Mathematical Analysis 1

Lecture #11

The Mean Value Theorems. L'Hôpital's Rule

Irena Penev

April 20, 2026

- This lecture has five parts:
    1. A review of local extrema
    2. Rolle's Theorem
    3. Lagrange's Mean Value Theorem
    4. Cauchy's Mean Value Theorem
    5. L'Hôpital's Rule

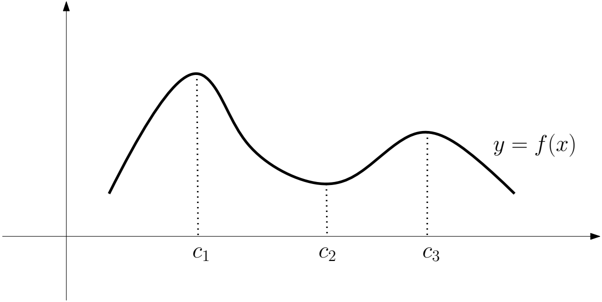
1. A review of local extrema

Definition

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and $c \in A$.

- We say that f reaches a *local maximum* at c if there exists some $\delta > 0$ s.t. for all $x \in A$, if $|x - c| < \delta$, then $f(c) \geq f(x)$.
- We say that f reaches a *local minimum* at c if there exists some $\delta > 0$ s.t. for all $x \in A$, if $|x - c| < \delta$, then $f(c) \leq f(x)$.
- We say that f reaches a *local extremum* at c if f reaches a local maximum or a local minimum at c.

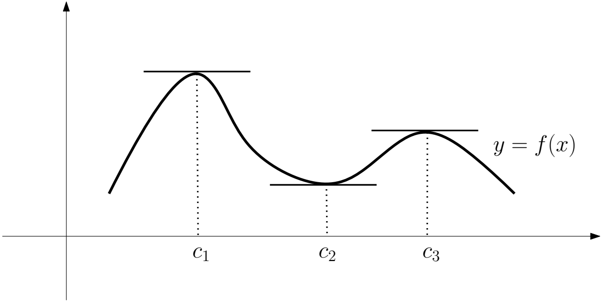
- For instance, in the picture below, the function f reaches local maxima at c_1 and c_3, and it reaches a local minimum at c_2.



Theorem 4.10.2

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and assume that f reaches a local extremum at a point $c_0 \in I \subseteq A$, where I is an open interval. If f is differentiable at c_0, then $f'(c_0) = 0$.

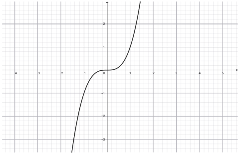
- **Remark:** Geometrically, Theorem 4.10.2 can be interpreted as follows: if a function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ reaches a local extremum at a point $c \in I \subseteq A$, where I is some open interval, and f is also differentiable at c, then the tangent to the curve $y = f(x)$ through the point $(c, f(c))$ is a horizontal line.



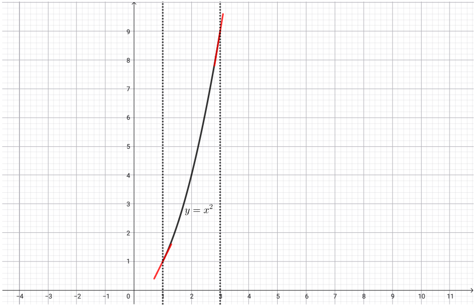
Theorem 4.10.2

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and assume that f reaches a local extremum at a point $c_0 \in I \subseteq A$, where I is an open interval. If f is differentiable at c_0, then $f'(c_0) = 0$.

- **Remark:** The converse of Theorem 4.10.2 is false.
  - Indeed, it is possible for the function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ to be differentiable at a point $c \in I \subseteq A$, where I is some open interval, and that $f'(c) = 0$, but that f does **not** reach a local extremum at c.
  - For instance, the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ for all $x \in \mathbb{R}$ is differentiable, and it satisfies $f'(0) = 0$, but f does **not** reach a local extremum at $c = 0$.



- **Remark:** Theorem 4.10.2 does **not** apply when a function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ reaches a local extremum at a point $c \in A$ that does **not** belong to any open interval $I \subseteq A$.
  - For instance, consider the function $f : [1, 3] \to \mathbb{R}$ given by $f(x) = x^2$ for all $x \in [1, 3]$.



  - Then f reaches a global (and therefore also local) minimum at $c_1 = 1$ and a global (and therefore also local) maximum at $c_2 = 3$.
  - However, $f'(c_1) = 2c_1 = 2$ and $f'(c_2) = 2c_2 = 6$.
  - Note that the tangents to the graph of f through the points $(1, f(1)) = (1, 1)$ and $(3, f(3)) = (3, 9)$ are **not** horizontal.

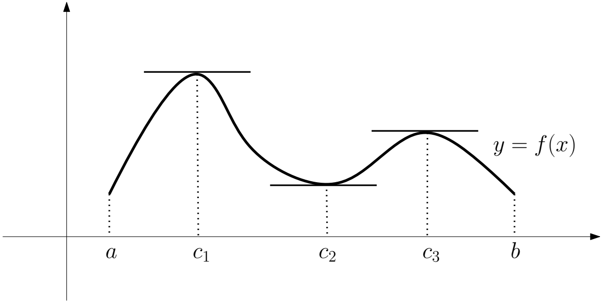
2. Rolle's Theorem

Rolle's Theorem

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a function that satisfies all the following:

- f is continuous on $[a, b]$;
- f is differentiable on (a, b);
- $f(a) = f(b)$.

Then there exists some $c \in (a, b)$ s.t. $f'(c) = 0$.



- The proof of Rolle's Theorem relies on the Extreme Value Theorem.
- Reminder:

The Extreme Value Theorem

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f reaches both a maximum and a minimum on $[a, b]$.

