Mathematical Analysis 1

Lecture #12

Local Extrema: the First and Second Derivative Test. Convexity and concavity

Irena Penev

April 27, 2026

- This lecture has five parts:
  1. Higher order derivatives
  2. Intermission: a review of local extrema and monotonicity
  3. Local exrema: the First and Second Derivative Tests
  4. Convexity and concavity
  5. Sketching graphs of functions

1. Higher order derivatives

- Suppose that $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is a function, and that A' is the set of all points in A that f is differentiable at.
  - Then we can treat $f' : A' \to \mathbb{R}$ as a function in its own right.
    - It is possible that $A' = \emptyset$. In this case, f' is simply the "empty function."
  - If A'' is the set of all points in A' that f' is differentiable at, then we can treat $f'' : A'' \to \mathbb{R}$ as a function in its own right, and we can repeat the process.
    - Here, $f'' = (f')'$.
  - The function f'' is called the *second derivative* of f, and likewise $f''' := (f'')'$ is the *third derivative* of f (the domain of f''' is the set of all points at which f'' is differentiable).
  - Meanwhile, f' is simply the *first derivative* of f.

Example 4.12.1

Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \sin x + \cos x$ for all $x \in \mathbb{R}$. Then for $x \in \mathbb{R}$, we have:

- $f'(x) = \cos x - \sin x$;
- $f''(x) = -\sin x - \cos x$;
- $f'''(x) = -\cos x + \sin x$.

- Beyond the third derivative, it is unusual to keep adding primes (visually, it is difficult to distinguish between f'''' and f''''').
  - So, we use the notation $f^{(n)}$ instead.
  - Let us be more formal and give a proper recursive definition.

Definition

Suppose that $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function. We define $f^{(0)} := f$. Next, for $n \in \mathbb{N}_0$, if B is the set of all points at which $f^{(n)}$ is differentiable, then we define $f^{(n+1)} : B \to \mathbb{R}$ to be $f^{(n+1)} = (f^{(n)})'$.

- Sometimes, instead of $f^{(n)}(x)$, we write $\frac{d^n}{x^n} f(x)$ or $\frac{d^n f}{x^n}(x)$. If we are given $y = f(x)$, then we may write $\frac{d^n y}{dx^x}$ instead of $f^{(n)}(x)$. Notation $\frac{d^n}{x^n} f(x)$, $\frac{d^n f}{x^n}(x)$, and $\frac{d^n y}{dx^x}$ is referred to as the *Leibniz notation.*

Definition

Suppose that $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is a function. For $n \in \mathbb{N}$, we say that f is *n times differentiable* at a point $a \in A$ if $f^{(n)}(a)$ exists. For $n \in \mathbb{N}$, f is *n times differentiable* if $f^{(n)}(a)$ exists for all $a \in A$. The function f is *infinitely differentiable* (or *smooth*) if f is n times differentiable for all $n \in \mathbb{N}$.

Definition

Suppose that $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is a function, and that I is an open interval. For $n \in \mathbb{N}$, we say that f is *n times differentiable on I* if $f^{(n)}(a)$ exists for all $a \in I$. The function f is *infinitely differentiable on I* (or *smooth on I*) if f is n times differentiable on I for all $n \in \mathbb{N}$.

Definition

A function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is *continuously differentiable* if it is differentiable,[a] and in addition, $f' : A \to \mathbb{R}$ is continuous. For $n \in \mathbb{N}$, f is *n times continuously differentiable* if f is n times differentiable, and moreover, $f^{(n)} : A \to \mathbb{R}$ is continuous.

---

[a] So, f is differentiable at all points in its domain A.

- **Remark:** By Theorem 4.3.1, all differentiable functions are continuous.
  - Thus, if a function f is $(n+1)$ times differentiable (for some $n \in \mathbb{N}$), then f must be n times continuously differentiable; however, the converse is false in general.

Example 4.12.2

Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) \quad := \quad \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

for all $x \in \mathbb{R}$. Show that f is differentiable, but not continuously differentiable (and consequently, f is not twice differentiable).

Solution. For $x \in \mathbb{R} \setminus \{0\}$, we have that

$$\begin{aligned} f'(x) &= \tfrac{d}{dx}\left(x^2 \sin\left(\tfrac{1}{x}\right)\right) = 2x \sin\left(\tfrac{1}{x}\right) + x^2 \cos\left(\tfrac{1}{x}\right) \cdot \left(-\tfrac{1}{x^2}\right) \\ &= 2x \sin\left(\tfrac{1}{x}\right) - \cos\left(\tfrac{1}{x}\right), \end{aligned}$$

whereas

$$f'(0) = \lim_{x \to 0} \tfrac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \tfrac{x^2 \sin\left(\frac{1}{x}\right) - 0}{x} = \lim_{x \to 0} x \sin\left(\tfrac{1}{x}\right) \overset{(*)}{=} 0,$$

where (*) follows via an easy application of the Squeeze Theorem. Thus, f is differentiable.

Example 4.12.2

Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) \quad := \quad \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

for all $x \in \mathbb{R}$. Show that f is differentiable, but not continuously differentiable (and consequently, f is not twice differentiable).

Solution (continued). Reminder: $f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$ for $x \in \mathbb{R} \setminus \{0\}$, $f'(0) = 0$.

However, note that $\lim\limits_{x \to 0} f'(x)$ does not exist.

- Indeed, $\lim\limits_{x \to 0} 2x \sin\left(\frac{1}{x}\right) = 0$ (via an easy application of the Squeeze Theorem), but $\cos\left(\frac{1}{x}\right)$ keeps oscillating as x approaches 0. (Try to give a fully formal proof!)

So, f' is not continuous at 0, and consequently, f' is not continuously differentiable. $\square$

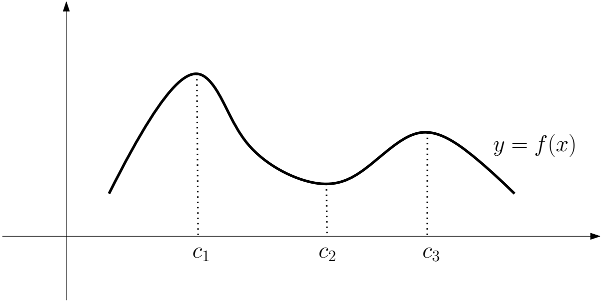
2. Intermission: a review of local extrema and monotonicity

Definition

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and $c \in A$.

- We say that f reaches a *local maximum* at c if there exists some $\delta > 0$ s.t. for all $x \in A$, if $|x - c| < \delta$, then $f(c) \geq f(x)$.
- We say that f reaches a *local minimum* at c if there exists some $\delta > 0$ s.t. for all $x \in A$, if $|x - c| < \delta$, then $f(c) \leq f(x)$.
- We say that f reaches a *local extremum* at c if f reaches a local maximum or a local minimum at c.

- For instance, in the picture below, the function f reaches local maxima at c_1 and c_3, and it reaches a local minimum at c_2.

