Mathematical Analysis 1

Lecture #10

The Chain Rule. The Inverse Function Theorem. Local extrema

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- This lecture has six parts:
  1. A brief review of differentiation
  2. The Chain Rule
  3. The Inverse Function Theorem
  4. A summary of important derivatives
  5. Implicit differentiation
  6. Local extrema

1. A brief review of differentiation

Definition

Suppose that $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is a function and $a_0 \in A$ is an accumulation point of A. Then the *derivative* of f at a_0 is defined to be

$$f'(a_0) \quad := \quad \lim_{a \to a_0} \frac{f(a) - f(a_0)}{a - a_0},$$

provided the limit exists (and is a real number). If the limit above exists, then we say that f is *differentiable* at a_0. We say that the function f is *differentiable* provided that f is differentiable at all points in its domain.

Definition

Suppose that $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is a function, and that $I \subseteq A$ is an open interval. (In particular, we are assuming that f is **defined** at all points in I.) We say that f is *differentiable on I* if f is differentiable at all points in I.

- **Remark:** In practice, we will most commonly deal with the situation where we have a function $f : A \subseteq \mathbb{R} \to \mathbb{R}$, and a point $a_0 \in I \subseteq A$, where I is some open interval.
  - In this case, f is defined on some open neighborhood (open interval) containing a_0, and in particular, a_0 belongs to A and is an accumulation point of A.
  - With this set-up, the definition from the previous slide applies, i.e. the derivative of f at a_0 is defined to be

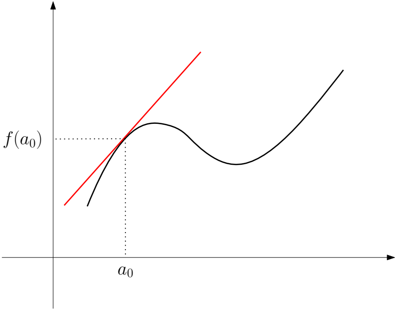
$$f'(a_0) \quad := \quad \lim_{a \to a_0} \tfrac{f(a) - f(a_0)}{a - a_0},$$

    provided the limit exists (as a real number).
  - Sometimes, it is convenient to rewrite the above limit as

$$f'(a_0) \quad := \quad \lim_{h \to 0} \tfrac{f(a_0 + h) - f(a_0)}{h},$$

    provided the limit exists (as a real number).

- Suppose $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is a function, differentiable at a point $a_0 \in A$ (in particular, a_0 is an accumulation point of A).
  - We define the *tangent line* (or simply *tangent*) to the graph of f though the point $\big(a_0, f(a_0)\big)$ to be the line though that point, and with slope $f'(a_0)$.



  - Note that the equation of this line (if it is exists) is

$$y = f'(a_0)(x - a_0) + f(a_0).$$

Definition

Suppose that $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is a function and $a_0 \in A$ is an accumulation point of A. Then the *derivative* of f at a_0 is defined to be

$$f'(a_0) \quad := \quad \lim_{a \to a_0} \frac{f(a) - f(a_0)}{a - a_0},$$

provided the limit exists (and is a real number). If the limit above exists, then we say that f is *differentiable* at a_0. We say that the function f is *differentiable* provided that f is differentiable at all points in its domain.

Theorem 4.3.1

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and let $a_0 \in A$ be an accumulation point of A. If f is differentiable at a_0, then f is continuous at a_0.

Proposition 4.3.3

Let $f, g : A \subseteq \mathbb{R} \to \mathbb{R}$ be functions, and let $a_0 \in A$ be an accumulation point of A. If f and g are both differentiable at a_0, then so is the function $f + g$, and moreover, we have that $(f+g)'(a_0) = f'(a_0) + g'(a_0)$.

Proposition 4.3.4

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, let $a_0 \in A$ be an accumulation point of A, and let $c \in \mathbb{R}$ be a constant. If f is differentiable at a_0, then so is the function cf, and moreover, we have that $(cf)'(a_0) = c\big(f'(a_0)\big)$.

The Product Rule

Let $f, g : A \subseteq \mathbb{R} \to \mathbb{R}$ be functions, let $a_0 \in A$ be an accumulation point of A, and assume that both f and g are differentiable at a_0. Then fg is also differentiable at a_0, and moreover, we have the following formula: $(fg)'(a_0) = f'(a_0)g(a_0) + f(a_0)g'(a_0)$.

- **Remark:** If $f, g : A \subseteq \mathbb{R} \to \mathbb{R}$ are differentiable functions (i.e. if they are differentiable at all points in their domain), then the Product Rule states that fg is also differentiable, and it gives us the formula $(fg)' = f'g + fg'$.

