
&= \lim_{x\to+\infty} \frac{\arctan x - \frac{\pi}{2}}{\frac{1}{x}} \\
&= \lim_{x\to+\infty} \frac{\frac{d}{dx}(\arctan x - \frac{\pi}{2})}{\frac{d}{dx}(\frac{1}{x})} \qquad \text{via L'Hôpital's Rule, since } \lim_{x\to+\infty} (\arctan x - \tfrac{\pi}{2}) = 0 \text{ and } \lim_{x\to+\infty} (\tfrac{1}{x}) = 0 \\
&= \lim_{x\to+\infty} \frac{\frac{1}{1+x^2}}{-\frac{1}{x^2}} \\
&= \lim_{x\to+\infty} \left(-\frac{x^2}{x^2+1}\right) \\
&= \lim_{x\to+\infty} \left(-\frac{1}{1+\frac{1}{x^2}}\right) \\
&= -1 =: b.
\end{aligned}$$

Thus, $y = \frac{\pi}{2}x - 1$ is a slant asymptote of f as $x \to +\infty$.

Example 4.15.3

Sketch the graph of the function $f(x) = x \arctan x$.

Solution (continued). Reminder: $y = \frac{\pi}{2}x - 1$ is a slant asymptote of f as $x \to +\infty$.

Now, the line obtained by reflecting $y = \frac{\pi}{2}x - 1$ about the y-axis is $y = -\frac{\pi}{2}x - 1$. Since f is even, it follows that $y = -\frac{\pi}{2}x - 1$ is a slant asymptote of f as $x \to -\infty$.

Note that $f(0) = 0$, and in fact, $f(x) = 0$ only for $x = 0$. Thus, the graph of f passes through the origin, and it does not intersect the x-axis or the y-axis anywhere else.

Example 4.15.3

Sketch the graph of the function $f(x) = x \arctan x$.

Solution (continued). Next, a straightforward calculation shows that

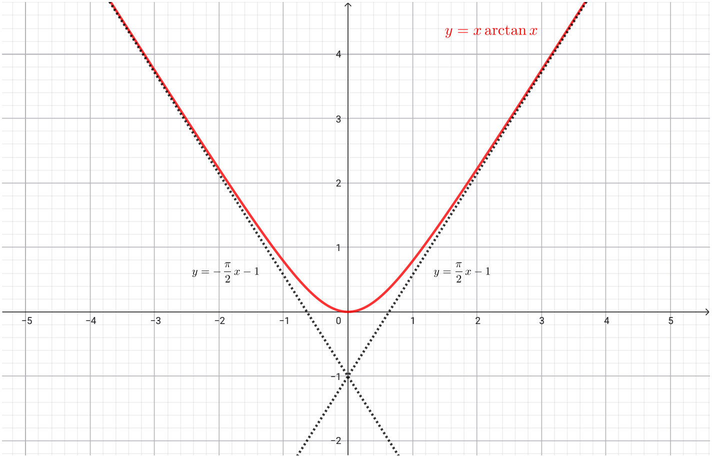
$$f'(x) = \arctan x + \tfrac{x}{1+x^2} \qquad \text{and} \qquad f''(x) = \tfrac{2}{(1+x^2)^2}$$

Note that $f'(0) = 0$, and that $f''(0) > 0$. Therefore, by the Second Derivative Test, f reaches a local minimum at $x = 0$, and clearly, we have that $f(0) = 0$. But in fact, $f''(x) > 0$ for all $x \in \mathbb{R}$, and consequently, f is strictly convex. Thus, the graph of f is as in the picture on the next slide.

Example 4.15.3

Sketch the graph of the function $f(x) = x \arctan x$.

Solution (continued).



Example 4.15.3

Sketch the graph of the function $f(x) = x \arctan x$.

Solution (continued).

- **Remark:** Note that the fact that $f''(x) > 0$ for all $x \in \mathbb{R}$ implies that f' is strictly increasing. Since $f'(0) = 0$, it follows that $f'(x) < 0$ for all $x \in (-\infty, 0)$, whereas $f'(x) > 0$ for all $x \in (0, +\infty)$. This yields the table below, which is consistent with the graph that we obtained above. Note, however, that the behavior of f' would have been quite difficult to analyze without using f''.

x	$(-\infty, 0)$	0	$(0, +\infty)$
$f'(x)$	$-$	0	$+$
$f(x)$	$\searrow$	min	$\nearrow$

$\square$