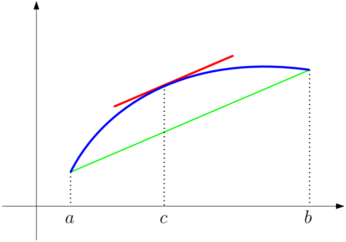


Theorem 4.10.2

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and assume that f reaches a local extremum at a point $c_0 \in I \subseteq A$, where I is an open interval. If f is differentiable at c_0, then $f'(c_0) = 0$.

Lagrange's Mean Value Theorem

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists some $c \in (a, b)$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$.



Corollary 4.10.3

Let I be an open interval in $\mathbb{R}$, and let $f : I \to \mathbb{R}$ be a differentiable function. Then all the following hold:

(a) if $f'(x) = 0$ for all $x \in I$, then f constant on I;

(b) if $f'(x) \geq 0$ for all $x \in I$, then f is non-decreasing on I;

(c) if $f'(x) > 0$ for all $x \in I$, then f is strictly increasing on I;

(d) if $f'(x) \leq 0$ for all $x \in I$, then f is non-increasing on I;

(e) if $f'(x) < 0$ for all $x \in I$, then f is strictly decreasing on I.

- Partial converse of Corollary 4.10.3:

Proposition 4.10.4

Let I be an open interval in $\mathbb{R}$, and let $f : I \to \mathbb{R}$ be a differentiable function. Then all the following hold:

(a) if f is constant on I, then $f'(x) = 0$ for all $x \in I$;
(b) if f is non-decreasing on I, then $f'(x) \geq 0$ for all $x \in I$;
(c) if f is non-increasing on I, then $f'(x) \leq 0$ for all $x \in I$.

- Let's slightly strengthen Corollary 4.10.3!

Definition

The *interior* of a set $A \subseteq \mathbb{R}$, denoted by A° or $\text{int}(A)$, is the set of all points $a \in A$ for which there exists some $\delta > 0$ s.t. $(a - \delta, a + \delta) \subseteq A$. In other words, a point $a \in A$ belongs to the interior of A if some open interval containing a is included in A.

- Any subset of $\mathbb{R}$ has a (possibly empty) interior.
- Here, we are interested in intervals (next slide).

Definition

The *interior* of a set $A \subseteq \mathbb{R}$, denoted by A° or $\text{int}(A)$, is the set of all points $a \in A$ for which there exists some $\delta > 0$ s.t. $(a-\delta, a+\delta) \subseteq A$. In other words, a point $a \in A$ belongs to the interior of A if some open interval containing a is included in A.

- For $a, b \in \mathbb{R}$ s.t. $a < b$, we have the following:
    - the interior of (a, b) is (a, b);
    - the interior of $[a, b)$ is (a, b);
    - the interior of $(a, b]$ is (a, b);
    - the interior of $[a, b]$ is (a, b);
    - the interior of $(-\infty, b)$ is $(-\infty, b)$;
    - the interior of $(-\infty, b]$ is $(-\infty, b)$;
    - the interior of $(a, +\infty)$ is $(a, +\infty)$;
    - the interior of $[a, +\infty)$ is $(a, +\infty)$;
    - the interior of $(-\infty, +\infty)$ is $(-\infty, +\infty)$.
- So, informally, the interior of an interval is the open interval that we obtain by removing the “edges” (if any) of the original interval.

Theorem 4.13.1

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and let $I \subseteq A$ be an interval. Assume that f is continuous on I and differentiable on I°. Then all the following hold:

- ⓐ if $f'(x) = 0$ for all $x \in I^\circ$, then f constant on I;
- ⓑ if $f'(x) \geq 0$ for all $x \in I^\circ$, then f is non-decreasing on I;
- ⓒ if $f'(x) > 0$ for all $x \in I^\circ$, then f is strictly increasing on I;
- ⓓ if $f'(x) \leq 0$ for all $x \in I^\circ$, then f is non-increasing on I;
- ⓔ if $f'(x) < 0$ for all $x \in I^\circ$, then f is strictly decreasing on I.

- The proof relies on Lagrange's Mean Value Thoerem, and it is nearly identical to the proof of Corollary 4.13.1.
- Details: Lecture Notes.

3. Local exrema: the First and Second Derivative Tests

The First Derivative Test

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and let $a, b, c \in \mathbb{R}$ be s.t. $a < c < b$ and $(a, b) \subseteq A$. Assume that f is continuous on the open interval (a, b) and differentiable on the open intervals (a, c) and (c, b).[a]

(a) If both the following hold:
    - $f'(x) \leq 0$ for all $x \in (a, c)$,
    - $f'(x) \geq 0$ for all $x \in (c, b)$,

    then f reaches a local minimum at c.

(b) If both the following hold:
    - $f'(x) \geq 0$ for all $x \in (a, c)$,
    - $f'(x) \leq 0$ for all $x \in (c, b)$,

    then f reaches a local maximum at c.

---

[a] The function f may or may not be differentiable on the point c itself. However, by hypothesis, f is defined and continuous at c.

The First Derivative Test

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and let $a, b, c \in \mathbb{R}$ be s.t. $a < c < b$ and $(a, b) \subseteq A$. Assume that f is continuous on the open interval (a, b) and differentiable on the open intervals (a, c) and (c, b).

ⓐ If both the following hold:
  - $f'(x) \leq 0$ for all $x \in (a, c)$,
  - $f'(x) \geq 0$ for all $x \in (c, b)$,

  then f reaches a local minimum at c.

Proof of (a). Suppose that $f'(x) \leq 0$ for all $x \in (a, c)$, and that $f'(x) \geq 0$ for all $x \in (c, b)$. Then Theorem 4.13.1 guarantees that f is non-increasing on $(a, c]$ and non-decreasing on $[c, b)$, and it follows that f reaches a local minimum at c. ☐

The First Derivative Test

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and let $a, b, c \in \mathbb{R}$ be s.t. $a < c < b$ and $(a, b) \subseteq A$. Assume that f is continuous on the open interval (a, b) and differentiable on the open intervals (a, c) and (c, b).

(b) If both the following hold:
  - $f'(x) \geq 0$ for all $x \in (a, c)$,
  - $f'(x) \leq 0$ for all $x \in (c, b)$,

  then f reaches a local maximum at c.

Proof of (b). Suppose that $f'(x) \geq 0$ for all $x \in (a, c)$, and that $f'(x) \leq 0$ for all $x \in (c, b)$. Then Theorem 4.13.1 guarantees that f is non-decreasing on $(a, c]$ and non-increasing on $[c, b)$, and it follows that f reaches a local maximum at c. ☐

The First Derivative Test

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and let $a, b, c \in \mathbb{R}$ be s.t. $a < c < b$ and $(a, b) \subseteq A$. Assume that f is continuous on the open interval (a, b) and differentiable on the open intervals (a, c) and (c, b).[a]

ⓐ If both the following hold:
  - $f'(x) \leq 0$ for all $x \in (a, c)$,
  - $f'(x) \geq 0$ for all $x \in (c, b)$,

  then f reaches a local minimum at c.