The Quotient Rule

Let $f, g : A \subseteq \mathbb{R} \to \mathbb{R}$ be functions, and let $a_0 \in A$ be an accumulation point of A. Assume that both f and g are differentiable at a_0 and that $g(a_0) \neq 0$. Then $\frac{f}{g}$ is also differentiable at a_0, and moreover, we have the following formula:

$$\left(\frac{f}{g}\right)'(a_0) = \frac{f'(a_0)g(a_0) - f(a_0)g'(a_0)}{(g(a_0))^2}.$$

- We derived the following formulas:

function $f(x)$	derivative $f'(x)$	differentiable for
$c = \text{const.}$	0	$x \in \mathbb{R}$
x^n (for a fixed $n \in \mathbb{N}$)	nx^{n-1}	$x \in \mathbb{R}$
e^x	e^x	$x \in \mathbb{R}$
$\ln x$	$\frac{1}{x}$	$x > 0$
$\sin x$	$\cos x$	$x \in \mathbb{R}$
$\cos x$	$-\sin x$	$x \in \mathbb{R}$
$\tan x$	$\frac{1}{\cos^2 x}$	$x \neq \frac{2k+1}{2}\pi,\ k \in \mathbb{Z}$

2. The Chain Rule

The Chain Rule

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and $g : B \subseteq \mathbb{R} \to \mathbb{R}$ be functions such that $\mathrm{Im}(f) \subseteq B$ (so that $g \circ f : A \to \mathbb{R}$ is defined). Assume that f is differentiable at a point $a_0 \in A$, and that g is differentiable at the point $b_0 := f(a_0)$. (In particular, a_0 belongs to and is an accumulation point of A, whereas b_0 belongs to and is an accumulation point of B.) Then $g \circ f$ is differentiable at a_0, and moreover, we have that

$$(g \circ f)'(a_0) \quad = \quad g'(b_0)f'(a_0).$$

- Proof: Lecture Notes.
- Let's take a look at some examples!

Example 4.6.1

Compute $\frac{d}{dx}((\sin x)^{100})$.

Solution. Set $h(x) = (\sin x)^{100}$, so that $h = g \circ f$, where $f(x) = \sin x$ and $g(x) = x^{100}$. We know that $f'(x) = \cos x$ and $g'(x) = 100x^{99}$. Therefore,

$$\begin{aligned} \frac{d}{dx}((\sin x)^{100}) &= h'(x) \\ &= g'(f(x))f'(x) \\ &= 100(\sin^{99} x)(\cos x). \end{aligned}$$

Remark: The idea is that we are differentiating the function $(\boxed{(\sin x)})^{100}$. For $\square = \boxed{(\sin x)}$, we first differentiate $\square^{100}$ to obtain $100\square^{99} = 100 \sin^{99} x$, and then we differentiate $\square$ to obtain $\cos x$; we multiply the two to obtain $100(\sin^{99} x)(\cos x)$. $\square$

Example 4.6.2

Compute $\frac{d}{dx}(\sin^3 x + 2)^2$.

Solution. By repeatedly applying the Chain Rule, we obtain:

$$\begin{aligned}
\frac{d}{dx}(\sin^3 x + 2)^2 &= 2(\sin^3 x + 2) \cdot \frac{d}{dx}(\sin^3 x + 2) \\
&= 2(\sin^3 x + 2)\left(\frac{d}{dx}(\sin^3 x) + \frac{d}{dx}(2)\right) \\
&= 2(\sin^3 x + 2)\left(3 \sin^2 x \frac{d}{dx}(\sin x) + 0\right) \\
&= 2(\sin^3 x + 2)(3(\sin^2 x)(\cos x)) \\
&= 6(\sin^2 x)(\cos x)(\sin^3 x + 2).
\end{aligned}$$

□

- **Remark:** When differentiating, we often use the following trick:

$$\square^{\Delta} \;=\; e^{\ln\left(\square^{\Delta}\right)} \;=\; e^{\Delta \ln \square}.$$

  This is useful for differentiating via the Chain Rule, further using the fact that $\frac{d}{dx}e^x = e^x$.

Proposition 4.6.3

Let $a \in (0,1) \cup (1,+\infty)$ be a fixed real number. Then:

(a) $\frac{d}{dx} a^x = a^x \ln a$ for $x \in \mathbb{R}$;

(b) $\frac{d}{dx} \log_a(x) = \frac{1}{x \ln a}$ for $x \in (0,+\infty)$.

Proof. First, recall that

- $\frac{d}{dx} e^x = e^x$ for $x \in \mathbb{R}$;
- $\frac{d}{dx} \ln x = \frac{1}{x}$ for $x \in (0,+\infty)$.

For (a), we compute (for $x \in \mathbb{R}$):

$$\begin{aligned} \frac{d}{dx} a^x &= \frac{d}{dx}\left(e^{\ln(a^x)}\right) \\ &= \frac{d}{dx}\left(e^{x \ln a}\right) \\ &= e^{x \ln a} \frac{d}{dx}(x \ln a) && \text{by the Chain Rule} \\ &= e^{\ln(a^x)} \ln a \\ &= a^x \ln a. \end{aligned}$$

Proposition 4.6.3

Let $a \in (0,1) \cup (1,+\infty)$ be a fixed real number. Then:

- (a) $\frac{d}{dx} a^x = a^x \ln a$ for $x \in \mathbb{R}$;
- (b) $\frac{d}{dx} \log_a(x) = \frac{1}{x \ln a}$ for $x \in (0,+\infty)$.

