Mathematical Analysis 1

Lecture #8

Limits of functions (part III)

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- This lecture has nine parts:
  1. A brief review of limits and continuity
  2. The continuity of inverse functions
  3. Infinite limits
  4. One-sided limits and vertical asymptotes
  5. Limits at infinity and horizontal asymptotes
  6. Slant asymptotes
  7. The Squeeze Theorem for functions
  8. The limit $\lim_{x \to 0} \frac{\sin x}{x}$
  9. Analogues of the Squeeze Theorem for infinite limits

1. A brief review of limits and continuity

Definition

An *accumulation point* of a set $A \subseteq \mathbb{R}$ is a point $a \in \mathbb{R}$ (note that a may or may not belong to A) s.t. for all real numbers $\varepsilon > 0$, there exists some $a' \in A$ s.t. $0 < |a' - a| < \varepsilon$.[a]

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[a] So, a is an accumulation point of A if and only one can find a points of A arbitrarily close to a (but distinct from a).

Definition

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, let $a \in \mathbb{R}$ be an accumulation point of A, and let $L \in \mathbb{R}$. We say that L *is the limit of* $f(x)$ *as* x *approaches* a, or that $f(x)$ *tends to* L *as* x *approaches* a, provided that the following holds:

> *for every* $\varepsilon > 0$*, there exists some* $\delta > 0$*, s.t. for all* $x \in A$*,*
> *if* $0 < |x - a| < \delta$*, then* $|f(x) - L| < \varepsilon$*.*

Under such circumstances, we write

$$L = \lim_{x \to a} f(x) \qquad \text{or} \qquad f(x) \to L \quad \text{as} \quad x \to a.$$

Definition

A function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is *continuous* at a point $a \in A$ if the following holds:

> *for all* $\varepsilon > 0$*, there exists some* $\delta > 0$*, s.t. for all* $x \in A$*, if* $|x - a| < \delta$*, then* $|f(x) - f(a)| < \varepsilon$*.*

- **Remark:** Suppose that $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is a function, and $a \in A$, as in the definition above.
    - Note that if a is an accumulation point of A, then f is continuous at a if and only if $\lim_{x \to a} f(x) = f(a)$ (and in particular, $\lim_{x \to a} f(x)$ exists).
    - If a is **not** an accumulation point of A, then f is automatically continuous at a.

Definition

If a function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is continuous at all points $a \in A$, then we simply say that f is *continuous*. If $I \subseteq A$ is an interval, then we say that f is *continuous* on I provided that $f \restriction I$ is continuous at all points in I.

- **Remark:** The intuition behind the concept of continuity is that a function is supposed to be continuous on an interval if its graph (restricted to the interval in question) can be drawn on a piece of paper without lifting the pen.

The Intermediate Value Theorem

Let a and b be real numbers s.t. $a < b$, let $f : [a, b] \to \mathbb{R}$ be a continuous function, and let $M \in \mathbb{R}$ be s.t. $\min\{f(a), f(b)\} < M < \max\{f(a), f(b)\}$.[a] Then $\exists c \in (a, b)$ s.t. $f(c) = M$.

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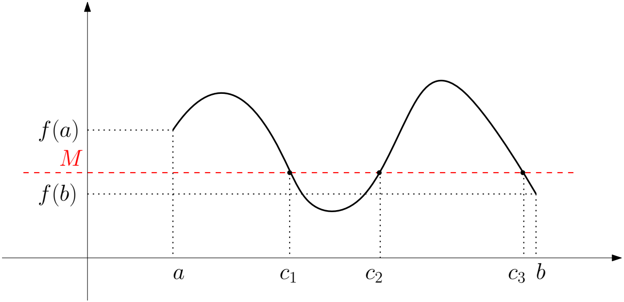
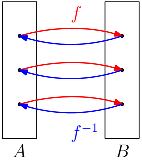
[a] So, we are assuming that $f(a) \neq f(b)$, and that either $f(a) < M < f(b)$ or $f(b) < M < f(a)$.



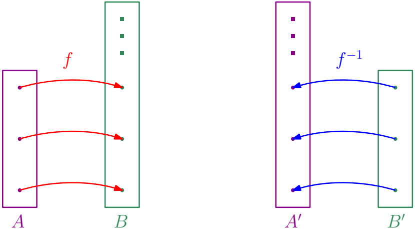
2. The continuity of inverse functions

- We know that if $f : A \to B$ is a bijection, then it has an *inverse function* $f^{-1} : B \to A$ that satisfies the following: for all $b \in B$, we have that $f^{-1}(b) = a$, where a is the unique member of A s.t. $f(a) = b$.

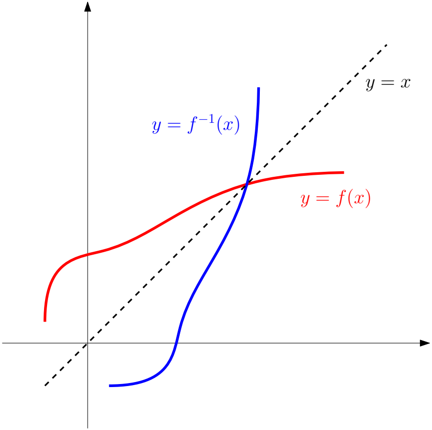


  - Let us try to generalize this.