ⓑ If both the following hold:
  - $f'(x) \geq 0$ for all $x \in (a, c)$,
  - $f'(x) \leq 0$ for all $x \in (c, b)$,

  then f reaches a local maximum at c.

---

[a] The function f may or may not be differentiable on the point c itself. However, by hypothesis, f is defined and continuous at c.

- **Remark:** Schematically, the First Derivative Test (and its proof) can be represented by the two tables below.

$a \qquad c \qquad b$

x	(a,c)	(c,b)
$f'(x)$	$-$	$+$
$f(x)$	$\searrow$ min	$\nearrow$

$a \qquad c \qquad b$

x	(a,c)	(c,b)
$f'(x)$	$+$	$-$
$f(x)$	$\nearrow$ max	$\searrow$

  - Technically, we would use this sort of table representation for the case when the inequalities from the First Derivative Test are strict, i.e. when we have $f'(x) > 0$ and $f'(x) < 0$, rather than merely $f'(x) \geq 0$ and $f'(x) \leq 0$.

Proposition 4.13.2

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and let $a, b, c \in R$ be s.t. $a < c < b$ and $(a, b) \subseteq A$. Assume that f is continuous on the open interval (a, b) and differentiable on the open intervals (a, c) and (c, b).

(a) If $f'(x) > 0$ for all $x \in (a, c) \cup (c, b)$, then f is strictly increasing on (a, b), and in particular, f does not reach a local extremum at c.

(b) If $f'(x) < 0$ for all $x \in (a, c) \cup (c, b)$, then f is strictly decreasing on (a, b), and in particular, f does not reach a local extremum at c.

- **Remark:** The proposition applies even if f is not differentiable at c. However, for the proposition to hold, it is indeed necessary for f to be continuous on the whole interval (a, b), and in particular, to be defined at and continuous at the point c.

Proposition 4.13.2

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and let $a, b, c \in R$ be s.t. $a < c < b$ and $(a, b) \subseteq A$. Assume that f is continuous on the open interval (a, b) and differentiable on the open intervals (a, c) and (c, b).

(a) If $f'(x) > 0$ for all $x \in (a, c) \cup (c, b)$, then f is strictly increasing on (a, b), and in particular, f does not reach a local extremum at c.

(b) If $f'(x) < 0$ for all $x \in (a, c) \cup (c, b)$, then f is strictly decreasing on (a, b), and in particular, f does not reach a local extremum at c.

Proof of (a). If $f'(x) > 0$ for all $x \in (a, c) \cup (c, b)$, then Theorem 4.13.1 guarantees that f is strictly increasing on the intervals $(a, c]$ and $[c, b)$, and consequently, it is strictly increasing on the interval (a, b) as well. ☐

Proposition 4.13.2

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and let $a, b, c \in R$ be s.t. $a < c < b$ and $(a, b) \subseteq A$. Assume that f is continuous on the open interval (a, b) and differentiable on the open intervals (a, c) and (c, b).

(a) If $f'(x) > 0$ for all $x \in (a, c) \cup (c, b)$, then f is strictly increasing on (a, b), and in particular, f does not reach a local extremum at c.

(b) If $f'(x) < 0$ for all $x \in (a, c) \cup (c, b)$, then f is strictly decreasing on (a, b), and in particular, f does not reach a local extremum at c.

Proof of (b). If $f'(x) < 0$ for all $x \in (a, c) \cup (c, b)$, then Theorem 4.13.1 guarantees that f is strictly decreasing on the intervals $(a, c]$ and $[c, b)$, and consequently, it is strictly decreasing on the interval (a, b) as well. $\square$

Proposition 4.13.2

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and let $a, b, c \in R$ be s.t. $a < c < b$ and $(a, b) \subseteq A$. Assume that f is continuous on the open interval (a, b) and differentiable on the open intervals (a, c) and (c, b).

(a) If $f'(x) > 0$ for all $x \in (a, c) \cup (c, b)$, then f is strictly increasing on (a, b), and in particular, f does not reach a local extremum at c.

(b) If $f'(x) < 0$ for all $x \in (a, c) \cup (c, b)$, then f is strictly decreasing on (a, b), and in particular, f does not reach a local extremum at c.

- **Remark:** Schematically, Proposition 4.13.2 (and its proof) can be represented by the table below.

$a \quad c \quad b$

x	(a,c)	(c,b)
$f'(x)$	$+$	$+$
$f(x)$	$\nearrow$	$\nearrow$

$a \quad c \quad b$

x	(a,c)	(c,b)
$f'(x)$	$-$	$-$
$f(x)$	$\searrow$	$\searrow$

- For a concrete example of this sort of behavior, consider the functions $f, g : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ and $g(x) = -x^3$ for all $x \in \mathbb{R}$.
  - Then $f'(x) = 2x^2$ and $g'(x) = -2x^2$.
  - So, for $c = 0$, we have that $f'(x) = g'(c) = 0$.
  - However, for $x \in \mathbb{R} \setminus \{0\}$, we have $f'(x) > 0$ and $g'(x) < 0$.
  - Consistently, with Proposition 4.13.2, the function f is strictly increasing, whereas the function g is strictly decreasing.

Example 4.13.3

Consider the function $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = 3x^5 - 5x^3 + 7$ for all $x \in \mathbb{R}$. Find all points x at which f reaches a local minimum or a local maximum, and compute $f(x)$ for those values of x. Does f reach a global maximum or minimum? What are the minimum and maximum values of f (if they exist)?

Solution. First of all, we observe that the function f is continuous and differentiable on $\mathbb{R}$, and that

$$f'(x) \;=\; 15x^4 - 15x^2 \;=\; 15x^2(x-1)(x+1)$$

for all $x \in \mathbb{R}$. Thus, $f'(x) = 0$ for $x \in \{-1, 0, 1\}$ (and for no other value of x). Thus, by Theorem 4.10.2, the only values of x at which f may possibly reach a local extremum are $x = -1$, $x = 0$, and $x = 1$ (and we must check at which, if any, of those values of x the function f indeed reaches a local minimum or a local maximum). Now, consider the following table (next slide).

Solution (continued). Reminder: $f(x) = 3x^5 - 5x^3 + 7$;
$f'(x) = 15x^2(x-1)(x+1)$; $f(x) = 0$ precisely for $x \in \{-1, 0, 1\}$.

x	$(-\infty, -1)$	-1	$(-1, 0)$	0	$(0, 1)$	1	$(1, +\infty)$
x^2	$+$		$+$	0	$+$		$+$
$x-1$	$-$		$-$		$-$	0	$+$
$x+1$	$-$	0	$+$		$+$		$+$
$f'(x)$	$+$	0	$-$	0	$-$	0	$+$
$f(x)$	$\nearrow$	max	$\searrow$		$\searrow$	min	$\nearrow$

- **Remark:** The local maximum and the local minimum in the table above were identified via the First Derivative Test. Meanwhile, by Proposition 4.13.2, f does **not** reach a local extremum at $x = 0$.