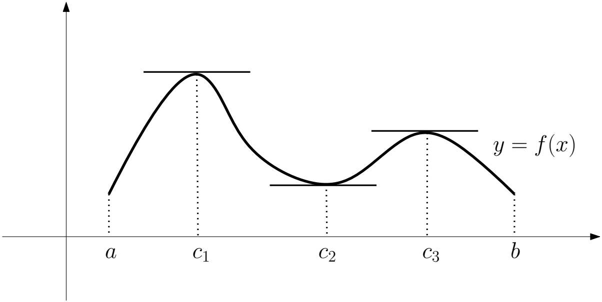
- Let's now prove Rolle's Theorem!

Rolle's Theorem

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a function that satisfies all the following:

- f is continuous on $[a, b]$;
- f is differentiable on (a, b);
- $f(a) = f(b)$.

Then there exists some $c \in (a, b)$ s.t. $f'(c) = 0$.



Proof. Since f is continuous on $[a, b]$, the Extreme Value Theorem guarantees that f reaches both a global minimum and a global maximum on $[a, b]$. Let $m, M \in [a, b]$ be s.t. f reaches a global minimum at m and a global maximum at M. Clearly, $f(m) \leq f(M)$.

If $f(m) = f(M)$, then f is a constant function, and so by Proposition 4.4.1, we have that $f'(c) = 0$ for all $c \in (a, b)$.

It remains to consider the case when $f(m) < f(M)$. Since $f(a) = f(b)$, we deduce that at least one of m, M belongs to (a, b).

- Indeed, since $f(m) < f(M)$ and $f(a) = f(b)$, we see that at most one of $f(m)$ and $f(M)$ may be equal to $f(a) = f(b)$. Thus, at most one of m and M belongs to $\{a, b\}$, and consequently, the other one belongs to the open interval (a, b).

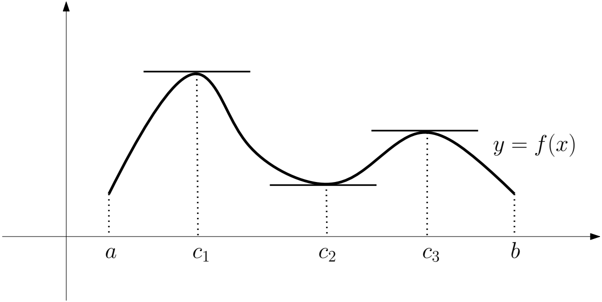
But then f reaches a local extremum at some point $c \in (a, b)$, and so by Theorem 4.10.2, we have that $f'(c) = 0$. $\square$

Rolle's Theorem

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a function that satisfies all the following:

- f is continuous on $[a, b]$;
- f is differentiable on (a, b);
- $f(a) = f(b)$.

Then there exists some $c \in (a, b)$ s.t. $f'(c) = 0$.



3. Lagrange's Mean Value Theorem

Lagrange's Mean Value Theorem

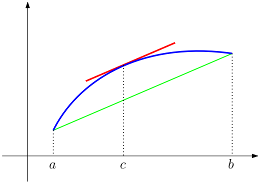
Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists some $c \in (a, b)$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$.

- **Remark:** Note that Rolle's Theorem is a special case of Lagrange's Mean Value Theorem for $f(a) = f(b)$.

Lagrange's Mean Value Theorem

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists some $c \in (a, b)$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$.

- **Remark:** Note that the line through the points $(a, f(a))$ and $(b, f(b))$ is given by the equation $y = \frac{f(b)-f(a)}{b-a}(x-a) + f(a)$, and in particular, the slope of this line is $\frac{f(b)-f(a)}{b-a}$.
  - So, the outcome of Lagrange's Mean Value Theorem is that $\exists c \in (a, b)$ s.t. the tangent to the graph of f through the point $(c, f(c))$ has the same slope as the line above, i.e. the line through $(a, f(a))$ and $(b, f(b))$.



Lagrange's Mean Value Theorem

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists some $c \in (a, b)$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Proof. Define the function $F : [a, b] \to \mathbb{R}$ by setting

$$F(x) \quad = \quad f(x) - \tfrac{f(b)-f(a)}{b-a}(x-a)$$

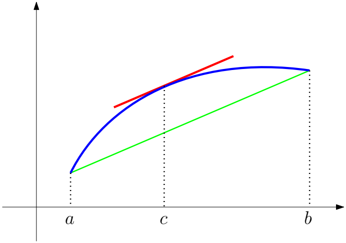
for all $x \in [a, b]$. Then F is continuous on $[a, b]$ and differentiable on (a, b). Furthermore, we have that $F(a) = f(a) = F(b)$. Thus, F satisfies the hypotheses of Rolle's Theorem, and so there exists some $c \in (a, b)$ s.t. $F'(c) = 0$. But by the construction of F, we have that

$$F'(x) = f'(x) - \tfrac{f(b)-f(a)}{b-a}$$

for all $x \in (a, b)$. Setting $x = c$, we get that $f'(c) = \frac{f(b)-f(a)}{b-a}$, which is what we needed. $\square$

Lagrange's Mean Value Theorem

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists some $c \in (a, b)$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$.



- **Terminology:** Lagrange's Mean Value Theorem is often referred to simply as the "Mean Value Theorem."

Definition

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function.

- We say that f is *non-decreasing* if for all $a, a' \in A$ s.t. $a \leq a'$, we have that $f(a) \leq f(a')$.
- We say that f is *strictly increasing* (or simply *increasing*) if for all $a, a' \in A$ s.t. $a < a'$, we have that $f(a) < f(a')$.
- We say that f is *non-increasing* if for all $a, a' \in A$ s.t. $a \leq a'$, we have that $f(a) \geq f(a')$.
- We say that f is *strictly decreasing* (or simply *decreasing*) if for all $a, a' \in A$ s.t. $a < a'$, we have that $f(a) > f(a')$.
- We say that f is *monotone* if it is non-decreasing or non-increasing.

For $A' \subseteq A$, we say that f is *non-decreasing* (resp. *increasing*, *non-increasing*, *decreasing*, *monotone*) on A' if the restriction $f \restriction A'$ is non-decreasing (resp. increasing, non-increasing, decreasing, monotone).