Proof (continued). Reminder: $\frac{d}{dx} \ln x = \frac{1}{x}$ for $x \in (0,+\infty)$.

For (b), we compute (for $x \in (0,+\infty)$):

$$\frac{d}{dx} \log_a(x) \quad = \quad \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) \quad = \quad \frac{1}{\ln a} \frac{d}{dx}(\ln x) \quad = \quad \frac{1}{\ln a} \cdot \frac{1}{x} \quad = \quad \frac{1}{x \ln a}.$$

□

Proposition 4.6.4

Let $\alpha \in \mathbb{R}$ be a fixed constant. Then $\frac{d}{dx}(x^\alpha) = \alpha x^{\alpha-1}$ for $x \in (0, +\infty)$.

Proof. For $x \in (0, +\infty)$, we compute:

$$\begin{aligned}
\frac{d}{dx}(x^\alpha) &= \frac{d}{dx}\left(e^{\ln(x^\alpha)}\right) \\
&= \frac{d}{dx}\left(e^{\alpha \ln x}\right) \\
&= e^{\alpha \ln x} \frac{d}{dx}(\alpha \ln x) \qquad \text{by the Chain Rule} \\
&= e^{\ln(x^\alpha)} \cdot \frac{\alpha}{x} \\
&= x^\alpha \cdot \frac{\alpha}{x} \\
&= \alpha x^{\alpha-1},
\end{aligned}$$

and we are done. □

Example 4.6.5

Compute $\frac{d}{dx}(x^x)$ for $x \in (0, +\infty)$.

Solution. For $x \in (0, +\infty)$, we compute:

$$\begin{aligned}
\frac{d}{dx}(x^x) &= \frac{d}{dx}\left(e^{\ln(x^x)}\right) \\
&= \frac{d}{dx}\left(e^{x \ln x}\right) \\
&= e^{x \ln x} \cdot \frac{d}{dx}(x \ln x) && \text{by the Chain Rule} \\
&= x^x \cdot \frac{d}{dx}(x \ln x) \\
&= x^x\left(\ln x + x \cdot \frac{1}{x}\right) && \text{by the Product Rule} \\
&= x^x(1 + \ln x).
\end{aligned}$$

□

3. The Inverse Function Theorem

- Suppose that $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is one-to-one and differentiable at a point $a_0 \in A$, and that f^{-1} is differentiable at the point $b_0 := f(a_0)$.
  - In particular, a_0 belongs to and is an accumulation point of A, whereas b_0 belongs to and is an accumulation point of $\mathrm{Im}(f)$.
  - Then by the Chain Rule, we have that

$$(f^{-1} \circ f)'(a_0) \;=\; (f^{-1})'\big(f(a_0)\big)f'(a_0) \;=\; (f^{-1})'(b_0)f'(a_0)$$

  - But note that $f^{-1} \circ f = \mathrm{Id}_A$, that is, $(f^{-1} \circ f) : A \to A$ is given by $(f^{-1} \circ f)(x) = x$ for all $x \in A$.
  - Therefore, $(f^{-1} \circ f)'(x) = 1$ for all $x \in A$ such that x is an accumulation point of A.
  - So, the computation above yields $1 = (f^{-1})'(b_0)f'(a_0)$.
  - Consequently, both $f'(a_0)$ and $(f^{-1})'(b_0)$ are non-zero, and moreover, we have that

$$(f^{-1})'(b_0) \;=\; \tfrac{1}{f'(a_0)}.$$

- The Inverse Function Theorem (see below) weakens the assumption for f^{-1} from the previous slide in the sense that it does **not** assume that f^{-1} is differentiable at $b_0 := f(a_0)$.
    - Instead, it strengthens the assumption for f in a suitable way, and it concludes (rather than assuming) that f^{-1} is differentiable at b_0, with the same formula for $(f^{-1})'(b_0)$ as the one that we obtained above.

The Inverse Function Theorem

Let I be an open interval in $\mathbb{R}$, and let $f : I \to \mathbb{R}$ be a one-to-one function that is differentiable on I. Let $a_0 \in I$, set $b_0 := f(a_0)$, and assume that $f'(a_0) \neq 0$. Then f^{-1} is differentiable at b_0, and moreover,

$$(f^{-1})'(b_0) = \frac{1}{f'(a_0)}.$$

The Inverse Function Theorem

Let I be an open interval in $\mathbb{R}$, and let $f : I \to \mathbb{R}$ be a one-to-one function that is differentiable on I. Let $a_0 \in I$, set $b_0 := f(a_0)$, and assume that $f'(a_0) \neq 0$. Then f^{-1} is differentiable at b_0, and moreover,

$$(f^{-1})'(b_0) \quad = \quad \tfrac{1}{f'(a_0)}.$$

Proof. First of all, since f is one-to-one, we know that $\forall a \in I \setminus \{a_0\}$: $f(a) \neq f(a_0)$, and consequently, $f(a) - f(a_0) \neq 0$.