- Suppose that $f : A \to B$ is a one-to-one function, and let $B' := \mathrm{Im}(f) = B' := \{f(x) \mid x \in A\}$.
    - If we restrict the codomain of f to B', then $f : A \to B'$ becomes a bijection, and it therefore has an inverse $f^{-1} : B' \to A$.
    - Sometimes, it is convenient to then extend the codomain of f^{-1} to a set $A' \supseteq A$, so that we get $f^{-1} : B' \to A'$.



- Note that if $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is a one-to-one function, and $B := \mathrm{Im}(f)$, then the graph of $f^{-1} : B \to \mathbb{R}$ is obtained by reflecting the graph of f about the $y = x$ line, as in the picture below.



- If a function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is one-to-one and continuous, its inverse need not be continuous (we will see an example below).
    - However, if the domain A happens to be an interval (open, closed, or half-open), then f^{-1} will, in fact, be continuous.
    - We state the relevant results, but we omit the proofs.
        - The proofs an be found in the Lecture Notes.
        - The Intermediate Value Theorem plays an important role.

Definition

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function.

- We say that f is *non-decreasing* if for all $a, a' \in A$ s.t. $a \leq a'$, we have that $f(a) \leq f(a')$.
- We say that f is *strictly increasing* (or simply *increasing*) if for all $a, a' \in A$ s.t. $a < a'$, we have that $f(a) < f(a')$.
- We say that f is *non-increasing* if for all $a, a' \in A$ s.t. $a \leq a'$, we have that $f(a) \geq f(a')$.
- We say that f is *strictly decreasing* (or simply *decreasing*) if for all $a, a' \in A$ s.t. $a < a'$, we have that $f(a) > f(a')$.
- We say that f is *monotone* if it is non-decreasing or non-increasing.

For $A' \subseteq A$, we say that f is *non-decreasing* (resp. *increasing*, *non-increasing*, *decreasing*, *monotone*) on A' if the restriction $f \restriction A'$ is non-decreasing (resp. increasing, non-increasing, decreasing, monotone).

Proposition 3.5.9

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous one-to-one function. Then the following hold:

- (a) if $f(a) < f(b)$, then f is strictly increasing, and $\text{Im}(f) = [f(a), f(b)]$;
- (b) if $f(a) > f(b)$, then f is strictly decreasing, and $\text{Im}(f) = [f(b), f(a)]$.

Proposition 3.5.10

Let I be an interval (open, closed, or half-open) in $\mathbb{R}$, and let $f : I \to \mathbb{R}$ be a continuous one-to-one function. Then f is either strictly increasing or strictly decreasing.

Proposition 3.5.11

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a one-to-one continuous function, where I is an interval. The $\text{Im}(f)$ is an interval of the same kind (open, closed, or half-open) as the interval I.

Theorem 3.5.12

Let I be an open interval, and let $f : I \to \mathbb{R}$ be a continuous one-to-one function. Then f^{-1} is also continuous.

Corollary 3.5.13

Let I be an interval (open, closed, or half-open) in $\mathbb{R}$, and let $f : I \to \mathbb{R}$ be a continuous one-to-one function. Then f^{-1} is also continuous.

- **Remark:** The formal proofs are given in the Lecture Notes, but here's the intuition.
    - Let us suppose that $f : I \subseteq \mathbb{R}$ is a continuous one-to-one function.
    - Now, our intuition is that the the graph of the continuous function f on the interval I can be drawn without lifting a pen.
    - So, the graph obtained by reflecting about the $y = x$ can also be drawn without lifting a pen, and this is precisely the graph of f^{-1}.
    - So, intuitively, f^{-1} should be continuous (and the formal proof is given below).

Theorem 3.5.12

Let I be an open interval, and let $f : I \to \mathbb{R}$ be a continuous one-to-one function. Then f^{-1} is also continuous.

Corollary 3.5.13

Let I be an interval (open, closed, or half-open) in $\mathbb{R}$, and let $f : I \to \mathbb{R}$ be a continuous one-to-one function. Then f^{-1} is also continuous.

- If $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is a one-to-one continuous function, but the domain A of f is **not** an interval, then f^{-1} need not be continuous.
    - Example: next slide!

Example 3.5.14

Consider the function $f : [0,1) \cup (1,2)$ given by

$$f(x) \;:=\; \begin{cases} x & \text{if} \quad x \in [0,1) \\ x-2 & \text{if} \quad x \in (1,2) \end{cases}$$

for all $x \in [0,1) \cup (1,2)$. Clearly, f is continuous and one-to-one.[a] But note that $\mathrm{Im}(f) = (-1,1)$, and that $f^{-1} : (-1,1) \to \mathbb{R}$ is given by

$$f^{-1}(x) \;=\; \begin{cases} x & \text{if} \quad x \in [0,1) \\ x+2 & \text{if} \quad x \in (-1,0) \end{cases}$$

for all $x \in (-1,1)$. Then f^{-1} is not continuous (because it is not continuous at 0). The graphs of f and f^{-1} are represented below.

