Mathematical Analysis 1

Lecture #9

Differentiation

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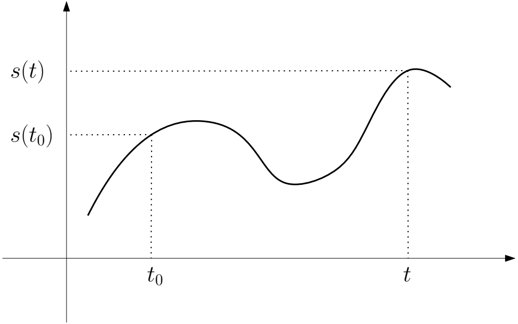
April 9, 2026

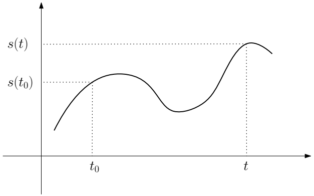
- This lecture has six parts:
  1. Motivating the derivative: instantaneous velocity and tangent to the curve
  2. The definition of the derivative
  3. Differentiability and continuity
  4. Sums and scalar multiples of differentiable functions
  5. The derivatives of some important functions
  6. The Product and Quotient Rules for derivatives

1. Motivating the derivative: instantaneous velocity and tangent to the curve

- Suppose a particle is traveling along a straight line (possibly changing speed and direction over time).
- Suppose we then plot the position of the particle against time.



- The horizontal axis (the "t-axis" in the picture above) denotes time, the vertical axis (the "s-axis") denotes position, and $s(t)$ denotes the position of the particle at time t.



- Now, consider two moments in time: t_0 and t_1 (for convenience, let us assume that $t_0 < t_1$).
- The average velocity of the particle during the time interval $[t_0, t_1]$ is

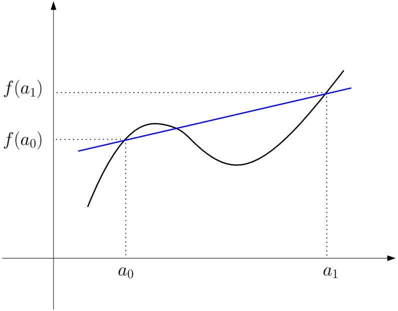
$$v_{[t_0,t_1]} \quad := \quad \frac{s(t_1)-s(t_0)}{t_1-t_0}.$$

- If $t_1 \approx t_0$, then the velocity above is something like the instantaneous velocity of the particle. More formally, we can take the limit! So, the instantaneous velocity of the particle at time t_0 is

$$v_{t_0} \quad := \quad \lim_{t \to t_0} \frac{s(t)-s(t_0)}{t-t_0},$$

if the limit exists.

- Suppose we are given a function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and distinct points $a_0, a_1 \in A$.

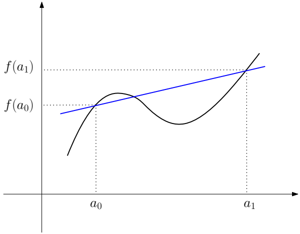


- Then the equation of the secant line to the graph of the function f through the points $(a_0, f(a_0))$ and $(a_1, f(a_1))$ is

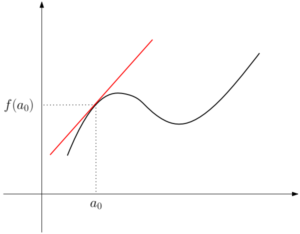
$$y = \frac{f(a_1)-f(a_0)}{a_1-a_0}(x - a_0) + f(a_0),$$

- The slope of this line (i.e. the coefficient in front of x) is $\frac{f(a_1)-f(a_0)}{a_1-a_0}$.

- Reminder: $y = \frac{f(a_1)-f(a_0)}{a_1-a_0}(x - a_0) + f(a_0)$.



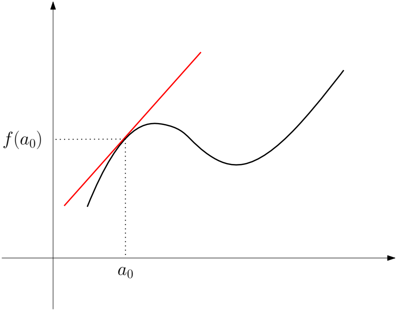
- Now, suppose we take $a_1 \approx a_0$. Then the secant line that we obtain gets very close to a tangent line.



- Reminder: $y = \frac{f(a_1)-f(a_0)}{a_1-a_0}(x - a_0) + f(a_0)$.
- The actual equation of the tangent to the graph of f through the point $(a_0, f(a_0))$ is obtained by taking the limit, as follows:

$$y := \left(\lim_{a\to a_0} \frac{f(a)-f(a_0)}{a-a_0}\right)(x - a_0) + f(a_0),$$

provided the limit exists. The slope of this tangent is $\lim_{a\to a_0} \frac{f(a)-f(a_0)}{a-a_0}$.



2. The definition of the derivative

Definition

Suppose that $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is a function and $a_0 \in A$ is an accumulation point of A. Then the *derivative* of f at a_0 is defined to be

$$f'(a_0) \quad := \quad \lim_{a \to a_0} \frac{f(a) - f(a_0)}{a - a_0},$$

provided the limit exists (and is a real number). If the limit above exists, then we say that f is *differentiable* at a_0. We say that the function f is *differentiable* provided that f is differentiable at all points in its domain.

Definition

Suppose that $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is a function, and that $I \subseteq A$ is an open interval. (In particular, we are assuming that f is **defined** at all points in I.) We say that f is *differentiable on I* if f is differentiable at all points in I.