Solution (continued). Reminder:

		-1		0		1	
x	$(-\infty,-1)$		$(-1,0)$		$(0,1)$		$(1,+\infty)$
x^2	$+$		$+$	0	$+$		$+$
$x-1$	$-$		$-$		$-$	0	$+$
$x+1$	$-$	0	$+$		$+$		$+$
$f'(x)$	$+$	0	$-$	0	$-$	0	$+$
$f(x)$	$\nearrow$	max	$\searrow$		$\searrow$	min	$\nearrow$

- We can now summarize:
  - f reaches a local maximum at $x = -1$, and we have that $f(-1) = 9$;
  - f reaches a local minimum at $x = 1$, and we have that $f(1) = 5$;
  - f has no other local extrema.
- Meanwhile, $\lim\limits_{x\to-\infty} f(x) = -\infty$ and $\lim\limits_{x\to+\infty} f(x) = +\infty$.
- Therefore, f reaches neither a global minimum nor a global maximum, and consequently, f has neither a minimum nor a maximum value.

Solution (continued).

- **Remark:** Using computer software, we can graph our function f (see below). Note that this graph is indeed consistent with our solution.

□

Example 4.13.4

Consider the function $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = \sqrt[3]{x^2}$. Find all points x at which f reaches a local minimum or a local maximum, and compute $f(x)$ for those values of x. Does f reach a global minimum or a global maximum? What are the minimum and maximum values of f (if they exist)?

Solution. The function f is defined and continuous on $\mathbb{R}$, and it is differentiable on $\mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, +\infty)$. For $x \in \mathbb{R} \setminus \{0\}$, we have that

$$f'(x) \quad = \quad \frac{2}{3\sqrt[3]{x}}$$

for all $x \in \mathbb{R}$. So, we have the table below.

x	$(-\infty, 0)$	0	$(0, +\infty)$
$f'(x)$	$-$	undef	$+$
$f(x)$	↘	min	↗

Solution (continued). Reminder:

x	$(-\infty, 0)$	0	$(0, +\infty)$
$f'(x)$	$-$	undef	$+$
$f(x)$	$\searrow$	min	$\nearrow$

So, f reaches a local minimum at $x = 0$, and we have that $f(0) = 0$. The function f has no other local extrema.

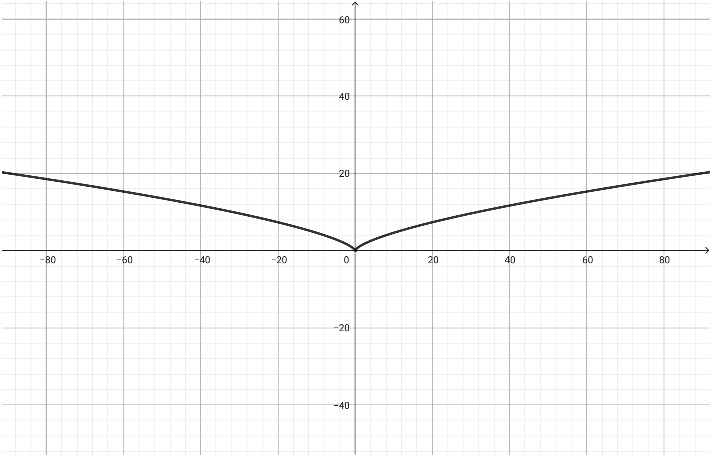
Moreover, the function f is strictly decreasing on $(-\infty, 0)$ and strictly increasing on $(0, +\infty)$.

So, f reaches a global minimum at $x = 0$, and the minimum value of f is $f(0) = 0$.

The function f does not reach a global maximum, and consequently, f does not have a maximum value.

Solution (continued).

- **Remark:** Using computer software, we can graph our function f (see below). Note that this graph is indeed consistent with our solution.



$\square$

The Second Derivative Test

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and let $a, b, c \in \mathbb{R}$ be s.t. $a < c < b$ and $(a, b) \subseteq A$. Assume that f is differentiable on the interval (a, b), that $f'(c) = 0$, and that $f''(c)$ exists.

(a) If $f''(c) > 0$, then f reaches a local minimum at c.

(b) If $f''(c) < 0$, then f reaches a local maximum at c.

Proof. We prove (a); the proof of (b) is similar. So, let us assume that $f''(c) > 0$. We will exhibit some $\delta > 0$ s.t. $(c - \delta, c + \delta) \subseteq (a, b)$, and s.t. the following hold:

- $f'(x) < 0$ for all $x \in (c - \delta, c)$;
- $f'(x) > 0$ for all $x \in (c, c + \delta)$.

This is enough, for the First Derivative Test will then imply that f reaches a local minimum at c, which is what we need to show.

Proof (continued). Reminder: $f'(c) = 0$; $f''(c) < 0$; WTS $\exists \delta > 0$ s.t. $(c - \delta, c + \delta) \subseteq (a, b)$, $f'(x) < 0\ \forall x \in (c - \delta, c)$, $f'(x) > 0\ \forall x \in (c, c + \delta)$.

First, note that $f''(c) \overset{(*)}{=} \lim_{x \to c} \frac{f'(x) - f'(c)}{x - c} \overset{(**)}{=} \lim_{x \to c} \frac{f'(x)}{x - c}$, where (*) follows from the definition of the derivative, whereas (**) follows from the fact that $f'(c) = 0$.

Now, set $\varepsilon := \frac{f''(c)}{2}$; since $f''(c) > 0$, we have that $\varepsilon > 0$. Fix $\delta > 0$ s.t. for all $x \in (a, b)$, if $0 < |x - c| < \delta$, then $|\frac{f'(x)}{x - c} - f''(c)| < \varepsilon$.

Since $c \in (a, b)$, WMA $(c - \delta, c + \delta) \subseteq (a, b)$, for otherwise, we simply choose a smaller δ. So, for all $x \in (c - \delta, c) \cup (c, c + \delta)$, we have that $|\frac{f'(x)}{x - c} - f''(c)| < \varepsilon$, which is equivalent to $\frac{f''(c)}{2} = f''(c) - \varepsilon < \frac{f'(x)}{x - c} < f''(c) + \varepsilon = \frac{3f''(c)}{2}$, which in turn implies that $\frac{f'(x)}{x - c} > \frac{f''(c)}{2} > 0$. Therefore, for any $x \in (c - \delta, c)$, we have that $f'(x) < 0$ (because $x - c < 0$), whereas for any $x \in (c, c + \delta)$, we have that $f'(x) > 0$ (because $x - c > 0$). $\square$

The Second Derivative Test

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and let $a, b, c \in \mathbb{R}$ be s.t. $a < c < b$ and $(a, b) \subseteq A$. Assume that f is differentiable on the interval (a, b), that $f'(c) = 0$, and that $f''(c)$ exists.

ⓐ If $f''(c) > 0$, then f reaches a local minimum at c.

ⓑ If $f''(c) < 0$, then f reaches a local maximum at c.