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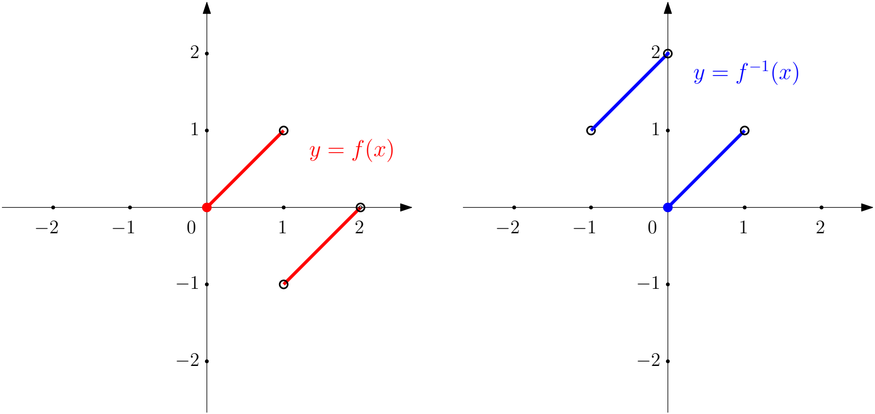
[a] The function f is continuous because it is continuous at all points in its domain. The non-existence of $\lim\limits_{x\to 1} f(x)$ is no obstacle to the continuity of f because 1 is not in the domain of the function f.


$y = f(x)$
$y = f^{-1}(x)$
−2
−1
0
1
2
−1
−2

3. Infinite limits

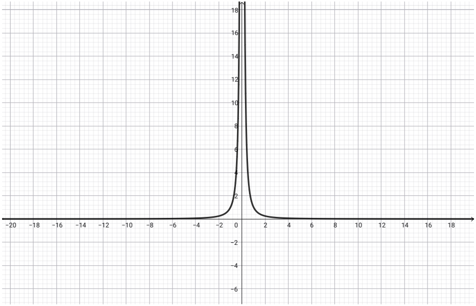
Definition

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and let $a \in \mathbb{R}$ be an accumulation point of A.

- We write
$$\lim_{x \to a} f(x) = +\infty$$
if the following holds:
for all $M \in \mathbb{R}$, *there exists some* $\delta > 0$, *s.t. for all* $x \in A$,
if $0 < |x - a| < \delta$, *then* $f(x) > M$.

- We write
$$\lim_{x \to a} f(x) = -\infty$$
if the following holds:
for all $M \in \mathbb{R}$, *there exists some* $\delta > 0$, *s.t. for all* $x \in A$,
if $0 < |x - a| < \delta$, *then* $f(x) < M$.

- For example, we have that $\lim\limits_{x \to 0} \frac{1}{x^2} = +\infty$ (the graph of this function is shown below).



4. One-sided limits and vertical asymptotes

- Sometimes, it is useful to consider “one-sided limits.”
  - Intuitively, we write $\lim_{x\to a^+} f(x) = L$ if, whenever x is very slightly greater than a, we have that $f(x) \approx L$.
  - Similarly, we have that $\lim_{x\to a^-} f(x) = L$ if, whenever x is very slightly smaller than a, we have that $f(x) \approx L$.
  - Let us now formalize this.
- **Remark:** Note that for any set $A \subseteq \mathbb{R}$, a point $a \in \mathbb{R}$ is an accumulation point of A if and only if it is an accumulation point of at least one of $A \cap (-\infty, a)$ and $A \cap (a, +\infty)$.

Definition [left-hand limits]

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and let $a \in \mathbb{R}$ be an accumulation point of $A \cap (-\infty, a)$.[a]

- For $L \in \mathbb{R}$, we write $\lim\limits_{x \to a^-} f(x) = L$ if the following holds:

  for every $\varepsilon > 0$, there exists some $\delta > 0$, s.t. for all $x \in A$, if $a - \delta < x < a$, then $|f(x) - L| < \varepsilon$.

- We write $\lim\limits_{x \to a^-} f(x) = +\infty$ if the following holds:

  for all $M \in \mathbb{R}$, there exists some $\delta > 0$ s.t. for all $x \in A$, if $a - \delta < x < a$, then $f(x) > M$.

- We write $\lim\limits_{x \to a^-} f(x) = -\infty$ if the following holds:

  for all $M \in \mathbb{R}$, there exists some $\delta > 0$ s.t. for all $x \in A$, if $a - \delta < x < a$, then $f(x) < M$.

---

[a] So, we are assuming that there are points of A to the left of a, and arbitrarily close to a. The typical situation is that the domain of f includes some open interval of the form $(a - \delta, a)$ for some small $\delta > 0$.

Definition [right-hand limits]

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and let $a \in \mathbb{R}$ be an accumulation point of $A \cap (a, +\infty)$.[a]

- For $L \in \mathbb{R}$, we write $\lim\limits_{x \to a^+} f(x) = L$ if the following holds:

  for every $\varepsilon > 0$, there exists some $\delta > 0$, s.t. for all $x \in A$, if $a < x < a + \delta$, then $|f(x) - L| < \varepsilon$.

- We write $\lim\limits_{x \to a^+} f(x) = +\infty$ if the following holds:

  for all $M \in \mathbb{R}$, there exists some $\delta > 0$ s.t. for all $x \in A$, if $a < x < a + \delta$, then $f(x) > M$.

- We write $\lim\limits_{x \to a^+} f(x) = -\infty$ if the following holds:

  for all $M \in \mathbb{R}$, there exists some $\delta > 0$ s.t. for all $x \in A$, if $a < x < a + \delta$, then $f(x) < M$.