- **Remark:** In practice, we will most commonly deal with the situation where we have a function $f : A \subseteq \mathbb{R} \to \mathbb{R}$, and a point $a_0 \in I \subseteq A$, where I is some open interval.
  - In this case, f is defined on some open neighborhood (open interval) containing a_0, and in particular, a_0 belongs to A and is an accumulation point of A.
  - With this set-up, the definition from the previous slide applies, i.e. the derivative of f at a_0 is defined to be

$$f'(a_0) \quad := \quad \lim_{a \to a_0} \frac{f(a) - f(a_0)}{a - a_0},$$

    provided the limit exists (as a real number).
  - Sometimes, it is convenient to rewrite the above limit as

$$f'(a_0) \quad := \quad \lim_{h \to 0} \frac{f(a_0 + h) - f(a_0)}{h},$$

    provided the limit exists (as a real number).

Example 4.2.1

Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ for all $x \in \mathbb{R}$. Consider any $a_0 \in \mathbb{R}$. Then

$$
\begin{aligned}
f'(a_0) &= \lim_{h \to 0} \frac{f(a_0+h) - f(a_0)}{h} \\
&= \lim_{h \to 0} \frac{(a_0+h)^2 - a_0^2}{h} \\
&= \lim_{h \to 0} \frac{(a_0^2 + 2a_0 h + h^2) - a_0^2}{h} \\
&= \lim_{h \to 0} \frac{2a_0 h + h^2}{h} \\
&= \lim_{h \to 0} (2a_0 + h) \\
&= 2a_0.
\end{aligned}
$$

- **Remark:** Suppose that $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is a function, and A' is the set of all points at which f is differentiable. Then $f' : A' \to \mathbb{R}$ is a function in its own right.

Example 4.2.2

Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ for all $x \in \mathbb{R}$. It then follows from Example 4.2.1 that the function f is differentiable, and the function $f' : \mathbb{R} \to \mathbb{R}$ is given by $f'(x) = 2x$ for all $x \in \mathbb{R}$.

- The notation $f'(x)$ that we introduced above is due to **Lagrange**.
- However, we sometimes use the following notation due to **Leibniz** instead:

$$\frac{df(x)}{dx} \qquad \text{or} \qquad \frac{df}{dx}(x) \qquad \text{or} \qquad \frac{d}{dx}f(x).$$

  - The three expressions above have exactly the same meaning as $f'(x)$.

- The Leibniz notation: $\frac{df(x)}{dx}$, $\frac{df}{dx}(x)$, $\frac{d}{dx}f(x)$.
- The Leibniz notation may be convenient when a function f is given by a formula, so that we may write $\frac{d}{dx}(x^4 - 3x^2 + 1)$ rather than $(x^4 - 3x^2 + 1)'$.
- The Leibniz notation is even more convenient when we need to differentiate an expression that involves more than one "letter," one of which is the variable with respect to which we are differentiating, whereas the others are treated as constants.
    - For instance, $\frac{d}{dx}(x^\alpha)$ tells us that we are differentiating with respect to the variable x, while treating α as a constant.
    - Similarly, $\frac{d}{dt}(ct^2)$ tells us that we are differentiating with respect to t, while treating c as a constant.

- The Leibniz notation: $\frac{df(x)}{dx}$, $\frac{df}{dx}(x)$, $\frac{d}{dx}f(x)$.
- We will also occasionally see equations of the form $y = f(x)$, in which case we may write $\frac{dy}{dx}$ instead of $\frac{df}{dx}(x)$ or $f'(x)$.
  - If we mean to write "the derivative of f evaluated at a_0," then we may write

$$\left.\frac{dy}{dx}\right|_{x=a_0}$$

    instead of $f'(a_0)$.

Definition

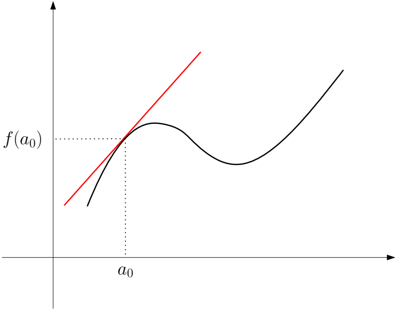
Suppose that $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is a function and $a_0 \in A$ is an accumulation point of A. Then the *derivative* of f at a_0 is defined to be

$$f'(a_0) \quad := \quad \lim_{a \to a_0} \frac{f(a) - f(a_0)}{a - a_0},$$

provided the limit exists (and is a real number). If the limit above exists, then we say that f is *differentiable* at a_0. We say that the function f is *differentiable* provided that f is differentiable at all points in its domain.

- **Remark:** Suppose that $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is a function and $a_0 \in A$ is an accumulation point of A, as in the definition of the derivative.
  - If f is differentiable at a_0, then we may think of $f'(a_0)$ as the "rate of change" at the point a_0 of the function f.

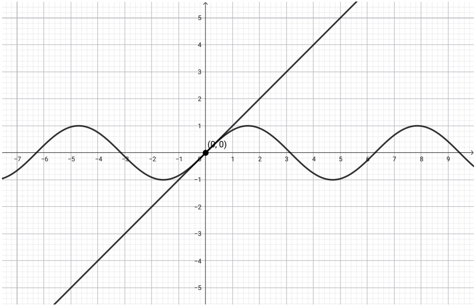
- Suppose $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is a function, differentiable at a point $a_0 \in A$ (in particular, a_0 is an accumulation point of A).
  - We define the *tangent line* (or simply *tangent*) to the graph of f though the point $\big(a_0, f(a_0)\big)$ to be the line though that point, and with slope $f'(a_0)$.