Lagrange's Mean Value Theorem

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists some $c \in (a, b)$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Corollary 4.10.3

Let I be an open interval in $\mathbb{R}$, and let $f : I \to \mathbb{R}$ be a differentiable function. Then all the following hold:

- (a) if $f'(x) = 0$ for all $x \in I$, then f constant on I;
- (b) if $f'(x) \geq 0$ for all $x \in I$, then f is non-decreasing on I;
- (c) if $f'(x) > 0$ for all $x \in I$, then f is strictly increasing on I;
- (d) if $f'(x) \leq 0$ for all $x \in I$, then f is non-increasing on I;
- (e) if $f'(x) < 0$ for all $x \in I$, then f is strictly decreasing on I.

Proof. We prove (a) and (b). The proofs of (c), (d), and (e) are similar to the proof of (b).

Corollary 4.10.3

Let I be an open interval in $\mathbb{R}$, and let $f : I \to \mathbb{R}$ be a differentiable function. Then all the following hold:

- (a) if $f'(x) = 0$ for all $x \in I$, then f constant on I;
- (b) if $f'(x) \geq 0$ for all $x \in I$, then f is non-decreasing on I;
- (c) if $f'(x) > 0$ for all $x \in I$, then f is strictly increasing on I;
- (d) if $f'(x) \leq 0$ for all $x \in I$, then f is non-increasing on I;
- (e) if $f'(x) < 0$ for all $x \in I$, then f is strictly decreasing on I.

Proof (continued). We first prove (a). Assume that $f'(x) = 0$ for all $x \in I$. Fix $a, b \in I$; WTS $f(a) = f(b)$. WMA $a \neq b$, for otherwise we are done. By symmetry, we may further assume that $a < b$. Then f restricted to $[a, b]$ satisfies the hypotheses of the Mean Value Theorem, and we deduce that $\exists c \in (a, b)$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$. Since $f'(c) = 0$, it follows that $\frac{f(b)-f(a)}{b-a} = 0$, and consequently, $f(a) = f(b)$. This proves (a).

Corollary 4.10.3

Let I be an open interval in $\mathbb{R}$, and let $f : I \to \mathbb{R}$ be a differentiable function. Then all the following hold:

- (a) if $f'(x) = 0$ for all $x \in I$, then f constant on I;
- (b) if $f'(x) \geq 0$ for all $x \in I$, then f is non-decreasing on I;
- (c) if $f'(x) > 0$ for all $x \in I$, then f is strictly increasing on I;
- (d) if $f'(x) \leq 0$ for all $x \in I$, then f is non-increasing on I;
- (e) if $f'(x) < 0$ for all $x \in I$, then f is strictly decreasing on I.

Proof (continued). We now prove (b). Assume that $f'(x) \geq 0$ for all $x \in I$. Fix $a, b \in I$ s.t. $a \leq b$; WTS $f(a) \leq f(b)$. WMA $a \neq b$ (and therefore $a < b$), for otherwise we are done. Clearly, f restricted to $[a, b]$ satisfies the hypotheses of the Mean Value Theorem, and so $\exists c \in (a, b)$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$. But $f'(c) \geq 0$ and $b - a > 0$; consequently, $f(b) - f(a) \geq 0$, and we deduce that $f(a) \leq f(b)$. This proves (b). $\square$

- We have the following partial converse of Corollary 4.10.3:

Proposition 4.10.4

Let I be an open interval in $\mathbb{R}$, and let $f : I \to \mathbb{R}$ be a differentiable function. Then all the following hold:

- ⓐ if f is constant on I, then $f'(x) = 0$ for all $x \in I$;
- ⓑ if f is non-decreasing on I, then $f'(x) \geq 0$ for all $x \in I$;
- ⓒ if f is non-increasing on I, then $f'(x) \leq 0$ for all $x \in I$.

- We have already proven part (a).
  - Reminder:

Proposition 4.4.1

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a constant function, and let $a_0 \in A$ be an accumulation point of A. Then $f'(a_0) = 0$.

Proposition 4.10.4

Let I be an open interval in $\mathbb{R}$, and let $f : I \to \mathbb{R}$ be a differentiable function. Then all the following hold:

- ⓐ if f is constant on I, then $f'(x) = 0$ for all $x \in I$;
- ⓑ if f is non-decreasing on I, then $f'(x) \geq 0$ for all $x \in I$;
- ⓒ if f is non-increasing on I, then $f'(x) \leq 0$ for all $x \in I$.

Proof. Part (a) follows immediately from Proposition 4.4.1. We now prove (b); the proof of (c) is similar. So, let us assume that f is non-decreasing on I, and fix an arbitrary $x_0 \in I$. Then

$$f'(x_0) \;=\; \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} \;=\; \lim_{x \to x_0^+} \frac{f(x)-f(x_0)}{x-x_0} \;\overset{(*)}{\geq}\; 0,$$

where (*) follows from the fact that f is non-decreasing, and so for any $x \in I$ s.t. $x > x_0$, we have that $f(x) \geq f(x_0)$, and consequently, $\frac{f(x)-f(x_0)}{x-x_0} \geq 0$. □

Corollary 4.10.3

Let I be an open interval in $\mathbb{R}$, and let $f : I \to \mathbb{R}$ be a differentiable function. Then all the following hold:

- (a) if $f'(x) = 0$ for all $x \in I$, then f constant on I;
- (b) if $f'(x) \geq 0$ for all $x \in I$, then f is non-decreasing on I;
- (c) if $f'(x) > 0$ for all $x \in I$, then f is strictly increasing on I;
- (d) if $f'(x) \leq 0$ for all $x \in I$, then f is non-increasing on I;
- (e) if $f'(x) < 0$ for all $x \in I$, then f is strictly decreasing on I.

- **Remark:** The converses of parts (c) and (e) of Corollary 4.10.4 are false.
  - For example, the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ for all $x \in \mathbb{R}$ is strictly increasing, but $f'(0) = 0$.
  - Similarly, the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = -x^3$ for all $x \in \mathbb{R}$ is strictly decreasing, but $g'(0) = 0$.

Example 4.10.5

Prove that $|\arctan x_1 - \arctan x_2| \leq |x_1 - x_2|$ for all $x_1, x_2 \in \mathbb{R}$.