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[a] So, we are assuming that there are points of A to the right of a, and arbitrarily close to a. The typical situation is that the domain of f includes some open interval of the form $(a, a + \delta)$ for some small $\delta > 0$.

Lemma 3.6.1

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and let $a \in \mathbb{R}$ be an accumulation point of A. Then a is an accumulation point of at least one of $A \cap (-\infty, a)$ and $A \cap (a, +\infty)$. Moreover:

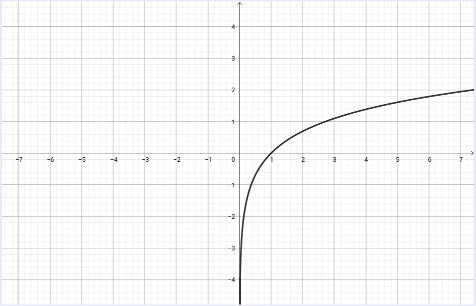
- (a) if $\lim\limits_{x \to a} f(x)$ exists (as a real number, or $+\infty$, or $-\infty$) and a is an accumulation point of $A \cap (-\infty, a)$, then $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a} f(x)$;
- (b) if $\lim\limits_{x \to a} f(x)$ exists (as a real number, or $+\infty$, or $-\infty$) and a is an accumulation point of $A \cap (a, +\infty)$, then $\lim\limits_{x \to a^+} f(x) = \lim\limits_{x \to a} f(x)$.

Proof. Exercise (follows from the appropriate definitions). ☐

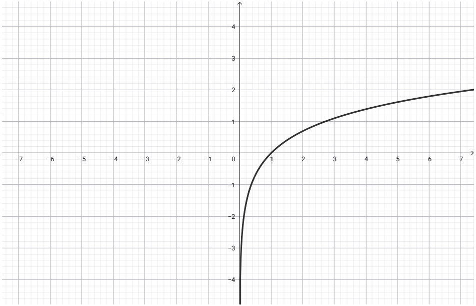
- **Remark:** Note that Lemma 3.6.1 in particular implies that if $\lim\limits_{x \to a} f(x)$ exists, then at least one of $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$ exists.

Example 3.6.2

The domain of the function $\ln x$ (whose graph is represented below) is $A := (0, +\infty)$. Clearly, 0 is an accumulation point of A. Moreover, 0 is an accumulation point of $A \cap (0, +\infty) = A$, but not of $A \cap (-\infty, 0) = \emptyset$. We have that $\lim_{x\to 0} \ln x = \lim_{x\to 0^+} \ln x = -\infty$, whereas $\lim_{x\to 0^-} \ln x$ does not exist. Finally, $x = 0$ is a vertical asymptote of $\ln x$.



- **Remark:** In Example 3.6.2, a crucial point is that 0 is a not an accumulation point of $A \cap (-\infty, 0)$, and that is the reason why $\lim_{x \to 0^-} \ln x$ does not exist, and yet this is no obstacle to the existence of $\lim_{x \to 0} \ln x$.



- **Remark:** However, if $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is a function and $a \in \mathbb{R}$, then:
  - if a is an accumulation point of $A \cap (-\infty, a)$ and yet $\lim\limits_{x \to a^-} f(x)$ does not exist, then $\lim\limits_{x \to a} f(x)$ does not exist either;
  - if a is an accumulation point of $A \cap (a, +\infty)$ and yet $\lim\limits_{x \to a^+} f(x)$ does not exist, then $\lim\limits_{x \to a} f(x)$ does not exist either.

Lemma 3.6.3

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$, let $a \in \mathbb{R}$ be an accumulation point both of $A \cap (-\infty, a)$ and of $A \cap (a, +\infty)$.[a] Then the following are equivalent:

1. (i) $\lim_{x \to a} f(x)$ exists (as a real number, or $+\infty$, or $-\infty$);
2. (ii) $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist (as real numbers, or $+\infty$, or $-\infty$) and are equal.

Moreover, in this case, we have that

$$\lim_{x \to a} f(x) = \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x).$$

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[a] The typical situation is that for some $\delta > 0$, we have that $(a - \delta, a) \cup (a, a + \delta) \subseteq A$, i.e. f is defined on some open interval containing a, except possibly at a itself.

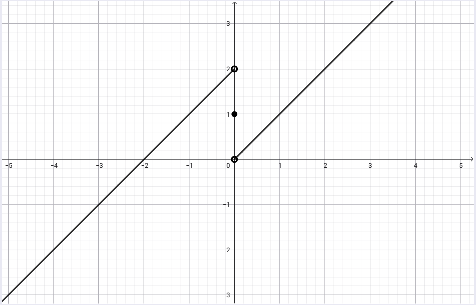
Proof. Exercise (follows from the appropriate definitions). $\square$

Example 3.6.4

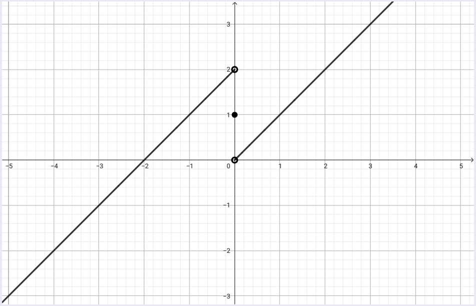
Consider the function $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x+2 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x = 0 \\ x & \text{if} \quad x > 0 \end{cases}$$

for all $x \in \mathbb{R}$. (The graph of this function is shown below.)