- We will implicitly use this fact in our computation below (ensuring that we are not dividing by zero, so that our terms are defined).

Since f is differentiable on I, Theorem 4.3.1 guarantees that f is continuous on I. Moreover, Proposition 3.5.11 guarantees that $J := \text{Im}(f)$ is an open interval (because the domain I of f is an open interval), and Theorem 3.5.12 guarantees that $f^{-1} : J \to \mathbb{R}$ is continuous. We now compute (next slide):

Proof (continued).

$$
\begin{aligned}
(f^{-1})'(b_0) &= \lim_{b \to b_0} \frac{f^{-1}(b) - f^{-1}(b_0)}{b - b_0} \\
&\overset{(*)}{=} \lim_{a \to a_0} \frac{a - a_0}{f(a) - f(a_0)} \\
&= \lim_{a \to a_0} \frac{1}{\frac{f(a) - f(a_0)}{a - a_0}} \\
&\overset{(**)}{=} \frac{1}{\lim\limits_{a \to a_0} \frac{f(a) - f(a_0)}{a - a_0}} = \frac{1}{f'(a_0)}.
\end{aligned}
$$

For (*), we used the substitution $a = f^{-1}(b)$, so that $b = f(a)$. This is valid by the continuity of f^{-1} at b_0 and the continuity of f at a_0, ensuring that

$$b \to b_0 \quad \Longleftrightarrow \quad a = f^{-1}(b) \to f^{-1}(b_0) = a_0.$$

Meanwhile, (**) uses the fact that $\lim\limits_{a \to a_0} \frac{f(a) - f(a_0)}{a - a_0} = f'(a_0) \neq 0$, which in particular means that we are not dividing by zero. □

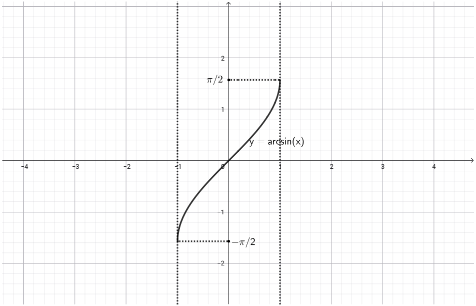
The Inverse Function Theorem

Let I be an open interval in $\mathbb{R}$, and let $f : I \to \mathbb{R}$ be a one-to-one function that is differentiable on I. Let $a_0 \in I$, set $b_0 := f(a_0)$, and assume that $f'(a_0) \neq 0$. Then f^{-1} is differentiable at b_0, and moreover,

$$(f^{-1})'(b_0) = \frac{1}{f'(a_0)}.$$

- Recall that
  - $\frac{d}{dx} \sin x = \cos x$ and $\frac{d}{dx} \cos x = -\sin x$ for $x \in \mathbb{R}$;
  - $\frac{d}{dx} \tan x = \frac{1}{\cos^2 x}$ for $x \in \mathbb{R} \setminus \left\{ \frac{2k+1}{2}\pi \mid k \in \mathbb{Z} \right\}$.
- Using these formulas, as well as the Inverse Function Theorem, we will now derive formulas for the derivatives of the inverse trigonometric functions $\arcsin x$ and $\arctan x$.
  - The derivative of $\arccos x$ is left as an exercise.

- The sine function is **not** one-to-one, and therefore, it does not have an inverse.
- However, we can turn it into a one-to-one function by restricting its domain to $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
- The arcsine function is simply the inverse function of the sine function restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
- The domain of the arcsine function is $[-1, 1]$, its image is $[-\frac{\pi}{2}, \frac{\pi}{2}]$, and its graph is represented below.



Proposition 4.7.1

For all $x \in [-1, 1]$, we have that $\sin(\arcsin x) = x$ and $\cos(\arcsin x) = \sqrt{1 - x^2}$.

Proof. Fix $x \in [-1, 1]$.

The fact that $\sin(\arcsin x) = x$ follows immediately from the definition of the arcsine function.

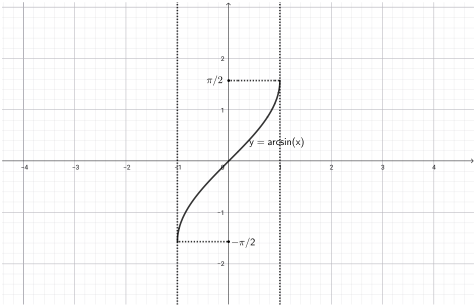
Since $\sin^2 y + \cos^2 y = 1$ for all $y \in \mathbb{R}$, it follows that

$$\cos^2(\arcsin x) \quad = \quad 1 - \sin^2(\arcsin x) \quad = \quad 1 - x^2.$$

But note that $\arcsin x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, and consequently, $\cos(\arcsin x) \geq 0$. Therefore,

$$\cos(\arcsin x) \quad = \quad \sqrt{1 - x^2},$$

and we are done. $\square$



Proposition 4.7.1

For all $x \in [-1, 1]$, we have that $\sin(\arcsin x) = x$ and $\cos(\arcsin x) = \sqrt{1 - x^2}$.