Solution. Fix $x_1, x_2 \in \mathbb{R}$. WMA $x_1 \neq x_2$ (for otherwise, the result is immediate), and by symmetry, we may further assume that $x_1 > x_2$. So, $|x_1 - x_2| = x_1 - x_2 > 0$. Moreover, the arctangent function is strictly increasing.

- Here, we may simply rely on the familiar fact from trigonometry that $\arctan x$ is a strictly increasing function.
- Alternatively, we may rely on the fact that for $x \in \mathbb{R}$, we have that $\frac{d}{dx} \arctan x = \frac{1}{1+x^2} > 0$, and so by Corollary 4.10.3(c), the function $\arctan x$ is strictly increasing.

So, $\arctan x_1 > \arctan x_2$, and consequently, we have that

$$\begin{array}{rcccl} |\arctan x_1 - \arctan x_2| & = & \arctan x_1 - \arctan x_2 & > & 0. \end{array}$$

Example 4.10.5

Prove that $|\arctan x_1 - \arctan x_2| \leq |x_1 - x_2|$ for all $x_1, x_2 \in \mathbb{R}$.

Solution (continued). Reminder: $x_1 > x_2$; $|x_1 - x_2| = x_1 - x_2 > 0$; $|\arctan x_1 - \arctan x_2| = \arctan x_1 - \arctan x_2 > 0$.

Now, the function $\arctan x$ is continuous on $[x_2, x_1]$ and differentiable on (x_2, x_1), and so by the Mean Value Theorem, $\exists y \in (x_2, x_1)$ s.t.

$$\arctan'(y) \quad = \quad \frac{\arctan x_1 - \arctan x_2}{x_1 - x_2}.$$

We now compute:

$$\frac{|\arctan x_1 - \arctan x_2|}{|x_1 - x_2|} \quad = \quad \frac{\arctan x_1 - \arctan x_2}{x_1 - x_2} \quad = \quad \arctan'(y) \quad = \quad \frac{1}{1+y^2} \quad \leq \quad 1.$$

Multiplying both sides by the positive number $|x_1 - x_2|$, we obtain

$$|\arctan x_1 - \arctan x_2| \quad \leq \quad |x_1 - x_2|,$$

which is what we needed to show. □

4. Cauchy's Mean Value Theorem

Cauchy's Mean Value Theorem

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b), with $g'(x) \neq 0$ for all $x \in (a, b)$. Then $g(a) \neq g(b)$, and there exists some $c \in (a, b)$ s.t.

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

- **Remark:** Cauchy's Mean Value Theorem generalizes Lagrange's Mean Value Theorem.
  - Indeed, when $g(x) = x$, then Cauchy's Mean Value Theorem becomes Lagrange's Mean Value Theorem.

Lagrange's Mean Value Theorem

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists some $c \in (a, b)$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Cauchy's Mean Value Theorem

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b), with $g'(x) \neq 0$ for all $x \in (a, b)$. Then $g(a) \neq g(b)$, and there exists some $c \in (a, b)$ s.t.

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Proof. We have that $g(a) \neq g(b)$, because otherwise, Rolle's Theorem would imply that $g'(c) = 0$ for some $c \in (a, b)$, a contradiction. Now, define $F : [a, b] \to \mathbb{R}$ by setting

$$F(x) = f(x) - \frac{f(b)-f(a)}{g(b)-g(a)} g(x) \qquad \forall x \in [a, b].$$

Clearly, F is continuous on $[a, b]$ and differentiable on (a, b), and furthermore, $F(a) = \frac{f(a)g(b)-f(b)g(a)}{g(b)-g(a)} = F(b)$. So, by Rolle's Theorem, there exists some $c \in (a, b)$ s.t. $F'(c) = 0$. But note that

$$F'(x) = f'(x) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(x)$$

for all $x \in (a, b)$. Setting $x = c$, we get that $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$. $\square$

5. L'Hôpital's Rule

- When evaluating limits, the following forms are "indeterminate," i.e. they can in principle be anything:

$$\frac{0}{0} \qquad \frac{\infty}{\infty} \qquad 0 \cdot \infty \qquad 1^{\infty} \qquad \infty - \infty \qquad 0^{0} \qquad \infty^{0}$$

  - In what follows, we cover L'Hôpital's Rule, which can (sometimes) be used to evaluate limits that look like indeterminate forms.
  - More precisely, L'Hôpital's Rule is (sometimes) used for evaluating indeterminate forms of types "$\frac{0}{0}$" and "$\frac{\infty}{\infty}$."
  - However, as we shall see, other types of indeterminate forms can sometimes be transformed into these two in convenient ways.

- There are several related results, all of which are referred to as "L'Hôpital's Rule."
- The proofs of these rules are somewhat complicated (here, we note that they rely on Cauchy's Mean Value Theorem).
- So, we first state the rules (without proof) and give some examples.
- Then, we give a partial proof of L'Hôpital's Rule(s).
  - The full proof is in the Lecture Notes.

L'Hôpital's Rule for limits at a point

Let $a_0 \in \mathbb{R}$ and $\delta > 0$, and assume that functions $f, g : (a_0 - \delta, a_0) \cup (a_0, a_0 + \delta) \to \mathbb{R}$ are both differentiable, and that $g'(x) \neq 0$ for all $x \in (a_0 - \delta, a_0) \cup (a_0, a_0 + \delta)$. Assume moreover that one of the following holds:

(i) $\lim\limits_{x \to a_0} f(x) = \lim\limits_{x \to a_0} g(x) = 0$;

(ii) $\lim\limits_{x \to a_0} f(x) = \pm\infty$ and $\lim\limits_{x \to a_0} g(x) = \pm\infty$.

Then, if $\lim\limits_{x \to a_0} \frac{f'(x)}{g'(x)}$ exists (either as a real number or as $\pm\infty$), then $\lim\limits_{x \to a_0} \frac{f(x)}{g(x)}$ exists as well, and in this case, we have that

$$\lim_{x \to a_0} \frac{f(x)}{g(x)} = \lim_{x \to a_0} \frac{f'(x)}{g'(x)}.$$

Example 4.11.1

Compute $\lim_{x\to 0} \frac{\tan x - x}{x - \sin x}$.