Example 3.6.4 (continued)



Then

- $\lim_{x \to 0^-} f(x) = 2$;
- $\lim_{x \to 0^+} f(x) = 0$;
- $\lim_{x \to 0} f(x)$ does not exist (because $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$).

Example 3.6.5

Consider the function $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x^2 & \text{if} \quad x \neq \pi \\ 100 & \text{if} \quad x = \pi \end{cases}$$

for all $x \in \mathbb{R}$. Then

- $\displaystyle\lim_{x \to \pi^-} f(x) = \pi^2$;
- $\displaystyle\lim_{x \to \pi^+} f(x) = \pi^2$;
- $\displaystyle\lim_{x \to \pi} f(x) = \pi^2$.

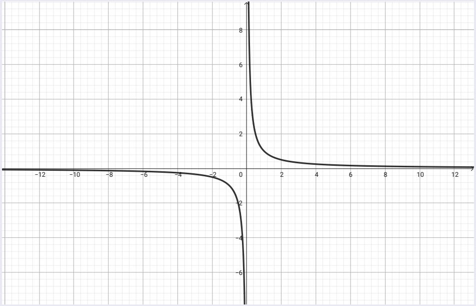
Definition

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function and let $a \in \mathbb{R}$. We say that $x = a$ is a *vertical asymptote* of f if at least one of $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$ exists and is equal to $+\infty$ or $-\infty$.

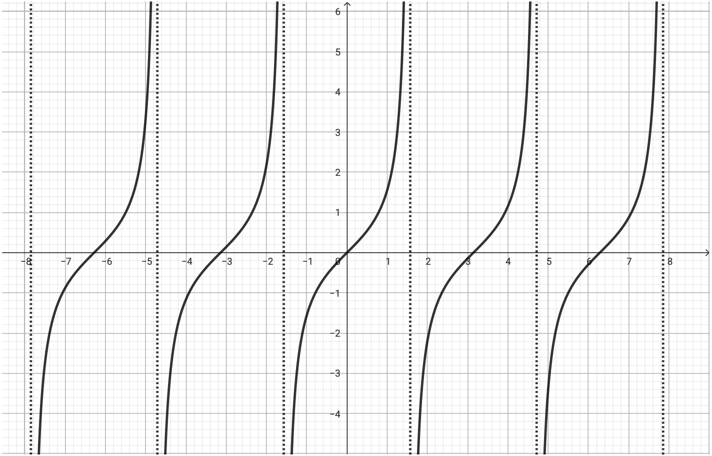
Example 3.6.6

Consider the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$ for all $x \in \mathbb{R} \setminus \{0\}$. (The graph of this function is shown below.) Then

- $\lim\limits_{x \to 0^-} f(x) = -\infty$ and $\lim\limits_{x \to 0^+} f(x) = +\infty$;
- $\lim\limits_{x \to 0} f(x)$ does not exist (because $\lim\limits_{x \to 0^-} f(x) \neq \lim\limits_{x \to 0^+} f(x)$);
- $x = 0$ is a vertical asymptote of $f(x)$.



- **Remark:** The function tan x is represented in the graph below. The dotted vertical lines are the vertical asymptotes, and they are all of the form $x = \frac{2k+1}{2}\pi$ for $k \in \mathbb{Z}$.



Example 3.6.7

For any integer k, the following hold:

- $\lim_{x \to \left(\frac{2k+1}{2}\pi\right)^-} \tan x = +\infty;$
- $\lim_{x \to \left(\frac{2k+1}{2}\pi\right)^+} \tan x = -\infty;$
- $\lim_{x \to \frac{2k+1}{2}\pi} \tan x$ does not exist;
- $x = \frac{2k+1}{2}\pi$ is a vertical asymptote of $\tan x$.

Example 3.6.8

Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) \; := \; \begin{cases} x^{-2} & \text{if} \quad x < 0 \\ \cos x & \text{if} \quad x \geq 0 \end{cases}$$

for all $x \in \mathbb{R}$. (The graph of this function is shown below.)

Example 3.6.8 (continued)

Then

- $\lim_{x \to 0^-} f(x) = +\infty$;
- $\lim_{x \to 0^+} f(x) = 1$;
- $\lim_{x \to 0} f(x)$ does not exist (because $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$);
- $x = 0$ is a vertical asymptote of $f(x)$.

Example 3.6.9

Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) \quad := \quad \begin{cases} x & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x \geq 0 \text{ and } x \in \mathbb{Q} \\ 1 & \text{if} \quad x \geq 0 \text{ and } x \notin \mathbb{Q} \end{cases}$$

for all $x \in \mathbb{R}$. Then

- $\lim\limits_{x \to 0^-} f(x) = 0$;
- $\lim\limits_{x \to 0^+} f(x)$ does not exist;
- $\lim\limits_{x \to 0} f(x)$ does not exist (because $\lim\limits_{x \to 0^+} f(x)$ does not exist, even though 0 is an accumulation point of $\mathbb{R} \cap (0, +\infty)$, where $\mathbb{R}$ is the domain of our function f).

Definition

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and let $a \in A$.