- **Remark:** For $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, we do in fact have that $\arcsin(\sin x) = x$.
  - However, this equality does **not** hold for $x \in \mathbb{R} \setminus [-\frac{\pi}{2}, \frac{\pi}{2}]$.
  - For example, note that $\arcsin(\sin \pi) = \arcsin(0) = 0 \neq \pi$.

Proposition 4.7.1

For all $x \in [-1, 1]$, we have that $\sin(\arcsin x) = x$ and $\cos(\arcsin x) = \sqrt{1 - x^2}$.

Proposition 4.7.2

$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$ for $x \in (-1, 1)$.

Proof. Recall that $\frac{d}{dx} \sin x = \cos x$ for $x \in \mathbb{R}$. Note the function $\sin \restriction (-\frac{\pi}{2}, \frac{\pi}{2}) : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is one-to-one and differentiable on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Now, fix any $y_0 \in (-1, 1)$, and set $x_0 := \arcsin y_0$, so that $y_0 = \sin x_0$. Then $x_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$, and so $\sin'(x_0) = \cos x_0 \neq 0$. We now compute:

$$\arcsin'(y_0) \overset{(*)}{=} \frac{1}{\sin'(x_0)} = \frac{1}{\cos(x_0)} = \frac{1}{\cos(\arcsin y_0)} \overset{(**)}{=} \frac{1}{\sqrt{1-y_0^2}},$$

where (*) follows from the Inverse Function Theorem, whereas (**) follows from Proposition 4.7.1 □

Proposition 4.7.2

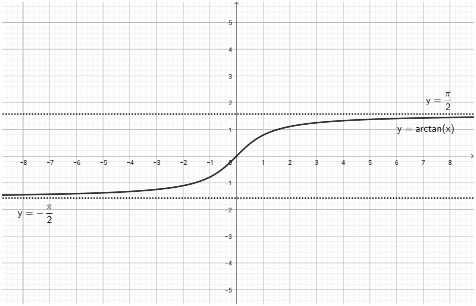
$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$ for $x \in (-1, 1)$.

- **Remark:** $\arcsin x$ is **not** differentiable at ± 1.
  - Certainly, the formula

    $$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$$

    from Proposition 4.7.2 would not work for $x = \pm 1$ (because we cannot divide by zero).
  - But in fact, no other formula will work for $x = \pm 1$, either, that is, $\arcsin x$ is simply not differentiable at ± 1.
    - Justification: Lecture Notes (easy).

- The tangent function is **not** one-to-one, and therefore, it does not have an inverse.
- However, we can turn it into a one-to-one function by restricting its domain to $(-\frac{\pi}{2}, \frac{\pi}{2})$.
- The arctangent function is simply the inverse function of the tangent function restricted to $(-\frac{\pi}{2}, \frac{\pi}{2})$.
- The domain of the arctagent function is $\mathbb{R}$, its range is $(-\frac{\pi}{2}, \frac{\pi}{2})$, and its graph is represented below.



- **Remark:** By the definition of the arctangent function, we have that $\tan(\arctan x) = x$ for all $x \in \mathbb{R}$.
  - Meanwhile, for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we have that $\arctan(\tan x) = x$.
  - However, this last equality fails for values of $x \notin (-\frac{\pi}{2}, \frac{\pi}{2})$.
    - For example, $\arctan(\tan \pi) = \arctan 0 = 0 \neq \pi$.

Proposition 4.7.3

$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$ for $x \in \mathbb{R}$.

Proof. First, we know that the function $\tan \upharpoonright (-\frac{\pi}{2}, \frac{\pi}{2}) : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is one-to-one and differentiable on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Now, fix $y_0 \in \mathbb{R}$, and set $x_0 := \arctan y_0$, so that $y_0 = \tan x_0$. Note that $\tan'(x_0) = \frac{1}{\cos^2 x_0} \neq 0$. We now compute:

$$\arctan'(y_0) \overset{(*)}{=} \frac{1}{\tan' x_0} = \frac{1}{\frac{1}{\cos^2 x_0}} \overset{(**)}{=} \frac{1}{1+\tan^2 x_0} = \frac{1}{1+y_0^2},$$

where (*) follows from the Inverse Function Theorem, whereas (**) follows from the following computation:

$$1 + \tan^2 x_0 = 1 + \frac{\sin^2 x_0}{\cos^2 x_0} = \frac{\cos^2 x_0 + \sin^2 x_0}{\cos^2 x_0} = \frac{1}{\cos^2 x_0}.$$