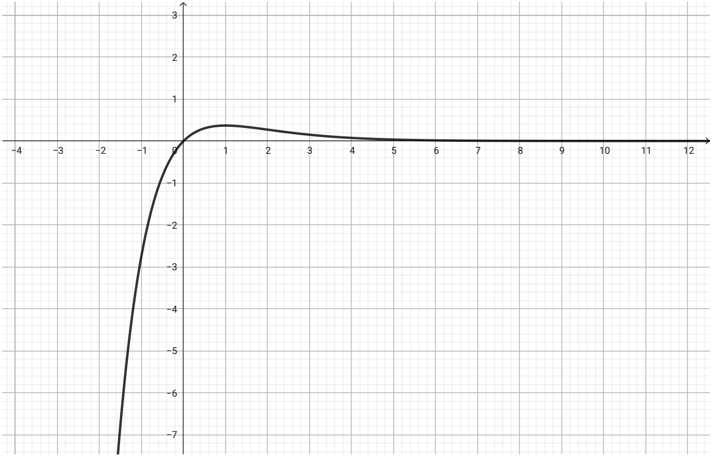
Example 4.13.5

Find all the local minima and local maxima (if any) of the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = xe^{-x}$ for all $x \in \mathbb{R}$.

Solution. Note that $f'(x) = (1 - x)e^{-x}$ and $f''(x) = (x - 2)e^{-x}$ for all $x \in \mathbb{R}$. Since $f'(x)$ is defined for all $x \in \mathbb{R}$, and since we have $f'(x) = 0$ only for $x = 1$, we see that the only “candidate” for a local extremum is $x = 1$. Since $f''(1) = -e^{-1} < 0$, the Second Derivative Test guarantees that f reaches a local maximum at $c = -1$ (with $f(-1) = -e^{-1}$).

Solution (continued).

- **Remark:** Using computer software, we can graph our function f (see below). Note that this graph is indeed consistent with our solution.



□

The First Derivative Test

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and let $a, b, c \in \mathbb{R}$ be s.t. $a < c < b$ and $(a, b) \subseteq A$. Assume that f is continuous on the open interval (a, b) and differentiable on the open intervals (a, c) and (c, b).[a]

(a) If both the following hold:
  - $f'(x) \leq 0$ for all $x \in (a, c)$,
  - $f'(x) \geq 0$ for all $x \in (c, b)$,

  then f reaches a local minimum at c.

(b) If both the following hold:
  - $f'(x) \geq 0$ for all $x \in (a, c)$,
  - $f'(x) \leq 0$ for all $x \in (c, b)$,

  then f reaches a local maximum at c.

---

[a] The function f may or may not be differentiable on the point c itself. However, by hypothesis, f is defined and continuous at c.

The Second Derivative Test

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and let $a, b, c \in \mathbb{R}$ be s.t. $a < c < b$ and $(a, b) \subseteq A$. Assume that f is differentiable on the interval (a, b), that $f'(c) = 0$, and that $f''(c)$ exists.

(a) If $f''(c) > 0$, then f reaches a local minimum at c.

(b) If $f''(c) < 0$, then f reaches a local maximum at c.

- **Remark:** In practice, the First Derivative Test is used much more frequently than the Second Derivative Test.

- Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function, where $a, b \in \mathbb{R}$ are s.t. $a < b$.
- Then by the Extreme Value Theorem, f reaches both a global maximum and a global minimum on $[a, b]$.
- While there is no algorithm for finding the points at which f reaches its global maximum and minimum, there is a strategy that will work if f is "reasonably well behaved," as follows.
  1. We find all the points $c \in (a, b)$ s.t. either $f'(c) = 0$ or $f'(c)$ does not exist. These points c are the "candidates" for points at which f might reach a local minimum or a local maximum on the interval (a, b).
     - This is the step where our strategy might fall apart if f is "poorly behaved." The implicit assumption is that there are only finitely many such points c, and that we know how to find them.
  2. We compute $f(c)$ for all values of c that we found in the previous step, and we compute $f(a)$ and $f(b)$.
  3. By comparing all the values that we computed in the previous step, we find the maximum and minimum values of f on the interval $[a, b]$.

Example 4.13.6

Find the maximum and the minimum value of the function $f(x) = \sin^3 x + \cos^3 x$ on the interval $[-\frac{\pi}{4}, \frac{3\pi}{4}]$.

Solution. Note that $f'(x) = 3 \sin x \cos x(\sin x - \cos x)$ for all $x \in \mathbb{R}$ (in particular, f is differentiable everywhere on $\mathbb{R}$). Note that, for $x \in (-\frac{\pi}{4}, \frac{3\pi}{4})$, we have that $f'(x) = 0$ iff $x \in \{0, \frac{\pi}{4}, \frac{\pi}{2}\}$. So, our "candidates" for points at which f might reach its global maximum or its global minimum are the following points: $-\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$. We now compute:

- $f(-\frac{\pi}{4}) = \sin^3(-\frac{\pi}{4}) + \cos^3(-\frac{\pi}{4}) = (-\frac{1}{\sqrt{2}})^3 + (\frac{1}{\sqrt{2}})^3 = 0$;
- $f(0) = \sin^3(0) + \cos^3(0) = 0^3 + 1^3 = 1$;
- $f(\frac{\pi}{4}) = \sin^3(\frac{\pi}{4}) + \cos^3(\frac{\pi}{4}) = (\frac{1}{\sqrt{2}})^3 + (\frac{1}{\sqrt{2}})^3 = \frac{1}{\sqrt{2}}$;
- $f(\frac{\pi}{2}) = \sin^3(\frac{\pi}{2}) + \cos^3(\frac{\pi}{2}) = 1^0 + 0^3 = 1$;
- $f(\frac{3\pi}{4}) = \sin^3(\frac{3\pi}{4}) + \cos^3(\frac{3\pi}{4}) = (\frac{1}{\sqrt{2}})^3 + (-\frac{1}{\sqrt{2}})^3 = 0$.

Example 4.13.6

Find the maximum and the minimum value of the function $f(x) = \sin^3 x + \cos^3 x$ on the interval $[-\frac{\pi}{4}, \frac{3\pi}{4}]$.

Solution (continued). So, f reaches a global minimum on the interval $[-\frac{\pi}{4}, \frac{3\pi}{4}]$ at the points $m_1 = -\frac{\pi}{4}$ and $m_2 = \frac{3\pi}{4}$, and the minimum value of f on this interval is $f(m_1) = f(m_2) = 0$. On the other hand, f reaches a global maximum on the interval $[-\frac{\pi}{4}, \frac{3\pi}{4}]$ at the points $M_1 = 0$ and $M_2 = \frac{\pi}{2}$, and the maximum value of f on this interval is $f(M_1) = f(M_2) = 1$.