- We
  say that f is *continuous at a from the left* if the following holds:
  for all $\varepsilon > 0$*, there exists some* $\delta > 0$*, s.t. for all* $x \in A$*, if* $a - \delta < x \leq a$*, then* $|f(x) - f(a)| < \varepsilon$*.*

- We say that f is *continuous at a from the right* if the following holds:
  for all $\varepsilon > 0$*, there exists some* $\delta > 0$*, s.t. for all* $x \in A$*, if* $a \leq x < a + \delta$*, then* $|f(x) - f(a)| < \varepsilon$*.*

- **Remark:** Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and let $a \in A$, as in the definition above. Then the following hold:
    - if a is an accumulation point of $A \cap (-\infty, a)$, then f is continuous at a from the left if and only if $\lim\limits_{x \to a^-} f(x) = f(a)$;
    - if a is an accumulation point of $A \cap (a, +\infty)$, then f is continuous at a from the right if and only if $\lim\limits_{x \to a^+} f(x) = f(a)$;
    - if a is an accumulation point both of $A \cap (-\infty, a)$ and of $A \cap (a, +\infty)$, then f is continuous at a if and only if it is continuous at a both from the left and from the right.

Example 3.6.10

Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) := \begin{cases} -x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ x + 7 & \text{if } x > 0 \end{cases}$$

for all $x \in \mathbb{R}$. Then

- $\lim_{x \to 0^-} f(x) = 0 = f(0)$, and so f is continuous at 0 from the left;
- $\lim_{x \to 0^+} f(x) = 7 \neq f(0)$, and so f is **not** continuous at 0 from the right;
- f is **not** continuous at 0 (because it is not continuous at 0 from the right).

5. Limits at infinity and horizontal asymptotes

- For a function $f : A \subseteq \mathbb{R} \to \mathbb{R}$, we would like to define $\lim\limits_{x \to +\infty} f(x)$ and $\lim\limits_{x \to -\infty} f(x)$.
- The idea is as follows:
  - $\lim\limits_{x \to +\infty} f(x)$ should be equal to a real number L if when $x \in A$ is very, very large, $f(x)$ is very, very close to L;
  - $\lim\limits_{x \to +\infty} f(x) = +\infty$ if when $x \in A$ is very, very large, $f(x)$ is very, very large;
  - $\lim\limits_{x \to +\infty} f(x) = -\infty$ if when $x \in A$ is very, very large, $f(x)$ is very, very large in the negative direction (i.e. $f(x)$ is a negative number whose absolute value is very large).
- The intuition for $\lim\limits_{x \to -\infty} f(x)$ is similar. Let us now formalize this.

Definition

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$, where the domain A of f is not bounded above.[a] Then:

- for $L \in \mathbb{R}$, we write $\lim_{x \to +\infty} f(x) = L$ if the following hold:

  for all real numbers $\varepsilon > 0$, there exists some $N \in \mathbb{R}$ s.t. for all $x \in A$, if $x > N$, then $|f(x) - L| < \varepsilon$;

- we write $\lim_{x \to +\infty} f(x) = +\infty$ if the following hold:

  for all $M \in \mathbb{R}$, there exists some $N \in \mathbb{R}$ s.t. for all $x \in A$, if $x > N$, then $f(x) > M$;

- we write $\lim_{x \to +\infty} f(x) = -\infty$ if the following hold:

  for all $M \in \mathbb{R}$, there exists some $N \in \mathbb{R}$ s.t. for all $x \in A$, if $x > N$, then $f(x) < M$.

---

[a] So, we are assuming that for all $N \in \mathbb{R}$, there exists some $x \in A$ s.t. $x > N$.

Definition

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$, where the domain A of f is not bounded below.[a] Then:

- for $L \in \mathbb{R}$, we write $\lim_{x \to -\infty} f(x) = L$ if the following hold:

  for all real numbers $\varepsilon > 0$, there exists some $N \in \mathbb{R}$ s.t. for all $x \in A$, if $x < N$, then $|f(x) - L| < \varepsilon$;

- we write $\lim_{x \to -\infty} f(x) = +\infty$ if the following hold:

  for all $M \in \mathbb{R}$, there exists some $N \in \mathbb{R}$ s.t. for all $x \in A$, if $x < N$, then $f(x) > M$;

- we write $\lim_{x \to -\infty} f(x) = -\infty$ if the following hold:

  for all $M \in \mathbb{R}$, there exists some $N \in \mathbb{R}$ s.t. for all $x \in A$, if $x < N$, then $f(x) < M$.

---

[a] So, we are assuming that for all $N \in \mathbb{R}$, there exists some $x \in A$ s.t. $x < N$.

Definition

For a function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and a real number $b \in \mathbb{R}$, we say that:

- $y = b$ is a *horizontal asymptote* of f as $x \to +\infty$ if $\lim\limits_{x \to +\infty} f(x) = b$;
- $y = b$ is a *horizontal asymptote* of f as $x \to -\infty$ if $\lim\limits_{x \to -\infty} f(x) = b$;
- y is a *horizontal asymptote* of f if at least one of $\lim\limits_{x \to +\infty} f(x)$ and $\lim\limits_{x \to -\infty} f(x)$ exists and is equal to b.

- **Remark:** The techniques and tricks that we use to compute limits of functions at $+\infty$ and $-\infty$ are similar to those that we used for computing the limits of sequences.