  - Note that the equation of this line (if it is exists) is

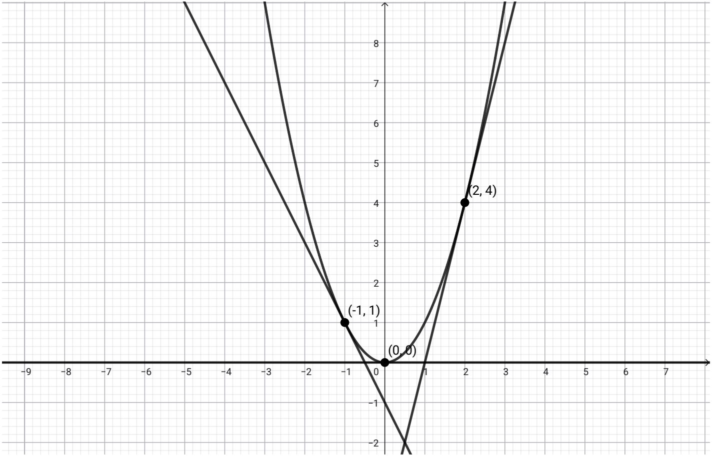
$$y = f'(a_0)(x - a_0) + f(a_0).$$

- **Remark:** Note that, according to our definition, the tangent to the graph of a function through a given point may possibly "pierce" the graph of the function (rather than touching it).
    - For instance, it can be shown that $\frac{d}{dx}(\sin x) = \cos x$ (proof: later!), and it readily follows that the equation of the tangent to $\sin x$ through the point $(0, \sin 0) = (0, 0)$ is $y = x$.
        - Indeed, the equation of the tangent to $\sin x$ through $(0, \sin 0)$ is $y = \sin'(0)(x - 0) + \sin 0$.
        - Since $\sin'(0) = \cos 0 = 1$ and $\sin 0 = 0$, it follows that the equation of our tangent is $y = x$, as we had claimed.



Example 4.2.3

Compute the equations of the tangents to the graph of the function $f(x) = x^2$ through each of the following points: $(0, 0)$, $(-1, 1)$, $(2, 4)$.



Example 4.2.3

Compute the equations of the tangents to the graph of the function $f(x) = x^2$ through each of the following points: $(0,0)$, $(-1,1)$, $(2,4)$.

Solution. By Example 4.2.2, we have that $f'(x) = 2x$ for all $x \in \mathbb{R}$. Now, by definition, for any point $a \in \mathbb{R}$, the equation of the tangent to the graph of f through the point $(a, f(a)) = (a, a^2)$ is $y = f'(a)(x-a) + f(a)$, that is, $y = 2a(x-a) + a^2$, i.e. $y = 2ax - a^2$. Therefore:

- the equation of the tangent to the graph of f through the point $(0,0)$ is $y = 0$ (i.e. the x-axis);
- the equation of the tangent to the graph of f through the point $(-1,1)$ is $y = -2x - 1$;
- the equation of the tangent to the graph of f through the point $(2,4)$ is $y = 4x - 4$. □

3. Differentiability and continuity

- Reminder:

Definition

A function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is *continuous* at a point $a \in A$ if the following holds:

> *for all $\varepsilon > 0$, there exists some $\delta > 0$, s.t. for all $x \in A$, if $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$.*

- **Remark:** Suppose that $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is a function, and $a \in A$, as in the definition above.
  - Note that if a is an accumulation point of A, then f is continuous at a iff $\lim_{x \to a} f(x) = f(a)$ (and in particular, $\lim_{x \to a} f(x)$ exists).
  - If a is **not** an accumulation point of A, then f is automatically continuous at a.

Theorem 4.3.1

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and let $a_0 \in A$ be an accumulation point of A. If f is differentiable at a_0, then f is continuous at a_0.

Proof. We assume that f is differentiable at a_0, so that

$$f'(a_0) = \lim_{a \to a_0} \frac{f(a) - f(a_0)}{a - a_0}$$

exists (as a real number). To prove that f is continuous at a_0, it suffices to show that $\lim_{a \to a_0} f(a) = f(a_0)$. For this, we compute:

$$\begin{aligned}
\lim_{a \to a_0} f(a) &= f(a_0) + \lim_{a \to a_0} (f(a) - f(a_0)) \\
&= f(a_0) + \lim_{a \to a_0} \left(\frac{f(a) - f(a_0)}{a - a_0} (a - a_0) \right) \\
&= f(a_0) + \left(\lim_{a \to a_0} \frac{f(a) - f(a_0)}{a - a_0} \right) \left(\lim_{a \to a_0} (a - a_0) \right) \\
&= f(a_0) + f'(a_0) \cdot 0 = f(a_0).
\end{aligned}$$

□

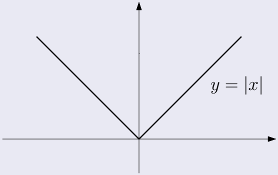
Theorem 4.3.1

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and let $a_0 \in A$ be an accumulation point of A. If f is differentiable at a_0, then f is continuous at a_0.

- **Remark:** The converse of Theorem 4.3.1 is false.
  - Indeed, as the example below shows, there are continuous functions that are not differentiable.