Solution. Define $f, g : (-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2})$ by setting $f(x) := \tan x - x$ and $g(x) := x - \sin x$ for all $x \in (-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2})$. Clearly, both f and g are differentiable, and moreover, for $x \in (-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2})$, we have that $g'(x) = \frac{d}{dx}(x - \sin x) = 1 - \cos(x) \neq 0$. Furthermore, $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$. Next, we now compute:

$$\begin{aligned}
\lim_{x\to 0} \frac{f'(x)}{g'(x)} &= \lim_{x\to 0} \frac{\frac{d}{dx}(\tan x - x)}{\frac{d}{dx}(x - \sin x)} = \lim_{x\to 0} \frac{\frac{1}{\cos^2 x} - 1}{1 - \cos x} \\
&= \lim_{x\to 0} \frac{1 - \cos^2 x}{\cos^2 x(1 - \cos x)} = \lim_{x\to 0} \frac{(1 - \cos x)(1 + \cos x)}{\cos^2 x(1 - \cos x)} \\
&= \lim_{x\to 0} \frac{1 + \cos x}{\cos^2 x} = 2.
\end{aligned}$$

So, by L'Hôpital's Rule, we have that $\lim_{x\to 0} \frac{f(x)}{g(x)} = 2$, that is,

$$\lim_{x\to 0} \frac{\tan x - x}{x - \sin x} = 2.$$

Example 4.11.1

Compute $\lim\limits_{x\to 0} \frac{\tan x - x}{x - \sin x}$.

Solution (continued). **Remark:** Typically, when computing limits using L'Hôpital's Rule, we do not include the level of detail that we included on the previous slide. Normally, we simply write something like this:

$$\begin{aligned}
\lim_{x\to 0} \frac{\tan x - x}{x - \sin x} &\overset{(*)}{=} \lim_{x\to 0} \frac{\frac{d}{dx}(\tan x - x)}{\frac{d}{dx}(x - \sin x)} = \lim_{x\to 0} \frac{\frac{1}{\cos^2 x} - 1}{1 - \cos x} \\
&= \lim_{x\to 0} \frac{1 - \cos^2 x}{\cos^2 x(1 - \cos x)} = \lim_{x\to 0} \frac{(1 - \cos x)(1 + \cos x)}{\cos^2 x(1 - \cos x)} \\
&= \lim_{x\to 0} \frac{1 + \cos x}{\cos^2 x} = 2,
\end{aligned}$$

where (*) follows from L'Hôpital's Rule, which we could apply because we have that $\lim\limits_{x\to 0}(\tan x - x) = 0$ and $\lim\limits_{x\to 0}(x - \sin x) = 0$. □

L'Hôpital's Rule for limits at a point

Let $a_0 \in \mathbb{R}$ and $\delta > 0$, and assume that functions $f, g : (a_0 - \delta, a_0) \cup (a_0, a_0 + \delta) \to \mathbb{R}$ are both differentiable, and that $g'(x) \neq 0$ for all $x \in (a_0 - \delta, a_0) \cup (a_0, a_0 + \delta)$. Assume moreover that one of the following holds:

(i) $\lim\limits_{x \to a_0} f(x) = \lim\limits_{x \to a_0} g(x) = 0$;

(ii) $\lim\limits_{x \to a_0} f(x) = \pm\infty$ and $\lim\limits_{x \to a_0} g(x) = \pm\infty$.

Then, if $\lim\limits_{x \to a_0} \frac{f'(x)}{g'(x)}$ exists (either as a real number or as $\pm\infty$), then $\lim\limits_{x \to a_0} \frac{f(x)}{g(x)}$ exists as well, and in this case, we have that

$$\lim_{x \to a_0} \frac{f(x)}{g(x)} = \lim_{x \to a_0} \frac{f'(x)}{g'(x)}.$$

- **Remark/Warning:** In order to apply L'Hôpital's Rule (the version above, or any version), we must make sure we really do have an indeterminate form of the form "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$," for otherwise, L'Hôpital's Rule does **not** apply.
  - Example: Next slide!

- For example,

  $$\lim_{x\to 0}\frac{1+x}{1+2x} \;=\; \frac{\lim\limits_{x\to 0}(1+x)}{\lim\limits_{x\to 0}(1+2x)} \;=\; 1,$$

  but

  $$\lim_{x\to 0}\frac{(1+x)'}{(1+2x)'} \;=\; \lim_{x\to 0}\frac{1}{2} \;=\; \frac{1}{2},$$

  and so

  $$\lim_{x\to 0}\frac{1+x}{1+2x} \;\neq\; \lim_{x\to 0}\frac{(1+x)'}{(1+2x)'}.$$

- The reason this we could not apply L'Hôpital's Rule in this case is because our original limit $\lim_{x\to 0}\frac{1+x}{1+2x}$ was not of the form "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$," i.e. neither (i) nor (ii) from the statement of the appropriate L'Hôpital's Rule was satisfied.

- **Remark/Warning:** For indeterminate forms of type "$\frac{0}{0}$" and "$\frac{\infty}{\infty}$," L'Hôpital's Rule gives us only a one-way implication, as follows:

$$\lim_{x\to a_0} \frac{f'(x)}{g'(x)} = L \qquad \underset{\not\Longleftarrow}{\Longrightarrow} \qquad \lim_{x\to a_0} \frac{f(x)}{g(x)} = L.$$

- This means that if $\lim_{x\to a_0} \frac{f'(x)}{g'(x)}$ does not exist (either as a real number or as $\pm\infty$), then we have no information about $\lim_{x\to a_0} \frac{f(x)}{g(x)}$, that is, $\lim_{x\to a_0} \frac{f(x)}{g(x)}$ may be a real number, may be $\pm\infty$, or may fail to exist.

- This further means that if, having verified that $\lim\limits_{x \to a_0} \frac{f(x)}{g(x)}$ an indeterminate form of type "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$," we attempt to compute

$$\lim_{x \to a_0} \frac{f(x)}{g(x)} \overset{\text{L'Hôpital}}{=} \lim_{x \to a_0} \frac{f'(x)}{g'(x)} = \dots,$$

our computation is only valid if $\lim\limits_{x \to a_0} \frac{f'(x)}{g'(x)}$ does, in fact, exist.