Example 3.6.11

Consider the function $f(x) = \frac{3x^2-2x+5}{7x^2-1}$. (Note that the roots of $7x^2 - 1 = 0$ are $\pm\frac{1}{\sqrt{7}}$, and so the domain of f is $\mathbb{R} \setminus \{\frac{1}{\sqrt{7}}, -\frac{1}{\sqrt{7}}\}$.) We now compute:

$$\begin{aligned}
\lim_{x\to+\infty} f(x) &= \lim_{x\to+\infty} \frac{3x^2-2x+5}{7x^2-1} \\
&= \lim_{x\to+\infty} \frac{x^2\left(3-\frac{2}{x}+\frac{5}{x^2}\right)}{x^2\left(7-\frac{1}{x^2}\right)} \\
&= \lim_{x\to+\infty} \frac{3-\overset{\nearrow 0}{\boxed{\frac{2}{x}}}+\overset{\nearrow 0}{\boxed{\frac{5}{x^2}}}}{7-\underset{\searrow 0}{\boxed{\frac{1}{x^2}}}} = \frac{3}{7}.
\end{aligned}$$

Example 3.6.11 (continued)

Similarly, we have the following:

$$\begin{aligned}
\lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty} \frac{3x^2-2x+5}{7x^2-1} \\
&= \lim_{x\to-\infty} \frac{x^2\left(3-\frac{2}{x}+\frac{5}{x^2}\right)}{x^2\left(7-\frac{1}{x^2}\right)} \\
&= \lim_{x\to-\infty} \frac{3-\overset{\nearrow 0}{\frac{2}{x}}+\overset{\nearrow 0}{\frac{5}{x^2}}}{7-\underset{\searrow 0}{\frac{1}{x^2}}} = \frac{3}{7}.
\end{aligned}$$

Note that this means that $y=\frac{3}{7}$ is a horizontal asymptote of f (both as $x\to+\infty$ and as $x\to-\infty$).

Example 3.6.12

Consider the function $f(x) = \frac{-x^3+x^2-x+1}{3x^2+1}$. (Note that $3x^2+1>0$ for all $x \in \mathbb{R}$, and so the domain of f is $\mathbb{R}$.) We compute:

$$\begin{aligned}
\lim_{x\to+\infty} f(x) &= \lim_{x\to+\infty} \frac{-x^3+x^2-x+1}{3x^2+1} \\
&= \lim_{x\to+\infty} \frac{x^3\left(-1+\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}\right)}{x^2\left(3+\frac{1}{x^2}\right)} \\
&= \lim_{x\to+\infty} \frac{\overset{\nearrow\,+\infty}{\bigcirc\!\!\!\!\!\!\;x}\;\left(-1+\overset{\nearrow\,0}{\frac{1}{x}}-\overset{\nearrow\,0}{\frac{1}{x^2}}+\overset{\nearrow\,0}{\frac{1}{x^3}}\right)}{3+\underset{\searrow\,0}{\frac{1}{x^2}}} \\
&= -\infty.
\end{aligned}$$

Example 3.6.12 (continued)

Similarly, we have the following:

$$
\begin{aligned}
\lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty} \frac{-x^3+x^2-x+1}{3x^2+1} \\
&= \lim_{x\to-\infty} \frac{x^3\left(-1+\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}\right)}{x^2\left(3+\frac{1}{x^2}\right)} \\
&= \lim_{x\to-\infty} \frac{\overset{\nearrow\,-\infty}{\boxed{x}}\left(-1+\overset{\nearrow\,0}{\boxed{\frac{1}{x}}}-\overset{\nearrow\,0}{\boxed{\frac{1}{x^2}}}+\overset{\nearrow\,0}{\boxed{\frac{1}{x^3}}}\right)}{3+\underset{\searrow\,0}{\boxed{\frac{1}{x^2}}}} \\
&= +\infty.
\end{aligned}
$$

So, f has no horizontal asymptotes.

Example 3.6.13

Consider the function $f(x) = \frac{x^4+x^3+1}{x^6-3x^4+3x-7}$. (It is not so simply to determine the domain of f, since there is no formula for computing the roots of polynomials of degree at least five. However, it is clear that $f(x)$ is defined when x is a real number, positive or negative, of very large absolute value.) We compute:

$$\begin{aligned}
\lim_{x\to+\infty} f(x) &= \lim_{x\to+\infty} \frac{x^4+x^3+1}{x^6-3x^4+3x-7} \\
&= \lim_{x\to+\infty} \frac{x^4\left(1+\frac{1}{x}+\frac{1}{x^4}\right)}{x^6\left(1-\frac{3}{x^2}+\frac{3}{x^5}-\frac{7}{x^6}\right)} \\
&= \lim_{x\to+\infty} \frac{1+\overset{\nearrow 0}{\frac{1}{x}}+\overset{\nearrow 0}{\frac{1}{x^4}}}{\underset{\searrow +\infty}{x^2}\left(1-\underset{\searrow 0}{\frac{3}{x^2}}+\underset{\searrow 0}{\frac{3}{x^5}}-\underset{\searrow 0}{\frac{7}{x^6}}\right)} = 0.
\end{aligned}$$

Example 3.6.13 (continued)