Example 4.3.2

Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x|$ for all $x \in \mathbb{R}$. Show that f is continuous at 0, but is not differentiable at 0.



Example 4.3.2

Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x|$ for all $x \in \mathbb{R}$. Show that f is continuous at 0, but is not differentiable at 0.

Solution. By definition, we have that

$$f(x) = \begin{cases} x & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

for all $x \in \mathbb{R}$. We then have that

- $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} x = 0$;
- $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} (-x) = 0$.

Since $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^-} f(x) = 0$, it follows that $\lim\limits_{x \to 0} f(x) = 0$, i.e. $\lim\limits_{x \to 0} f(x) = f(0)$. So, f is continuous at 0.

Example 4.3.2

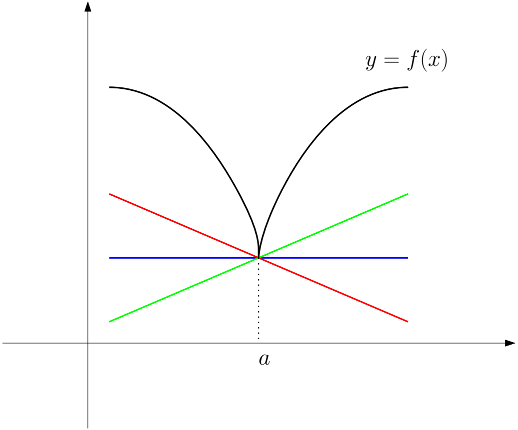
Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x|$ for all $x \in \mathbb{R}$. Show that f is continuous at 0, but is not differentiable at 0.

Solution (continued). It remains to show that f is **not** differentiable at 0. Note that:

- $\lim\limits_{x \to 0^+} \frac{f(x)-f(0)}{x-0} = \lim\limits_{x \to 0^+} \frac{x-0}{x-0} = 1$;
- $\lim\limits_{x \to 0^-} \frac{f(x)-f(0)}{x-0} = \lim\limits_{x \to 0^-} \frac{-x-0}{x-0} = -1$.

Thus, $\lim\limits_{x \to 0^+} \frac{f(x)-f(0)}{x-0} \neq \lim\limits_{x \to 0^-} \frac{f(x)-f(0)}{x-0}$. Consequently, $\lim\limits_{x \to 0} \frac{f(x)-f(0)}{x-0}$ does not exist, that is, $f'(0)$ does not exist. So, f is **not** differentiable at 0. $\square$

- **Remark:** Intuitively, a function that is continuous at a point a, but has a "sharp corner" at a, is not differentiable at a. Note that the presences of "sharp corners" prevent the existence of a tangent.



4. Sums and scalar multiples of differentiable functions

Proposition 4.3.3

Let $f, g : A \subseteq \mathbb{R} \to \mathbb{R}$ be functions, and let $a_0 \in A$ be an accumulation point of A. If f and g are both differentiable at a_0, then so is the function $f + g$, and moreover, we have that $(f + g)'(a_0) = f'(a_0) + g'(a_0)$.

Proposition 4.3.4

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, let $a_0 \in A$ be an accumulation point of A, and let $c \in \mathbb{R}$ be a constant. If f is differentiable at a_0, then so is the function cf, and moreover, we have that $(cf)'(a_0) = c(f'(a_0))$.

- We prove Proposition 4.3.3.
- The proof of Proposition 4.3.4 is similar (see the Lecture Notes).

Proposition 4.3.3

Let $f, g : A \subseteq \mathbb{R} \to \mathbb{R}$ be functions, and let $a_0 \in A$ be an accumulation point of A. If f and g are both differentiable at a_0, then so is the function $f + g$, and moreover, we have that $(f+g)'(a_0) = f'(a_0) + g'(a_0)$.

Proof. Assume that f and g are both differentiable at a_0. Then

$$\begin{aligned}
(f+g)'(a_0) &= \lim_{a \to a_0} \frac{(f+g)(a) - (f+g)(a_0)}{a - a_0} \\
&= \lim_{a \to a_0} \frac{f(a) + g(a) - f(a_0) - g(a_0)}{a - a_0} \\
&= \lim_{a \to a_0} \frac{\big(f(a) - f(a_0)\big) + \big(g(a) - g(a_0)\big)}{a - a_0} \\
&= \left(\lim_{a \to a_0} \frac{f(a) - f(a_0)}{a - a_0}\right) + \left(\lim_{a \to a_0} \frac{g(a) - g(a_0)}{a - a_0}\right) \\
&= f'(a_0) + g'(a_0).
\end{aligned}$$

In particular, $f + g$ is indeed differentiable at a_0. $\square$

5. The derivatives of some important functions

- A function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is *constant* if there exists some $c \in \mathbb{R}$ such that for all $x \in A$, we have that $f(x) = c$.

Proposition 4.4.1

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a constant function, and let $a_0 \in A$ be an accumulation point of A. Then $f'(a_0) = 0$.

Proof. We compute:

$$f'(a_0) \quad = \quad \lim_{a \to a_0} \frac{f(a) - f(a_0)}{a - a_0} \quad \overset{(*)}{=} \quad \lim_{a \to a_0} \frac{0}{a - a_0} \quad = \quad 0,$$

where (*) follows form the fact that f is constant, and so $f(a) = f(a_0)$ for all $a \in A$. □

Proposition 4.4.2

Let n be a positive integer. Then $\frac{d}{dx}(x^n) = nx^{n-1}$ for $x \in \mathbb{R}$.