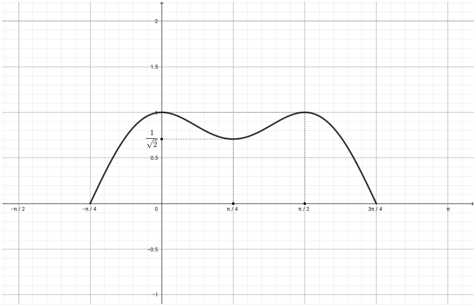
The minimum value of f on the interval $[-\frac{\pi}{4}, \frac{3\pi}{4}]$ is 0, and the maximum value of f on this interval is 1.

Example 4.13.6

Find the maximum and the minimum value of the function $f(x) = \sin^3 x + \cos^3 x$ on the interval $[-\frac{\pi}{4}, \frac{3\pi}{4}]$.

Solution (continued).

- **Remark:** Using computer software, we get the graph below. This is consistent with our calculation above.



□

4. Convexity and concavity

- Recall from algebra that for $x_1, x_2 \in \mathbb{R}$ s.t. $x_1 < x_2$, the closed interval $[x_1, x_2]$ is given by

$$[x_1, x_2] \quad = \quad \{\lambda_1 x_1 + \lambda_2 x_2 \mid \lambda_1, \lambda_2 \in \mathbb{R},\ \lambda_1, \lambda_2 \geq 0,\ \lambda_1 + \lambda_2 = 1\}.$$

  - In fact, for $x \in [x_1, x_2]$, we have that

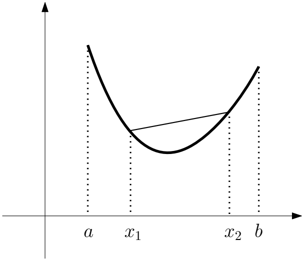
$$x \quad = \quad \tfrac{x_2 - x}{x_2 - x_1} x_1 + \tfrac{x - x_1}{x_2 - x_1} x_2,$$

    and so if we set

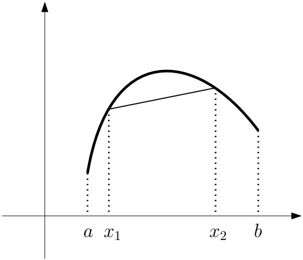
$$\lambda_1 := \tfrac{x_2 - x}{x_2 - x_1} \qquad \text{and} \qquad \lambda_2 := \tfrac{x - x_1}{x_2 - x_1},$$

    then we get $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = 1$, and $x = \lambda_1 x_1 + \lambda_2 x_2$.

- The idea behind convexity and concavity is in the picture below.



convex



concave

- Now let's give a proper definition!

Definition

For a function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and an interval $I \subseteq A$, we say that:

- f is *convex* on the interval I provided that for all $x_1, x_2 \in I$, and all real numbers $\lambda_1, \lambda_2 \geq 0$ s.t. $\lambda_1 + \lambda_2 = 1$, we have that

$$f(\lambda_1 x_1 + \lambda_2 x_2) \quad \leq \quad \lambda_1 f(x_1) + \lambda_2 f(x_2);$$

- f is *strictly convex* on the interval I provided that for all distinct $x_1, x_2 \in I$, and all real numbers $\lambda_1, \lambda_2 > 0$ s.t. $\lambda_1 + \lambda_2 = 1$, we have that

$$f(\lambda_1 x_1 + \lambda_2 x_2) \quad < \quad \lambda_1 f(x_1) + \lambda_2 f(x_2);$$

Definition (continued)

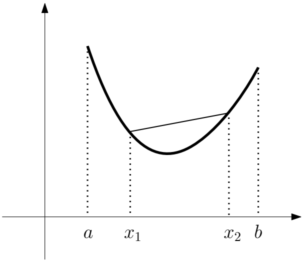
For a function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and an interval $I \subseteq A$, we say that:

- f is *concave* on the interval I provided that for all $x_1, x_2 \in I$, and all real numbers $\lambda_1, \lambda_2 \geq 0$ s.t. $\lambda_1 + \lambda_2 = 1$, we have that

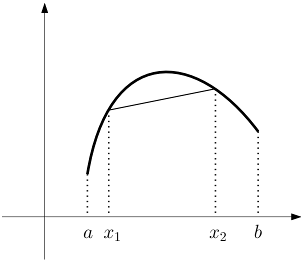
$$f(\lambda_1 x_1 + \lambda_2 x_2) \quad \geq \quad \lambda_1 f(x_1) + \lambda_2 f(x_2);$$

- f is *strictly concave* on the interval I provided that for all distinct $x_1, x_2 \in I$, and all real numbers $\lambda_1, \lambda_2 > 0$ s.t. $\lambda_1 + \lambda_2 = 1$, we have that

$$f(\lambda_1 x_1 + \lambda_2 x_2) \quad > \quad \lambda_1 f(x_1) + \lambda_2 f(x_2).$$



convex



concave

Definition

A function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is *affine* on an interval $I \subseteq A$ if there exist real numbers $m, b \in \mathbb{R}$ s.t.

$$f(x) \;:=\; mx + b \qquad \forall x \in I.$$

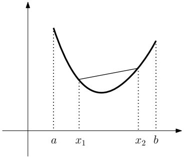
Proposition 4.14.1

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and let $I \subseteq A$ be an interval. Then the following are equivalent:

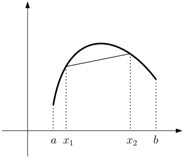
- f is both convex and concave on I;
- f is affine on I.

Moreover, in this case, f is neither strictly convex nor strictly concave.

- Proof: Lecture Notes.



convex



concave

Theorem 4.14.2

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, differentiable on an open interval $I \subseteq A$. Then all the following hold:

(a) f' is non-decreasing on I iff f is convex on I;
(b) f' is strictly increasing on I iff f is strictly convex on I;
(c) f' is non-increasing on I iff f is concave on I;
(d) f' is strictly decreasing on I iff f is strictly concave on I.

- Proof: Lecture Notes.

Corollary 4.14.3

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, differentiable on an open interval $I \subseteq A$. Then all the following hold:

(a) f is convex on I iff $f''(x) \geq 0$ for all $x \in I$;

(b) if $f''(x) > 0$ for all $x \in I$, then f is strictly convex on I;

(c) f is concave on I iff $f''(x) \leq 0$ for all $x \in I$.

(d) if $f''(x) < 0$ for all $x \in I$, then f is strictly concave on I.

- **Remark:** The converses of (b) and (d) are false. For instance, $f_1(x) = x^4$ is strictly convex on any open interval, but $f_1''(0) = 0$. Similarly, $f_2(x) = -x^4$ is strictly concave on any open interval, but $f_2''(0) = 0$.

Corollary 4.14.3

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, differentiable on an open interval $I \subseteq A$. Then all the following hold:

(a) f is convex on I iff $f''(x) \geq 0$ for all $x \in I$;

(b) if $f''(x) > 0$ for all $x \in I$, then f is strictly convex on I;

(c) f is concave on I iff $f''(x) \leq 0$ for all $x \in I$.

(d) if $f''(x) < 0$ for all $x \in I$, then f is strictly concave on I.

Proof. We prove (a) and (b); the proofs of (c) and (d) are analogous.