  - If $\lim\limits_{x \to a_0} \frac{f'(x)}{g'(x)}$ does **not** exist, then we **cannot** conclude that $\lim\limits_{x \to a_0} \frac{f(x)}{g(x)}$ does not exist, either.
  - In this case, we need to scratch out the whole computation

$$\xcancel{\lim_{x \to a_0} \frac{f(x)}{g(x)} \overset{\text{L'Hôpital}}{=} \lim_{x \to a_0} \frac{f'(x)}{g'(x)} = \dots}$$

    and start over: L'Hôpital's Rule will not help us, and we need to find some other way to compute the limit $\lim\limits_{x \to a_0} \frac{f(x)}{g(x)}$ (or to show that the limit does not exist).
    - Example: Lecture Notes.
  - Similar remarks apply to other versions of L'Hôpital's Rule.

L'Hôpital's Rule for limits at a point

Let $a_0 \in \mathbb{R}$ and $\delta > 0$, and assume that functions $f, g : (a_0 - \delta, a_0) \cup (a_0, a_0 + \delta) \to \mathbb{R}$ are both differentiable, and that $g'(x) \neq 0$ for all $x \in (a_0 - \delta, a_0) \cup (a_0, a_0 + \delta)$. Assume moreover that one of the following holds:

1. $\lim\limits_{x \to a_0} f(x) = \lim\limits_{x \to a_0} g(x) = 0$;
2. $\lim\limits_{x \to a_0} f(x) = \pm\infty$ and $\lim\limits_{x \to a_0} g(x) = \pm\infty$.

Then, if $\lim\limits_{x \to a_0} \frac{f'(x)}{g'(x)}$ exists (either as a real number or as $\pm\infty$), then $\lim\limits_{x \to a_0} \frac{f(x)}{g(x)}$ exists as well, and in this case, we have that

$$\lim_{x \to a_0} \frac{f(x)}{g(x)} = \lim_{x \to a_0} \frac{f'(x)}{g'(x)}.$$

- Let's now state the remaining versions of L'Hôpital's Rule!

L'Hôpital's Rule for right-hand limits

Let $a_0 \in \mathbb{R}$ and $\delta > 0$, and assume that functions $f, g : (a_0, a_0 + \delta) \to \mathbb{R}$ are both differentiable, and that $g'(x) \neq 0$ for all $x \in (a_0, a_0 + \delta)$. Assume moreover that one of the following holds:

1. (i) $\displaystyle\lim_{x \to a_0^+} f(x) = \lim_{x \to a_0^+} g(x) = 0$;
2. (ii) $\displaystyle\lim_{x \to a_0^+} f(x) = \pm\infty$ and $\displaystyle\lim_{x \to a_0^+} g(x) = \pm\infty$.

Then, if $\displaystyle\lim_{x \to a_0^+} \frac{f'(x)}{g'(x)}$ exists (either as a real number or as $\pm\infty$), then $\displaystyle\lim_{x \to a_0^+} \frac{f(x)}{g(x)}$ exists as well, and in this case, we have that

$$\lim_{x \to a_0^+} \frac{f(x)}{g(x)} = \lim_{x \to a_0^+} \frac{f'(x)}{g'(x)}.$$

L'Hôpital's Rule for left-hand limits

Let $a_0 \in \mathbb{R}$ and $\delta > 0$, and assume that functions $f, g : (a_0 - \delta, a_0) \to \mathbb{R}$ are both differentiable, and that $g'(x) \neq 0$ for all $x \in (a_0 - \delta, a_0)$. Assume moreover that one of the following holds:

- (i) $\lim\limits_{x \to a_0^-} f(x) = \lim\limits_{x \to a_0^-} g(x) = 0$;
- (ii) $\lim\limits_{x \to a_0^-} f(x) = \pm\infty$ and $\lim\limits_{x \to a_0^-} g(x) = \pm\infty$.

Then, if $\lim\limits_{x \to a_0^-} \frac{f'(x)}{g'(x)}$ exists (either as a real number or as $\pm\infty$), then $\lim\limits_{x \to a_0^-} \frac{f(x)}{g(x)}$ exists as well, and in this case, we have that

$$\lim_{x \to a_0^-} \frac{f(x)}{g(x)} = \lim_{x \to a_0^-} \frac{f'(x)}{g'(x)}.$$

L'Hôpital's Rule for limits at $+\infty$

Let $a_0 \in \mathbb{R}$, and assume that functions $f, g : (a_0, +\infty) \to \mathbb{R}$ are both differentiable, and that $g'(x) \neq 0$ for all $x \in (a_0, +\infty)$. Assume moreover that one of the following holds:

1. $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} g(x) = 0$;
2. $\lim\limits_{x \to +\infty} f(x) = \pm\infty$ and $\lim\limits_{x \to +\infty} g(x) = \pm\infty$.

Then, if $\lim\limits_{x \to +\infty} \frac{f'(x)}{g'(x)}$ exists (either as a real number or as $\pm\infty$), then $\lim\limits_{x \to +\infty} \frac{f(x)}{g(x)}$ exists as well, and in this case, we have that

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = \lim_{x \to +\infty} \frac{f'(x)}{g'(x)}.$$

L'Hôpital's Rule for limits at $-\infty$

Let $a_0 \in \mathbb{R}$, and assume that functions $f, g : (-\infty, a_0) \to \mathbb{R}$ are both differentiable, and that $g'(x) \neq 0$ for all $x \in (-\infty, a_0)$. Assume moreover that one of the following holds:

(i) $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} g(x) = 0$;

(ii) $\lim\limits_{x\to-\infty} f(x) = \pm\infty$ and $\lim\limits_{x\to-\infty} g(x) = \pm\infty$.