Proof. For $x \in \mathbb{R}$, we compute:

$$\begin{aligned}
\tfrac{d}{dx}(x^n) &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h\to 0} \frac{\left(\sum_{i=0}^{n} \binom{n}{i} x^{n-i} h^i\right) - x^n}{h} \\
&= \lim_{h\to 0} \frac{\left(x^n h^0 + \sum_{i=1}^{n} \binom{n}{i} x^{n-i} h^i\right) - x^n}{h} = \lim_{h\to 0} \frac{\sum_{i=1}^{n} \binom{n}{i} x^{n-i} h^i}{h} \\
&= \lim_{h\to 0} \left(\sum_{i=1}^{n} \binom{n}{i} x^{n-i} h^{i-1} \right) = \sum_{i=1}^{n} \binom{n}{i} x^{n-i} 0^{i-1} \\
&\overset{(*)}{=} \binom{n}{1} x^{n-1} = nx^{n-1},
\end{aligned}$$

where (*) follows from the fact that $0^0 = 1$, whereas $0^k = 0$ for all $k \in \mathbb{N}$. $\square$

Proposition 4.4.1

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a constant function, and let $a_0 \in A$ be an accumulation point of A. Then $f'(a_0) = 0$.

Proposition 4.4.2

Let n be a positive integer. Then $\frac{d}{dx}(x^n) = nx^{n-1}$ for $x \in \mathbb{R}$.

Example 4.4.3

Consider the polynomial function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 4x^3 + 7x^2 - 5x + 3$ for all $x \in \mathbb{R}$. Then

$$\begin{aligned}
f'(x) &= \tfrac{d}{dx}\left(4x^3 + 7x^2 - 5x + 3\right) \\
&= 4\tfrac{d}{dx}(x^3) + 7\tfrac{d}{dx}(x^2) - 5\tfrac{d}{dx}(x) + \tfrac{d}{dx}(3) && \text{by Propositions 4.3.3 and 4.3.4} \\
&= 4(3x^2) + 7(2x) - 5(1) + (0) && \text{by Propositions 4.4.1 and 4.4.2} \\
&= 12x^2 + 14x - 5.
\end{aligned}$$

Proposition 4.4.2

Let n be a positive integer. Then $\frac{d}{dx}(x^n) = nx^{n-1}$ for $x \in \mathbb{R}$.

- **Remark:** Once we develop more theory, we will see that the formula from Proposition 4.4.2 generalizes to **real** exponents α (in place of the positive integer exponent n), though we may possibly require $x > 0$ to ensure that all terms are defined.
- More precisely, for a fixed real number α, we will get the formula

$$\frac{d}{dx}(x^\alpha) = \alpha x^{\alpha-1} \qquad \text{for } x \in (0, +\infty).$$

- We cannot yet derive this formula in its full generality.
- We do, however, have the following proposition for n-th roots (next slide).
  - Recall that for an integer $n \geq 2$, we have that $x^{1/n} = \sqrt[n]{x}$.

Proposition 4.4.4

Let $n \geq 2$ be an integer. Then:

(a) if n is odd, then $\frac{d}{dx}(x^{1/n}) = \frac{1}{n}x^{(1/n)-1} = \frac{1}{n}x^{-(n-1)/n}$ for $x \in \mathbb{R} \setminus \{0\}$;

(b) if n is even, then $\frac{d}{dx}(x^{1/n}) = \frac{1}{n}x^{(1/n)-1} = \frac{1}{n}x^{-(n-1)/n}$ for $x \in (0, +\infty)$.

- Proof: Lecture Notes.
- **Remark:** Under our assumption that $n \geq 2$, the expression $0^{-(n-1)/n}$ is undefined (regardless of the parity of n), an in fact, the function $x^{1/n}$ is **not** differentiable at 0.

- We now turn to the exponential function e^x and the logarithmic function $\ln x$.
- Recall that the sequence $\{(1+\frac{1}{n})^n\}_{n=1}^{\infty}$ converges, and we defined Euler's number e to be the limit of this sequence, that is,
$$e \;:=\; \lim_{n\to\infty}(1+\tfrac{1}{n})^n.$$
- It can be shown (but it is not simple, and we omit the details) that we also have the following **function** limit:
$$e \;=\; \lim_{x\to\infty}(1+\tfrac{1}{x})^x.$$
In the above, x may take real values (not just integer values).
- In what follows, we will use the above fact without proof.

Proposition 4.4.5

All the following hold:

(a) $\lim_{x \to 0}(1+x)^{1/x} = e$; (b) $\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$; (c) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$.

Proof. (a) First, we compute:

$$\lim_{x \to 0^+}(1+x)^{1/x} \overset{(*)}{=} \lim_{y \to +\infty}(1+\tfrac{1}{y})^y = e,$$

where for (*), we used the substitution $y = \frac{1}{x}$. Next, we compute:

$$\begin{aligned}
\lim_{x \to 0^-}(1+x)^{1/x} &\overset{(*)}{=} \lim_{y \to +\infty}(1-\tfrac{1}{y})^{-y} = \lim_{y \to +\infty}(\tfrac{y}{y-1})^y \\
&\overset{(**)}{=} \lim_{z \to +\infty}(\tfrac{z+1}{z})^{z+1} = \lim_{z \to +\infty}(1+\tfrac{1}{z})^{z+1} \\
&= \Big(\lim_{z \to +\infty}(1+\tfrac{1}{z})^z\Big)\Big(\lim_{z \to +\infty}(1+\tfrac{1}{z})\Big) = e \cdot 1 = e,
\end{aligned}$$

where for (*), we used the substitution $y = -\frac{1}{x}$, and for (**), we used the substitution $z = y - 1$. We have now shown that $\lim_{x \to 0^+}(1+x)^{1/x} = \lim_{x \to 0^-}(1+x)^{1/x} = e$, and (a) follows.