□

4. A summary of important derivatives

function $f(x)$	derivative $f'(x)$	differentiable for
$c =$ const.	0	$x \in \mathbb{R}$
x^α (for a fixed const. $\alpha \in \mathbb{R}$)	$\alpha x^{\alpha-1}$	$x > 0$
e^x	e^x	$x \in \mathbb{R}$
$\ln x$	$\frac{1}{x}$	$x > 0$
$\sin x$	$\cos x$	$x \in \mathbb{R}$
$\cos x$	$-\sin x$	$x \in \mathbb{R}$
$\tan x$	$\frac{1}{\cos^2 x}$	$x \neq \frac{2k+1}{2}\pi,\ k \in \mathbb{Z}$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$x \in (-1,1)$
$\arctan x$	$\frac{1}{1+x^2}$	$x \in \mathbb{R}$

- The formulas from the table above should be memorized.

function $f(x)$	derivative $f'(x)$	differentiable for
x^α (for a fixed const. $\alpha \in \mathbb{R}$)	$\alpha x^{\alpha-1}$	$x > 0$

- **Remark:** As stated in our table, for a fixed constant $\alpha \in \mathbb{R}$, the formula $\frac{d}{dx}(x^\alpha) = \alpha x^{\alpha-1}$ is valid for $x \in (0, +\infty)$.
  - However:
    - for the special case when α is a positive integer, the formula $\frac{d}{dx}(x^\alpha) = \alpha x^{\alpha-1}$ is valid for $x \in \mathbb{R}$;
    - for the special case when $\alpha = \frac{1}{n}$ for some **odd** $n \in \mathbb{N}$, the formula $\frac{d}{dx}(x^\alpha) = \alpha x^{\alpha-1}$ is valid for $x \in \mathbb{R} \setminus \{0\}$;
    - for the special case when α is a negative integer or when $\alpha = -\frac{1}{n}$ for some **odd** $n \in \mathbb{N}$, the formula $\frac{d}{dx}(x^\alpha) = \alpha x^{\alpha-1}$ is valid for $x \in \mathbb{R} \setminus \{0\}$.

  Indeed, the first two bullet points above follow from Propositions 4.4.2 and 4.4.4(a), respectively, and the third bullet point follows from an argument similar to the one given in those two propositions.

function $f(x)$	derivative $f'(x)$	differentiable for
$c = \text{const.}$	0	$x \in \mathbb{R}$
x^α (for a fixed const. $\alpha \in \mathbb{R}$)	$\alpha x^{\alpha-1}$	$x > 0$
e^x	e^x	$x \in \mathbb{R}$
$\ln x$	$\frac{1}{x}$	$x > 0$
$\sin x$	$\cos x$	$x \in \mathbb{R}$
$\cos x$	$-\sin x$	$x \in \mathbb{R}$
$\tan x$	$\frac{1}{\cos^2 x}$	$x \neq \frac{2k+1}{2}\pi$, $k \in \mathbb{Z}$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$x \in (-1, 1)$
$\arctan x$	$\frac{1}{1+x^2}$	$x \in \mathbb{R}$

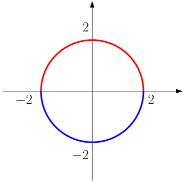
- **Remark:** Consider $a \in (0,1) \cup (1,+\infty)$.
  - In addition to the formulas from the table above, you should either memorize the following formals (from Proposition 4.6.3), or be able to derive them very quickly:
    - $\frac{d}{dx}(a^x) = a^x \ln a$ for $x \in \mathbb{R}$;
    - $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$ for $x \in (0,+\infty)$.

5. Implicit differentiation

- Sometimes a function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is given "implicitly," without an explicit formula for $f(x)$. For instance, a function may be given by an equation such as

$$x^2 + y^2 = 4 \qquad \text{or} \qquad x^3 + y^3 = 6xy,$$

where we think of y as a function of x.
- Technically, an equation of the type given above might define more than one function, as is the case with the equation $x^2 + y^2 = 4$.
  - This equation defines defines the circle of radius 2, centered at the origin, and it in fact defines two distinct functions:
    - $y = \sqrt{4 - x^2}$ for $x \in [-2, 2]$;
    - $y = -\sqrt{4 - x^2}$ for $x \in [-2, 2]$.



- **Remark:** It is also possible that an equation does not define **any** function (with a non-empty domain).
  - For example, this is the case for the equation $x^2 + y^2 = -1$.
  - However, such equations are not interesting.
- Sometimes, it may be very difficult (or even impossible) to find an actual formula for y in terms of x.
  - Luckily, it is not necessary to find such a formula in order to compute the derivative of our implicitly defined function.
  - Instead, the idea is to differentiate both sides of the equation, and then solve for $\frac{dy}{dx}$ in terms of x and y.
  - Here, is it important to remember that y is a function of x, and so we must not forget to apply the Chain Rule.
  - For instance, we have:
    - $\frac{d}{dx}(y^7) = 7y^6 \frac{dy}{dx}$;
    - $\frac{d}{dx}(\sin y) = (\cos y) \frac{dy}{dx}$.