Then, if $\lim\limits_{x\to-\infty} \frac{f'(x)}{g'(x)}$ exists (either as a real number or as $\pm\infty$), then $\lim\limits_{x\to-\infty} \frac{f(x)}{g(x)}$ exists as well, and in this case, we have that

$$\lim_{x\to-\infty} \frac{f(x)}{g(x)} = \lim_{x\to-\infty} \frac{f'(x)}{g'(x)}.$$

Example 4.11.3

Compute $\lim\limits_{x\to+\infty} \frac{2^x}{x}$.

Solution. We compute:

$$\begin{aligned}
\lim_{x\to+\infty} \frac{\overset{\nearrow\,+\infty}{\textcircled{2^x}}}{\underset{\searrow\,+\infty}{\textcircled{x}}} &= \lim_{x\to+\infty} \frac{\frac{d}{dx}(2^x)}{\frac{d}{dx}(x)} && \text{by L'Hôpital's Rule} \\
&= \lim_{x\to+\infty} \frac{2^x \ln 2}{1} \\
&= \lim_{x\to+\infty} (2^x \ln 2) \\
&= +\infty.
\end{aligned}$$

$\square$

Example 4.11.4

Compute $\lim\limits_{x\to+\infty} \frac{x^2}{e^x}$.

Solution. We compute:

$$\lim_{x\to+\infty} \frac{\overset{\nearrow\,+\infty}{\enclose{circle}{x^2}}}{\underset{\searrow\,+\infty}{\enclose{circle}{e^x}}} \overset{\text{L'Hôpital}}{=} \lim_{x\to+\infty} \frac{\frac{d}{dx}x^2}{\frac{d}{dx}e^x} = \lim_{x\to+\infty} \frac{\overset{\nearrow\,+\infty}{\enclose{circle}{2x}}}{\underset{\searrow\,+\infty}{\enclose{circle}{e^x}}}$$

$$\overset{\text{L'Hôpital}}{=} \lim_{x\to+\infty} \frac{\frac{d}{dx}(2x)}{\frac{d}{dx}e^x} = \lim_{x\to+\infty} \frac{2}{e^x} = 0.$$

$\square$

Example 4.11.5

Compute $\lim\limits_{x\to 0^+}(x\ln x)$.

Solution We compute:

$$\begin{aligned}
\lim_{x\to 0^+}(x\ln x) &= \lim_{x\to 0^+}\frac{\overset{\nearrow\,-\infty}{\ln x}}{\underset{\searrow\,+\infty}{\frac{1}{x}}} \\
&= \lim_{x\to 0^+}\frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}\left(\frac{1}{x}\right)} && \text{by L'Hôpital's Rule} \\
&= \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}} \\
&= \lim_{x\to 0^+}(-x) = 0.
\end{aligned}$$

□

Example 4.11.6

Compute $\lim\limits_{x\to+\infty} x^{1/x}$.

Solution. We compute:

$$\begin{aligned}
\lim_{x\to+\infty} x^{1/x} &= \lim_{x\to+\infty} e^{\frac{1}{x}\ln x} \\
&= e^{\lim\limits_{x\to+\infty} \frac{\ln x}{x}} \\
&\overset{(*)}{=} e^{\lim\limits_{x\to+\infty} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x)}} \\
&= e^{\lim\limits_{x\to+\infty} \frac{\frac{1}{x}}{1}} \\
&= e^{\lim\limits_{x\to+\infty} \frac{1}{x}} \\
&= e^{0} \;=\; 1,
\end{aligned}$$

where in (*), we applied l'Hôpital's Rule, which was applicable because $\lim\limits_{x\to+\infty} \ln x = +\infty$ and $\lim\limits_{x\to+\infty} x = +\infty$. $\square$

Example 4.11.7

Compute $\lim\limits_{x \to +\infty} (x \ln(x+1) - x \ln x)$.

Solution. We compute:

$$\begin{aligned}
\lim_{x \to +\infty} \left(x \ln(x+1) - x \ln x\right) &= \lim_{x \to +\infty} x\Big(\ln(x+1) - \ln x\Big) \\
&= \lim_{x \to +\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \\
&\overset{(*)}{=} \lim_{x \to +\infty} \frac{\frac{d}{dx}\ln\left(1+\frac{1}{x}\right)}{\frac{d}{dx}\left(\frac{1}{x}\right)} \\
&= \lim_{x \to +\infty} \frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} \\
&= \lim_{x \to +\infty} \left(\frac{1}{1+\frac{1}{x}}\right) = 1,
\end{aligned}$$

where (*) was obtained via L'Hôpital's Rule, which applies since $\lim\limits_{x \to +\infty} \ln(1+\frac{1}{x}) = \ln 1 = 0$ and $\lim\limits_{x \to +\infty} \left(\frac{1}{x}\right) = 0$. □

- Let's give a partial proof L'Hôpital's Rules!
- We will prove a subcase of L'Hôpital's Rule for right-hand limits (we will use Cauchy's Mean Value Theorem).
  - The rule for left-hand limits can be derived from the one for right-hand limits.
  - Meanwhile, the rule for limits at a point follows immediately from the rules for one-sided limits and from Lemma 3.6.3.
  - L'Hôpital's Rules for limits at $\pm\infty$ can be derived from those for one-sided limits.

Lemma 3.6.3

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$, let $a \in \mathbb{R}$ be an accumulation point both of $A \cap (-\infty, a)$ and of $A \cap (a, +\infty)$. Then the following are equivalent:

(i) $\lim_{x \to a} f(x)$ exists (as a real number, or $+\infty$, or $-\infty$);

(ii) $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist (as real numbers, or $+\infty$, or $-\infty$) and are equal.

Moreover, in this case, we have that $\lim_{x \to a} f(x) = \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$.

- First, a technical lemma.

Lemma 4.11.8

Let $f : A \subseteq \mathbb{R}$ be a function, differentiable on some open interval $I \subseteq A$ of $\mathbb{R}$. Assume that $f'(x) \neq 0$ for all $x \in I$. Then $f \restriction I$ is one-to-one, and in particular, there exists at most one point $x \in I$ s.t. $f(x) = 0$.

- **Notation:** As usual, $f \restriction I$ denotes the restriction of f to I.