Proposition 4.4.5

All the following hold:

(a) $\lim_{x \to 0} (1+x)^{1/x} = e$; (b) $\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$; (c) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$.

Proof (continued). (b) We compute:

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = \lim_{x \to 0} \ln\left((1+x)^{1/x}\right) \overset{(*)}{=} \ln\left(\lim_{x \to 0}(1+x)^{1/x}\right) \overset{(a)}{=} \ln e = 1,$$

where (*) follows from the continuity of the logarithmic functions.

(c) We compute:

$$\lim_{x \to 0} \frac{e^x - 1}{x} \overset{(*)}{=} \lim_{y \to 0} \frac{y}{\ln(1+y)} = \lim_{y \to 0} \frac{1}{\frac{\ln(1+y)}{y}} = \frac{1}{\lim_{y \to 0} \frac{\ln(1+y)}{y}} \overset{(b)}{=} \frac{1}{1} = 1,$$

where (*) was obtained via substitution $y = e^x - 1$. □

Proposition 4.4.5

All the following hold:

(a) $\lim_{x \to 0} (1+x)^{1/x} = e$; (b) $\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$; (c) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$.

Proposition 4.4.6

$\frac{d}{dx}(e^x) = e^x$ for $x \in \mathbb{R}$.

Proof. For $x \in \mathbb{R}$, we compute:

$$\frac{d}{dx}(e^x) = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} \frac{e^x(e^h - 1)}{h} = e^x \Big(\lim_{h \to 0} \frac{e^h - 1}{h} \Big) \overset{(*)}{=} e^x,$$

where (*) follows from Proposition 4.4.5(c). □

Proposition 4.4.5

All the following hold:

(a) $\lim_{x\to 0}(1+x)^{1/x}=e$; (b) $\lim_{x\to 0}\frac{\ln(1+x)}{x}=1$; (c) $\lim_{x\to 0}\frac{e^x-1}{x}=1$.

Proposition 4.4.7

$\frac{d}{dx}(\ln x)=\frac{1}{x}$ for $x\in(0,+\infty)$.

Proof. For $x\in(0,+\infty)$, we compute:

$$\begin{aligned}\frac{d}{dx}(\ln x) &= \lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h} = \lim_{h\to 0}\frac{\ln\left(\frac{x+h}{x}\right)}{h} = \frac{1}{x}\lim_{h\to 0}\frac{\ln\left(1+\frac{h}{x}\right)}{\frac{h}{x}} \\ &\overset{(*)}{=} \frac{1}{x}\lim_{y\to 0}\frac{\ln(1+y)}{y} \overset{(**)}{=} \frac{1}{x}\cdot 1 = \frac{1}{x}.\end{aligned}$$

where (*) is obtained via substitution $y=\frac{h}{x}$ (with x treated as a constant and h as a variable inside the limit), whereas (**) follows from Proposition 4.4.5(b). □

- We now turn to the sine and cosine functions.
- Recall from trigonometry that $\sin^2 x + \cos^2 x = 1 \ \forall x \in \mathbb{R}$.
- Furthermore, we have the following addition formulas for sine and cosine:
  - $\sin(x + y) = \sin x \cos y + \sin y \cos x$,
  - $\cos(x + y) = \cos x \cos y - \sin x \sin y$,

  for all $x, y \in \mathbb{R}$.
- Reminder:

Theorem 3.7.2

$\lim_{x \to 0} \frac{\sin x}{x} = 1.$

Proposition 4.4.8

$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0.$

- Proof: Lecture Notes.

Proposition 4.4.9

$\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$ for $x \in \mathbb{R}$.

Proof. For $x \in \mathbb{R}$, we compute:

$$\begin{aligned}
\frac{d}{dx}(\sin x) &= \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x\cos h+\sin h\cos x-\sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin h\cos x-\sin x(1-\cos h)}{h} \\
&= \lim_{h\to 0} \left(\cos x\frac{\sin h}{h} - \sin x\frac{1-\cos h}{h}\right) \\
&= (\cos x)\left(\lim_{h\to 0} \frac{\sin h}{h}\right) - (\sin x)\left(\lim_{h\to 0} \frac{1-\cos h}{h}\right) \\
&\overset{(*)}{=} (\cos x)\cdot 1 - (\sin x)\cdot 0 \;=\; \cos x,
\end{aligned}$$

where (*) follows from Theorem 3.7.2 and Proposition 4.4.8.

Proposition 4.4.9

$\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$ for $x \in \mathbb{R}$.

Proof (continued). For $x \in \mathbb{R}$, we compute:

$$\begin{aligned}
\frac{d}{dx}(\cos x) &= \lim_{h\to 0} \frac{\cos(x+h)-\cos x}{h} = \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\
&= -\lim_{h\to 0} \frac{-\cos x \cos h + \sin x \sin h + \cos x}{h} \\
&= -\lim_{h\to 0} \frac{\sin x \sin h + \cos x(1-\cos h)}{h} \\
&= -\lim_{h\to 0} \left(\sin x \frac{\sin h}{h} + \cos x \frac{1-\cos h}{h} \right) \\
&= -(\sin x)\left(\lim_{h\to 0} \frac{\sin h}{h} \right) - (\cos x)\left(\lim_{h\to 0} \frac{1-\cos h}{h} \right) \\
&\overset{(*)}{=} -(\sin x)\cdot 1 - (\cos x)\cdot 0 = -\sin x,
\end{aligned}$$

where (*) follows from Theorem 3.7.2 and Proposition 4.4.8. □

- Summary:

Proposition 4.4.1

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a constant function, and let $a_0 \in A$ be an accumulation point of A. Then $f'(a_0) = 0$.