Example 4.9.1

Compute $\frac{dy}{dx}$ for $x^2 + y^2 = 4$. Then, compute the equation of the tangent to the curve defined by this equation passing through the point $(1, \sqrt{3})$.

Solution. We differentiate both sides of the equation to obtain

$$\frac{d}{dx}(x^2 + y^2) \quad = \quad \frac{d}{dx}(4),$$

which yields

$$2x + 2y\frac{dy}{dx} \quad = \quad 0.$$

Solving for $\frac{dy}{dx}$, we obtain

$$\frac{dy}{dx} \quad = \quad -\frac{x}{y}.$$

Example 4.9.1

Compute $\frac{dy}{dx}$ for $x^2 + y^2 = 4$. Then, compute the equation of the tangent to the curve defined by this equation passing through the point $(1, \sqrt{3})$.

Solution (continued). Reminder: $\frac{dy}{dx} = -\frac{x}{y}$.

For $(x_0, y_0) = (1, \sqrt{3})$, we get

$$\left.\frac{dy}{dx}\right|_{(x,y)=(1,\sqrt{3})} = -\frac{1}{\sqrt{3}}.$$

So, the tangent to our curve through the point $(x_0, y_0) = (1, \sqrt{3})$ has slope $-\sqrt{3}$, and therefore, the equation of this tangent is

$$y = -\frac{1}{\sqrt{3}}(x-1) + \sqrt{3},$$

or equivalently,

$$y = -\frac{x}{\sqrt{3}} + \sqrt{3} + \frac{1}{\sqrt{3}}.$$

□

Example 4.9.2

Compute $\frac{dy}{dx}$ for $x^3 + y^3 = 6xy$. Then, compute the tangent to this curve through the point $(x_0, y_0) = (3, 3)$.

Solution. We differentiate both sides of the equation:

$$\frac{d}{dx}(x^3 + y^3) \quad = \quad \frac{d}{dx}(6xy),$$

which yields

$$3x^2 + 3y^2\frac{dy}{dx} \quad = \quad 6y + 6x\frac{dy}{dx}$$

Dividing both sides by 3, we obtain

$$x^2 + y^2\frac{dy}{dx} \quad = \quad 2y + 2x\frac{dy}{dx}.$$

This implies that

$$y^2\frac{dy}{dx} - 2x\frac{dy}{dx} \quad = \quad 2y - x^2,$$

which yields

$$\frac{dy}{dx} \quad = \quad \frac{2y - x^2}{y^2 - 2x}.$$

Example 4.9.2

Compute $\frac{dy}{dx}$ for $x^3 + y^3 = 6xy$. Then, compute the tangent to this curve through the point $(x_0, y_0) = (3, 3)$.

Solution (continued). Reminder: $\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}$.

For $(x_0, y_0) = (3, 3)$, we obtain

$$\left.\frac{dy}{dx}\right|_{(x,y)=(3,3)} = -1.$$

Therefore, the equation of the tangent to our curve through the point $(x_0, y_0) = (3, 3)$ is

$$y = -(x - 3) + 3,$$

that is,

$$y = -x + 6.$$

□

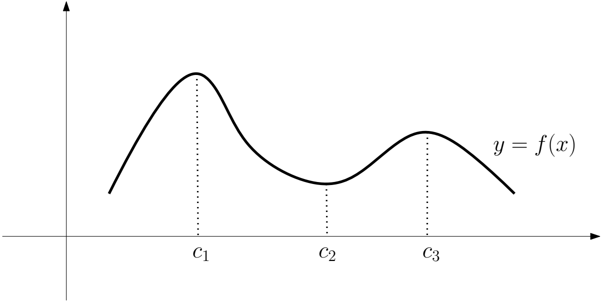
6. Local extrema

Definition

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and $c \in A$.

- We say that f reaches a *local maximum* at c if there exists some $\delta > 0$ such that for all $x \in A$, if $|x - c| < \delta$, then $f(c) \geq f(x)$.
- We say that f reaches a *local minimum* at c if there exists some $\delta > 0$ such that for all $x \in A$, if $|x - c| < \delta$, then $f(c) \leq f(x)$.
- We say that f reaches a *local extremum* at c if f reaches a local maximum or a local minimum at c.

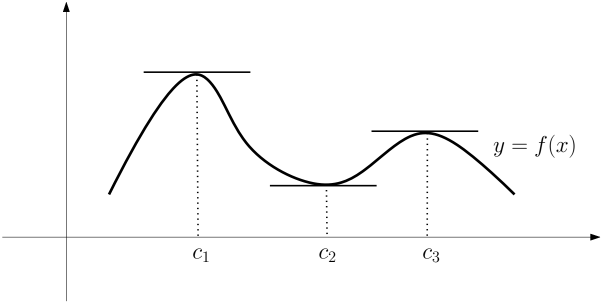
- For instance, in the picture below, the function f reaches local maxima at c_1 and c_3, and it reaches a local minimum at c_2.