Proof. Suppose otherwise, and fix distinct points $a, b \in I$ s.t. $f(a) = f(b)$. By symmetry, we may assume that $a < b$. Since f is differentiable on I, it is also continuous on I (by Theorem 4.3.1). So, f is continuous on $[a, b]$ and differentiable on (a, b). Since $f(a) = f(b)$, Rolle's theorem guarantees that there exists some $c \in (a, b) \subseteq I$ s.t. $f'(c) = 0$, a contradiction. $\square$

L'Hôpital's Rule for right-hand limits

Let $a_0 \in \mathbb{R}$ and $\delta > 0$, and assume that functions $f, g : (a_0, a_0 + \delta) \to \mathbb{R}$ are both differentiable, and that $g'(x) \neq 0$ for all $x \in (a_0, a_0 + \delta)$. Assume moreover that one of the following holds:

1. (i) $\lim\limits_{x \to a_0^+} f(x) = \lim\limits_{x \to a_0^+} g(x) = 0$;
2. (ii) $\lim\limits_{x \to a_0^+} f(x) = \pm\infty$ and $\lim\limits_{x \to a_0^+} g(x) = \pm\infty$.

Then, if $\lim\limits_{x \to a_0^+} \frac{f'(x)}{g'(x)}$ exists (either as a real number or as $\pm\infty$), then $\lim\limits_{x \to a_0^+} \frac{f(x)}{g(x)}$ exists as well, and in this case, we have that

$$\lim_{x \to a_0^+} \frac{f(x)}{g(x)} \quad = \quad \lim_{x \to a_0^+} \frac{f'(x)}{g'(x)}.$$

Partial proof. First of all, in view of Lemma 4.11.8, there exists at most one point $x \in (a_0, a_0 + \delta)$ s.t. $g(x) = 0$. But then we may simply assume that $g(x) \neq 0$ for all $x \in (a_0, a_0 + \delta)$, for otherwise, we simply choose a suitable smaller δ.

We consider the case (i) holds and $L := \lim\limits_{x \to a_0^+} \frac{f'(x)}{g'(x)}$ is a real number. We extend the domain of f and g to a_0 by setting $f(a_0) := 0$ and $g(a_0) := 0$. Then $\lim\limits_{x \to a_0^+} f(x) = 0 = f(a_0)$ and $\lim\limits_{x \to a_0^+} g(x) = 0 = g(a_0)$, and so f and g are both continuous at a_0 from the right.

Meanwhile, f and g are differentiable, and therefore continuous (by Theorem 4.3.1) on the interval $(a_0, a_0 + \delta)$. Thus, f and g are continuous on $[a_0, a_0 + \delta)$ and differentiable on $(a_0, a_0 + \delta)$. Note that this implies that for all $b \in (a_0, a_0 + \delta)$, we may apply Cauchy's Mean Value Theorem to the functions f and g and the interval $[a_0, b]$.

Partial proof (continued). Reminder: $\lim\limits_{x \to a_0^+} f(x) = \lim\limits_{x \to a_0^+} g(x) = 0$;

$L := \lim\limits_{x \to a_0^+} \frac{f'(x)}{g'(x)}$ is a real number.

We fix an arbitrary $\varepsilon > 0$, and using the fact that $\lim\limits_{x \to a_0^+} \frac{f'(x)}{g'(x)} = L$, we fix $\delta_0 \in (0, \delta)$ s.t. for all $x \in (a_0, a_0 + \delta_0)$, we have that $|\frac{f'(x)}{g'(x)} - L| < \varepsilon$. Now, fix an arbitrary $b_0 \in (a_0, a_0 + \delta_0)$; WTS $|\frac{f(b_0)}{g(b_0)} - L| < \varepsilon$. Note that this will immediately imply that $\lim\limits_{x \to a_0^+} \frac{f(x)}{g(x)} = L$.

Partial proof (continued). Reminder: $\lim\limits_{x \to a_0^+} f(x) = \lim\limits_{x \to a_0^+} g(x) = 0$;

$L := \lim\limits_{x \to a_0^+} \frac{f'(x)}{g'(x)}$ is a real number.

First, we apply Cauchy's Mean Value Theorem to the functions f and g and the interval $[a_0, b_0]$ to obtain a point $c_0 \in (a_0, b_0)$ s.t.

$$\frac{f'(c_0)}{g'(c_0)} \quad = \quad \frac{f(b_0)-f(a_0)}{g(b_0)-g(a_0)} \quad \overset{(*)}{=} \quad \frac{f(b_0)}{g(b_0)},$$

where (*) follows from the fact that $f(a_0) = g(a_0) = 0$. But now

$$|\frac{f(b_0)}{g(b_0)} - L| \quad = \quad |\frac{f'(c_0)}{g'(c_0)} - L| \quad \overset{(*)}{<} \quad \varepsilon,$$

where (*) follows from the fact that $c_0 \in (a_0, b_0) \subseteq (a_0, a_0 + \delta_0)$. This proves that $\lim\limits_{x \to a_0^+} \frac{f(x)}{g(x)} = L$, which is what we needed to show. $\square$

L'Hôpital's Rule for right-hand limits

Let $a_0 \in \mathbb{R}$ and $\delta > 0$, and assume that functions $f, g : (a_0, a_0 + \delta) \to \mathbb{R}$ are both differentiable, and that $g'(x) \neq 0$ for all $x \in (a_0, a_0 + \delta)$. Assume moreover that one of the following holds:

1. (i) $\lim\limits_{x \to a_0^+} f(x) = \lim\limits_{x \to a_0^+} g(x) = 0$;
2. (ii) $\lim\limits_{x \to a_0^+} f(x) = \pm\infty$ and $\lim\limits_{x \to a_0^+} g(x) = \pm\infty$.

Then, if $\lim\limits_{x \to a_0^+} \frac{f'(x)}{g'(x)}$ exists (either as a real number or as $\pm\infty$), then $\lim\limits_{x \to a_0^+} \frac{f(x)}{g(x)}$ exists as well, and in this case, we have that

$$\lim_{x \to a_0^+} \frac{f(x)}{g(x)} = \lim_{x \to a_0^+} \frac{f'(x)}{g'(x)}.$$

- The remaining details are in the Lecture Notes.