Proposition 4.4.2

Let n be a positive integer. Then $\frac{d}{dx}(x^n) = nx^{n-1}$ for $x \in \mathbb{R}$.

Proposition 4.4.6

$\frac{d}{dx}(e^x) = e^x$ for $x \in \mathbb{R}$.

Proposition 4.4.7

$\frac{d}{dx}(\ln x) = \frac{1}{x}$ for $x \in (0, +\infty)$.

Proposition 4.4.9

$\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$ for $x \in \mathbb{R}$.

6. The Product and Quotient Rules for derivatives

- We will prove the so called "Product Rule" and "Quotient Rule" for derivatives.
  - First, we state them both.
  - Then, we show some applications of the two rules.
  - Finally, we prove them both.

The Product Rule

Let $f, g : A \subseteq \mathbb{R} \to \mathbb{R}$ be functions, let $a_0 \in A$ be an accumulation point of A, and assume that both f and g are differentiable at a_0. Then fg is also differentiable at a_0, and moreover, we have the following formula: $(fg)'(a_0) = f'(a_0)g(a_0) + f(a_0)g'(a_0)$.

- **Remark:** If $f, g : A \subseteq \mathbb{R} \to \mathbb{R}$ are differentiable functions (i.e. if they are differentiable at all points in their domain), then the Product Rule states that fg is also differentiable, and it gives us the formula $(fg)' = f'g + fg'$.

Example 4.5.1

Consider the function $f : (0, +\infty) \to \mathbb{R}$ given by $f(x) = x^5 \ln x$ for all $x \in (0, +\infty)$. Then, for $x \in \mathbb{R}$, we compute:

$$f'(x) = \frac{d}{dx}(x^5 \ln x) \overset{(*)}{=} (5x^4)\ln x + x^5 \frac{1}{x} = 5x^4 \ln x + x^4,$$

where (*) was obtained via the Product Rule.

The Quotient Rule

Let $f, g : A \subseteq \mathbb{R} \to \mathbb{R}$ be functions, and let $a_0 \in A$ be an accumulation point of A. Assume that both f and g are differentiable at a_0 and that $g(a_0) \neq 0$. Then $\frac{f}{g}$ is also differentiable at a_0, and moreover, we have the following formula:

$$\left(\frac{f}{g}\right)'(a_0) = \frac{f'(a_0)g(a_0) - f(a_0)g'(a_0)}{(g(a_0))^2}.$$

- Using the Quotient Rule, we can prove the following proposition.

Proposition 4.5.2

$\frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x}$ for $x \in \mathbb{R} \setminus \{\frac{2k+1}{2}\pi \mid k \in \mathbb{Z}\}$.

Proposition 4.5.2

$\frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x}$ for $x \in \mathbb{R} \setminus \{\frac{2k+1}{2}\pi \mid k \in \mathbb{Z}\}$.

Proof. First, recall that by Proposition 4.4.9, we have that

$$\frac{d}{dx}(\sin x) = \cos x \qquad \text{and} \qquad \frac{d}{dx}(\cos x) = -\sin x$$

for $x \in \mathbb{R}$. Now, for $x \in \mathbb{R} \setminus \{\frac{2k+1}{2}\pi \mid k \in \mathbb{Z}\}$, we compute:

$$\begin{aligned}
\frac{d}{dx}(\tan x) &= \left(\frac{\sin x}{\cos x}\right)' \\
&= \frac{(\sin x)' \cdot (\cos x) - (\sin x) \cdot (\cos x)'}{\cos^2 x} && \text{by the Quotient Rule} \\
&= \frac{(\cos x) \cdot (\cos x) - (\sin x) \cdot (-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x}.
\end{aligned}$$

□

- Let's now prove the Product Rule and the Quotient Rule!

The Product Rule

Let $f, g : A \subseteq \mathbb{R} \to \mathbb{R}$ be functions, let $a_0 \in A$ be an accumulation point of A, and assume that both f and g are differentiable at a_0. Then fg is also differentiable at a_0, and moreover, we have the following formula: $(fg)'(a_0) = f'(a_0)g(a_0) + f(a_0)g'(a_0)$.

Proof. First of all, since f and g are differentiable at a_0, Theorem 4.3.1 implies that they are continuous at a_0, and consequently,

$$\lim_{a \to a_0} f(a) = f(a_0) \qquad \text{and} \qquad \lim_{a \to a_0} g(a) = g(a_0).$$