We first prove (a). By Theorem 4.14.2, f is convex on I iff f' is non-decreasing on I. But by Corollary 4.10.3 and Proposition 4.10.4, f' is non-decreasing on I iff $f''(x) \geq 0$ for all $x \in I$. This proves (a).

For (b), we observe that if $f''(x) > 0$ for all $x \in I$, then Corollary 4.10.3 implies that f' is strictly increasing, and so by Theorem 4.14.2, f is strictly convex. $\square$

Corollary 4.14.3

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, differentiable on an open interval $I \subseteq A$. Then all the following hold:

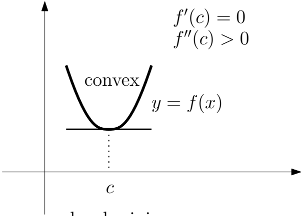
(a) f is convex on I iff $f''(x) \geq 0$ for all $x \in I$;

(b) if $f''(x) > 0$ for all $x \in I$, then f is strictly convex on I;

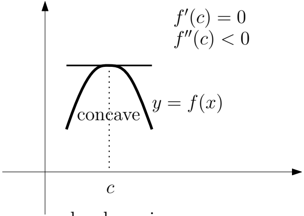
(c) f is concave on I iff $f''(x) \leq 0$ for all $x \in I$.

(d) if $f''(x) < 0$ for all $x \in I$, then f is strictly concave on I.

- **Remark:** Now that we have linked the second derivative to convexity and concavity, the Second Derivative Test should make intuitive sense (see the picture below).

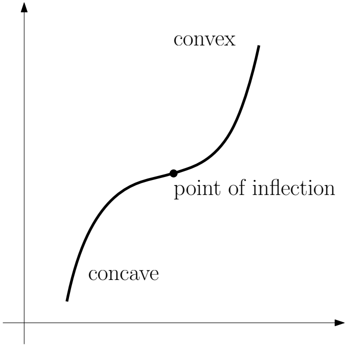


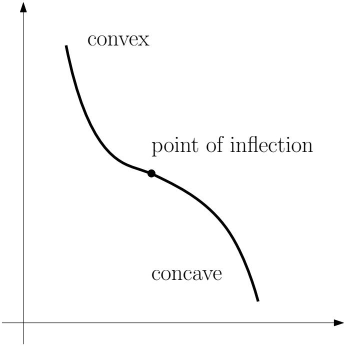
local minimum



local maximum

- A “point of inflection” is a point at which a function switches from being convex to being concave, or vice versa.





- Let us be more precise.

Definition

For a function $f : A \subseteq \mathbb{R} \to \mathbb{R}$, a point $c \in A$ is said to be a *point of inflection* (or an *inflection point*) of f if f is continuous at c, and there exists some $\delta > 0$ s.t. all the following hold:

- $(c - \delta, c + \delta) \subseteq A$;
- f is convex on one of the open intervals $(c - \delta, c)$ and $(c, c + \delta)$, and is concave on the other;
- for all $\delta' \in (0, \delta)$, f is affine neither on $(c - \delta', c)$ nor on $(c, c + \delta')$.

- In view of Proposition 4.14.1, the third bullet point simply states that f is not simultaneously convex and concave on any small interval on one side (to the left or to the right) of c. So, points of inflection mark a "true switch" from convexity to concavity or vice versa.

Proposition 4.14.4

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, let $c \in A$, and let $\delta > 0$ be s.t. $(c - \delta, c + \delta) \subseteq A$. Assume that f is continuous at c, and is twice differentiable on $(c - \delta, c)$ and $(c, c + \delta)$. If one of the following holds:

1. $f''(x) > 0$ for all $x \in (c - \delta, c)$, and $f''(x) < 0$ for all $x \in (c, c + \delta)$,
2. $f''(x) < 0$ for all $x \in (c - \delta, c)$, and $f''(x) > 0$ for all $x \in (c, c + \delta)$,

then c is a point of inflection of f.

Proof. In view of Theorem 4.14.3, we have the following:

- if (i) holds, then f switches from being strictly convex to being strictly concave at c;
- if (ii) holds, then f switches from being strictly concave to being strictly convex at c.

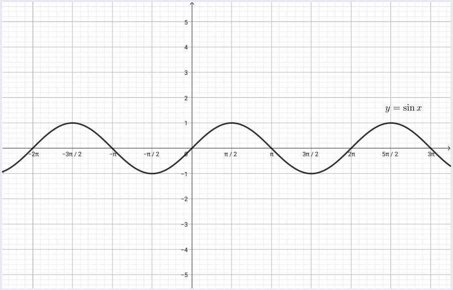
In either case, c is a point of inflection of f. □

Example 4.14.5

Consider the function $\sin x$. Note that $\sin'(x) = \cos x$ and $\sin''(x) = -\sin x$ for all $x \in \mathbb{R}$. Thus, for $k \in \mathbb{Z}$:

- $\sin''(k\pi) = 0$;
- $\sin''(x) > 0$ for all $x \in \big((2k-1)\pi, 2k\pi\big)$;
- $\sin''(x) < 0$ for all $x \in \big(2k\pi, (2k+1)\pi\big)$.

The points of inflection of $\sin x$ are all the points of the form $x = k\pi$ ($k \in \mathbb{Z}$).



5. Sketching graphs of functions

- To sketch the graph of a function $y = f(x)$, we need to do all the following:
    1. Determine the domain of the function (i.e. the largest subset of $\mathbb{R}$ on which the function is defined).
    2. Examine the behavior of the function near any "gaps" or "edges" of the domain, at and near any points at which f is discontinuous, as well as at $+\infty$ and $-\infty$. In particular, determine whether the function has any vertical, horizontal, or slant asymptotes.
        - A "gap" in the domain would be a point c at which f is undefined, whereas it is defined near c (both to the left and to the right). An "edge" of the domain would be a point c s.t. f defined near c to the left or to the right, but not both.

- To sketch the graph of a function $y = f(x)$, we need to do all the following (continued):
  3. Determine whether the function is periodic (and if so, find its period), and whether it is even and/or odd.
     - A function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is *periodic* if there exists a real number $p > 0$ s.t. for all $a \in A$ and all $k \in \mathbb{Z}$, we have that $a + kp \in A$ and $f(a + kp) = f(a)$. The smallest such p, if it exists, is called the *period* of the function.
     - A function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is *even* if for all $x \in A$, we have that $-x \in A$ and $f(-x) = f(x)$. Meanwhile, f is *odd* if for all $x \in A$, we have that $-x \in A$ and $f(-x) = -f(x)$. Graphs of even functions are symmetric about the y-axis, and graphs of odd functions are symmetric about the origin.
  4. Find the x- and y-intercepts of f. Determine intervals at which $f(x) > 0$ and at which $f(x) < 0$.
     - An *x-intercept* of f is any value of x s.t. $f(x) = 0$ (i.e. the value at which f intersects the x-axis). The y-intercept of f is the value $y = f(0)$ (i.e. the value at which f intersects the y-axis), if it exists.