Theorem 4.10.2

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and assume that f reaches a local extremum at a point $c_0 \in I \subseteq A$, where I is an open interval. If f is differentiable at c_0, then $f'(c_0) = 0$.

- **Remark:** Geometrically, Theorem 4.10.2 can be interpreted as follows: if a function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ reaches a local extremum at a point $c \in I \subseteq A$, where I is some open interval, and f is also differentiable at c, then the tangent to the curve $y = f(x)$ through the point $(c, f(c))$ is a horizontal line.



Theorem 4.10.2

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and assume that f reaches a local extremum at a point $c_0 \in I \subseteq A$, where I is an open interval. If f is differentiable at c_0, then $f'(c_0) = 0$.

Proof. We prove the theorem for the case when f reaches a local maximum at c_0; the proof for the case when f reaches a local minimum at c_0 is analogous.

So, let us assume f reaches a local maximum at c_0, and that f is differentiable at c_0. Fix $\delta > 0$ such that for all $c \in A$, if $|c - c_0| < \delta$, then $f(c_0) \geq f(c)$. WMA that $(c_0 - \delta, c_0 + \delta) \subseteq I \subseteq A$, for otherwise, we simply choose a smaller δ. Therefore, we in fact have that for all $c \in \mathbb{R}$, if $|c - c_0| < \delta$, then $f(c_0) \geq f(c)$.

Proof (continued). Reminder: for all $c \in \mathbb{R}$, if $|c - c_0| < \delta$, then $f(c_0) \geq f(c)$; WTS $f'(c_0) = 0$.

Now, by definition, $f'(c_0) = \lim_{c \to c_0} \frac{f(c)-f(c_0)}{c-c_0}$. Since f is defined on an open interval containing c_0, we deduce that

$$f'(c_0) = \lim_{c \to c_0^-} \frac{f(c)-f(c_0)}{c-c_0} \qquad \text{and} \qquad f'(c_0) = \lim_{c \to c_0^+} \frac{f(c)-f(c_0)}{c-c_0}.$$

Now, for all $c \in (c_0 - \delta, c_0)$, we have that $c < c_0$ and $f(c) \leq f(c_0)$, which implies that $\frac{f(c)-f(c_0)}{c-c_0} \geq 0$, and consequently, $f'(c_0) = \lim_{c \to c_0^-} \frac{f(c)-f(c_0)}{c-c_0} \geq 0$.

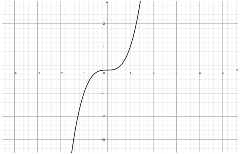
On the other hand, for all $c \in (c_0, c_0 + \delta)$, we have that $c > c_0$ and $f(c) \leq f(c_0)$, which implies that $\frac{f(c)-f(c_0)}{c-c_0} \leq 0$, and consequently, $f'(c_0) = \lim_{c \to c_0^-} \frac{f(c)-f(c_0)}{c-c_0} \leq 0$.

We have now shown that $f'(c_0) \geq 0$ and $f'(c_0) \leq 0$, and it follows that $f'(c_0) = 0$. □

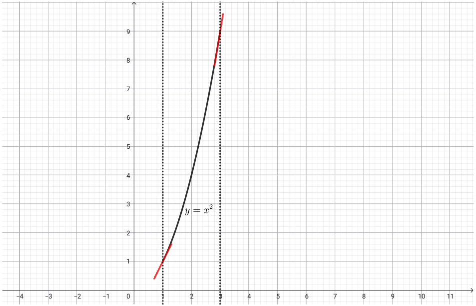
Theorem 4.10.2

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and assume that f reaches a local extremum at a point $c_0 \in I \subseteq A$, where I is an open interval. If f is differentiable at c_0, then $f'(c_0) = 0$.

- **Remark:** The converse of Theorem 4.10.2 is false.
  - Indeed, it is possible for the function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ to be differentiable at a point $c \in I \subseteq A$, where I is some open interval, and that $f'(c) = 0$, but that f does **not** reach a local extremum at c.
  - For instance, the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ for all $x \in \mathbb{R}$ is differentiable, and it satisfies $f'(0) = 0$, but f does **not** reach a local extremum at $c = 0$.



- **Remark:** Theorem 4.10.2 does **not** apply when a function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ reaches a local extremum at a point $c \in A$ that does **not** belong to any open interval $I \subseteq A$.
  - For instance, consider the function $f : [1, 3] \to \mathbb{R}$ given by $f(x) = x^2$ for all $x \in [1, 3]$.



  - Then f reaches a global (and therefore also local) minimum at $c_1 = 1$ and a global (and therefore also local) maximum at $c_2 = 3$.
  - However, $f'(c_1) = 2c_1 = 2$ and $f'(c_2) = 2c_2 = 6$.
  - Note that the tangents to the graph of f through the points $(1, f(1)) = (1, 1)$ and $(3, f(3)) = (3, 9)$ are **not** horizontal.