Proof (continued). We now compute:

$$
\begin{aligned}
(fg)'(a_0) &= \lim_{a\to a_0} \frac{(fg)(a)-(fg)(a_0)}{a-a_0} \\
&= \lim_{a\to a_0} \frac{f(a)g(a)-f(a_0)g(a_0)}{a-a_0} \\
&= \lim_{a\to a_0} \frac{f(a)g(a)-f(a_0)g(a)+f(a_0)g(a)-f(a_0)g(a_0)}{a-a_0} \\
&= \lim_{a\to a_0} \frac{\big(f(a)-f(a_0)\big)g(a)+f(a_0)\big(g(a)-g(a_0)\big)}{a-a_0} \\
&= \lim_{a\to a_0} \left(\frac{f(a)-f(a_0)}{a-a_0}g(a)\right) + \lim_{a\to a_0} \left(f(a_0)\frac{g(a)-g(a_0)}{a-a_0}\right) \\
&= \left(\lim_{a\to a_0} \frac{f(a)-f(a_0)}{a-a_0}\right)\left(\lim_{a\to a_0} g(a)\right) + f(a_0) \lim_{a\to a_0} \left(\frac{g(a)-g(a_0)}{a-a_0}\right) \\
&= f'(a_0)\left(\lim_{a\to a_0} g(a)\right) + f(a_0)g'(a_0) \\
&\overset{(*)}{=} f'(a_0)g(a_0) + f(a_0)g'(a_0),
\end{aligned}
$$

where (*) follows from the continuity of g at a_0. □

The Quotient Rule

Let $f, g : A \subseteq \mathbb{R} \to \mathbb{R}$ be functions, and let $a_0 \in A$ be an accumulation point of A. Assume that both f and g are differentiable at a_0 and that $g(a_0) \neq 0$. Then $\frac{f}{g}$ is also differentiable at a_0, and moreover, we have the following formula:

$$\left(\frac{f}{g}\right)'(a_0) \quad = \quad \frac{f'(a_0)g(a_0) - f(a_0)g'(a_0)}{(g(a_0))^2}.$$

Proof. We begin with a technical claim.

> ***Claim.*** *The function $\frac{1}{g}$ is differentiable at a_0, and moreover, we have the following formula:* $\left(\frac{1}{g}\right)'(a_0) = -\frac{g'(a_0)}{(g(a_0))^2}$.

Proof of the Claim. First, since g is differentiable at a_0, Theorem 4.3.1 guarantees that g is also continuous at a_0, and consequently, $\lim\limits_{a \to a_0} g(a) = g(a_0)$.

Claim. *The function $\frac{1}{g}$ is differentiable at a_0, and moreover, we have the following formula:* $\left(\frac{1}{g}\right)'(a_0) = -\frac{g'(a_0)}{(g(a_0))^2}$.

Proof of the Claim (continued). We now compute:

$$\begin{aligned}
\left(\frac{1}{g}\right)'(a_0) &= \lim_{a\to a_0} \frac{\frac{1}{g(a)} - \frac{1}{g(a_0)}}{a - a_0} \\
&= \lim_{a\to a_0} \frac{g(a_0) - g(a)}{\big(g(a)g(a_0)\big)(a - a_0)} \\
&= \Big(\lim_{a\to a_0} \frac{1}{g(a)g(a_0)}\Big)\Big(\lim_{a\to a_0} \frac{g(a_0) - g(a)}{a - a_0}\Big) \\
&= \frac{1}{(\lim_{a\to a_0} g(a))g(a_0)}\Big(-\lim_{a\to a_0} \frac{g(a) - g(a_0)}{a - a_0}\Big) \\
&= \frac{1}{(g(a_0))^2}(-g'(a_0)) \;=\; -\frac{g'(a_0)}{(g(a_0))^2}.
\end{aligned}$$

This proves the Claim. ♦

The Quotient Rule (abbreviated)

$$\left(\frac{f}{g}\right)'(a_0) \quad = \quad \frac{f'(a_0)g(a_0)-f(a_0)g'(a_0)}{(g(a_0))^2}.$$

Proof (continued).

Claim. *The function $\frac{1}{g}$ is differentiable at a_0, and moreover, we have the following formula:* $\left(\frac{1}{g}\right)'(a_0) = -\frac{g'(a_0)}{(g(a_0))^2}$.

The result now readily follows from the Product Rule and the above Claim, applied to the functions f and $\frac{1}{g}$. Indeed, we compute:

$$\begin{aligned}
\left(\frac{f}{g}\right)'(a_0) &= (f \cdot \tfrac{1}{g})'(a_0) \\
&= f'(a_0)\big((\tfrac{1}{g})(a_0)\big) + f(a_0)\big((\tfrac{1}{g})'(a_0)\big) && \text{by the Product Rule} \\
&= \frac{f'(a_0)}{g(a_0)} + f(a_0)\big(-\frac{g'(a_0)}{(g(a_0))^2}\big) && \text{by the Claim} \\
&= \frac{f'(a_0)g(a_0)-f(a_0)g'(a_0)}{(g(a_0))^2}.
\end{aligned}$$

□

- We have proven:

The Product Rule

Let $f, g : A \subseteq \mathbb{R} \to \mathbb{R}$ be functions, let $a_0 \in A$ be an accumulation point of A, and assume that both f and g are differentiable at a_0. Then fg is also differentiable at a_0, and moreover, we have the following formula: $(fg)'(a_0) = f'(a_0)g(a_0) + f(a_0)g'(a_0)$.

The Quotient Rule

Let $f, g : A \subseteq \mathbb{R} \to \mathbb{R}$ be functions, and let $a_0 \in A$ be an accumulation point of A. Assume that both f and g are differentiable at a_0 and that $g(a_0) \neq 0$. Then $\frac{f}{g}$ is also differentiable at a_0, and moreover, we have the following formula:

$$\left(\frac{f}{g}\right)'(a_0) = \frac{f'(a_0)g(a_0) - f(a_0)g'(a_0)}{(g(a_0))^2}.$$