- To sketch the graph of a function $y = f(x)$, we need to do all the following (continued):
    5. Find f', and using this, determine at which intervals the function increases or decreases. Find all local maxima and minima.
        - Sometimes, local maxima and minima can more easily be found using f''.
    6. Find f'', and using this, determine the intervals at which the function is convex or concave, and find all its inflection points.

Example 4.15.1

Sketch the graph of the function $f(x) = \frac{(x-1)^3}{(x+1)^2}$.

Solution. The domain of the function f is $\mathbb{R} \setminus \{-1\}$, i.e. the function is defined for all real numbers $x \neq -1$. Moreover, it is clear that f is continuous (at all points in its domain). Note that

$$\lim_{x \to -1} f(x) = -\infty,$$

and so $x = -1$ is a vertical asymptote of $f(x)$. Furthermore, the graph of the function approaches the line $x = -1$ "downward" both from the left and from the right. Next, note that

$$\lim_{x \to +\infty} f(x) = +\infty \qquad \text{and} \qquad \lim_{x \to -\infty} f(x) = -\infty,$$

and so f has no horizontal asymptotes.

Example 4.15.1

Sketch the graph of the function $f(x) = \frac{(x-1)^3}{(x+1)^2}$.

Solution (continued).

Example 4.15.1

Sketch the graph of the function $f(x) = \frac{(x-1)^3}{(x+1)^2}$.

Solution (continued). Further, note that

$$\lim_{x\to+\infty} \frac{f(x)}{x} \quad = \quad \lim_{x\to+\infty} \frac{(x-1)^3}{x(x+1)^2} \quad = \quad 1 \quad =: \quad a,$$

and that

$$\lim_{x\to+\infty} \Big(f(x) - ax\Big) \quad = \quad \lim_{x\to+\infty} \frac{-5x^2+2x-1}{x^2+2x+1} \quad = \quad -5 \quad =: \quad b.$$

So, $y = x - 5$ is a slant asymptote of $y = f(x)$ as $x \to +\infty$. A similar calculation shows that $y = x - 5$ is a slant asymptote of $y = f(x)$ as $x \to -\infty$.

Example 4.15.1

Sketch the graph of the function $f(x) = \frac{(x-1)^3}{(x+1)^2}$.

Solution (continued). The function f is neither even nor odd (e.g. because it is defined as 1, but not at -1). The function is also not periodic because, otherwise, there would be some constant $p > 0$ (the period of the function) s.t. f is undefined at $-1 + kp$ for all $k \in \mathbb{Z}$.

Note that $f(0) = -1$ (so, $y = f(x)$ intersects the y axis at $y = -1$), and that $f(x) = 0$ only for $x = 1$ (so, $y = f(x)$ intersects the x-axis at $x = 1$).

Example 4.15.1

Sketch the graph of the function $f(x) = \frac{(x-1)^3}{(x+1)^2}$.

Solution (continued). Further, we have the following table, showing where f is positive, negative, or zero.

		-1		1	
x	$(-\infty, -1)$		$(-1, 1)$		$(1, +\infty)$
$x - 1$	$-$		$-$	0	$+$
$x + 1$	$-$	0	$+$		$+$
$f(x)$	$-$	undef	$-$	0	$+$

Example 4.15.1

Sketch the graph of the function $f(x) = \frac{(x-1)^3}{(x+1)^2}$.

Solution (continued). Further, a straightforward calculation shows that

$$f'(x) = \frac{(x-1)^2(x+5)}{(x+1)^3} \quad \text{and} \quad f''(x) = 24\frac{x-1}{(x+1)^4}$$

for all $x \neq -1$. So, for the first derivative, we get the table below.

		-5		-1		1	
x	$(-\infty,-5)$		$(-5,-1)$		$(-1,1)$		$(1,+\infty)$
$x-1$	$-$		$-$		$-$	0	$+$
$x+1$	$-$		$-$	0	$+$		$+$
$x+5$	$-$	0	$+$		$+$		$+$
$f'(x)$	$+$	0	$-$	undef	$+$	0	$+$
$f(x)$	$\nearrow$	max	$\searrow$	undef	$\nearrow$		$\nearrow$

Example 4.15.1

Sketch the graph of the function $f(x) = \frac{(x-1)^3}{(x+1)^2}$.

Solution (continued). Reminder:

$$f'(x) = \frac{(x-1)^2(x+5)}{(x+1)^3} \quad \text{and} \quad f''(x) = 24\frac{x-1}{(x+1)^4}$$

for all $x \neq -1$.

For the second derivative, we get the table below. (Here, “$\smile$” means “convex,” and “$\frown$” means “concave.”)

x	$(-\infty,-1)$	-1	$(-1,1)$	1	$(1,+\infty)$
$x-1$	$-$		$-$	0	$+$
$x+1$	$-$	0	$+$		$+$
$f''(x)$	$-$	undef	$-$	0	$+$
$f(x)$	$\frown$	undef	$\frown$	inflec	$\smile$

Example 4.15.1

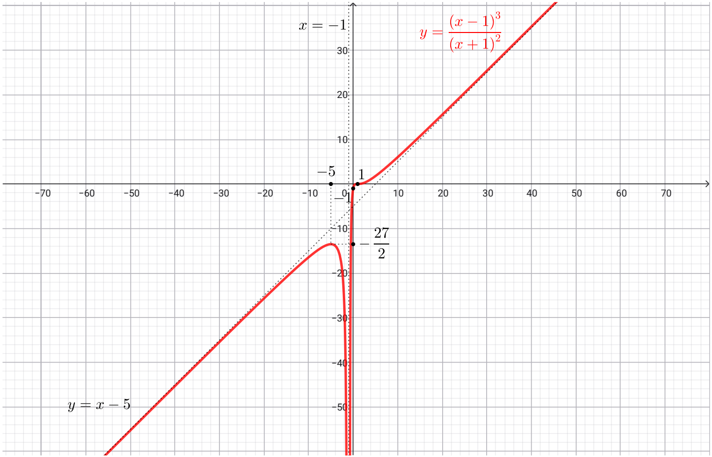
Sketch the graph of the function $f(x) = \frac{(x-1)^3}{(x+1)^2}$.

Solution (continued). We now combine the three tables above to obtain the table below.

x	$(-\infty,-5)$	-5	$(-5,-1)$	-1	$(-1,1)$	1	$(1,+\infty)$
$f'(x)$	$+$	0	$-$	undef	$+$	0	$+$
$f''(x)$	$-$		$-$	undef	$-$	0	$+$
$f(x)$	$-$ ↗ ⌢	max	$-$ ↘ ⌢	undef	$-$ ↗ ⌢	0 inflec	$+$ ↗ ⌣

Note that f reaches a local maximum at $x = -5$, and we have that $f(-5) = -\frac{27}{2}$. (There are no other local extrema.) Furthermore, the only point of inflection is $x = 1$, and we have that $f(1) = 0$.

Solution (continued). Putting all this together, we get the following picture.



□