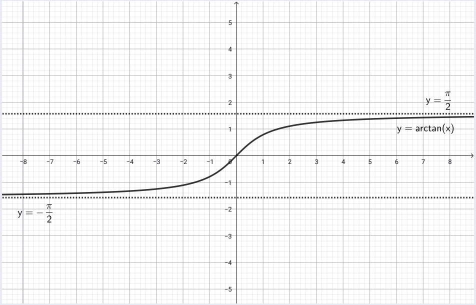
Similarly, we have the following:

$$\begin{aligned}
\lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty} \frac{x^4+x^3+1}{x^6-3x^4+3x-7} \\
&= \lim_{x\to-\infty} \frac{x^4\left(1+\frac{1}{x}+\frac{1}{x^4}\right)}{x^6\left(1-\frac{3}{x^2}+\frac{3}{x^5}-\frac{7}{x^6}\right)} \\
&= \lim_{x\to-\infty} \frac{1+\overset{\nearrow 0}{\frac{1}{x}}+\overset{\nearrow 0}{\frac{1}{x^4}}}{\underset{\searrow +\infty}{x^2}\left(1-\underset{\searrow 0}{\frac{3}{x^2}}+\underset{\searrow 0}{\frac{3}{x^5}}-\underset{\searrow 0}{\frac{7}{x^6}}\right)} = 0.
\end{aligned}$$

So, $y=0$ is a horizontal asymptote of f (both as $x\to+\infty$ and as $x\to-\infty$).

Example 3.6.14

Note that $\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$ and $\lim_{x\to\infty} \arctan x = -\frac{\pi}{2}$ (see the picture below). So, $y = \frac{\pi}{2}$ is a horizontal asymptote of $\arctan x$ as $x \to +\infty$, and $y = -\frac{\pi}{2}$ is a horizontal asymptote of $\arctan x$ as $x \to -\infty$.



6. Slant asymptotes

Definition

For a function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and real numbers $a \neq 0$ and b, we say that:

- $y = ax + b$ is a *slant asymptote* of f as $x \to +\infty$ if $\lim\limits_{x \to +\infty} (f(x) - (ax + b)) = 0$;[a]
- $y = ax + b$ is a *slant asymptote* of f as $x \to -\infty$ if $\lim\limits_{x \to -\infty} (f(x) - (ax + b)) = 0$;[b]
- $y = ax + b$ is a *slant asymptote* of f as $x \to -\infty$ if either $\lim\limits_{x \to +\infty} (f(x) - (ax + b)) = 0$ or $\lim\limits_{x \to -\infty} (f(x) - (ax + b)) = 0$.

---

[a] So, for very large x, we have that $f(x) \approx ax + b$.

[b] So, for a negative real number x of very large absolute value, we have that $f(x) \approx ax + b$.

Proposition 3.6.15

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and let $a \neq 0$ and b be real numbers. Then:

- (a) $y = ax + b$ is a slant asymptote of f as $x \to +\infty$ if and only if the following hold:

$$\lim_{x\to+\infty} \frac{f(x)}{x} = a \qquad \text{and} \qquad \lim_{x\to+\infty} (f(x) - ax) = b;$$

- (b) $y = ax + b$ is a slant asymptote of f as $x \to -\infty$ if and only if the following hold:

$$\lim_{x\to-\infty} \frac{f(x)}{x} = a \qquad \text{and} \qquad \lim_{x\to-\infty} (f(x) - ax) = b.$$

- Proof: Lecture Notes.

Example 3.6.16

Determine whether the function $f(x) = \frac{x^2+x+1}{x-2}$ has any slant asymptotes, and if so, compute them.

Solution. First, we note that

$$a_+ = \lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{x^2+x+1}{x^2-2x} = \lim_{x\to+\infty} \frac{1+\frac{1}{x}+\frac{1}{x^2}}{1-\frac{2}{x}} = 1$$

and

$$\begin{aligned} b_+ &= \lim_{x\to+\infty} \Big(f(x) - a_+ x\Big) \\ &= \lim_{x\to+\infty} \left(\frac{x^2+x+1}{x-2} - x\right) \\ &= \lim_{x\to+\infty} \frac{3x+1}{x-2} \\ &= \lim_{x\to+\infty} \frac{3+\frac{1}{x}}{1-\frac{2}{x}} = 3. \end{aligned}$$

So, by Proposition 3.6.15, $y = x + 3$ is a slant asymptote of f as $x \to +\infty$.

Example 3.6.16

Determine whether the function $f(x) = \frac{x^2+x+1}{x-2}$ has any slant asymptotes, and if so, compute them.

Solution (continued). Similarly, we can compute

$$a_- = \lim_{x\to-\infty} \frac{f(x)}{x} = 1 \quad \text{and} \quad b_- = \lim_{x\to-\infty} \Big(f(x) - a_- x\Big) = 3,$$

and so Proposition 3.6.15 guarantees that $y = x + 3$ is a slant asymptote of f as $x \to -\infty$. ☐

⑦ The Squeeze Theorem for functions

The Squeeze Theorem for functions

Let $f, g, h : A \subseteq \mathbb{R} \to \mathbb{R}$ be functions s.t. $f(x) \leq g(x) \leq h(x)\ \forall x \in A$, let $a \in \mathbb{R}$ be an accumulation point of A, and let $L \in \mathbb{R}$. If $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$.

Proof. Assume that $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, and fix $\varepsilon > 0$.

Using the fact that $\lim_{x \to a} f(x) = L$, we fix $\delta_1 > 0$ s.t. $\forall x \in A$, if $0 < |x - a| < \delta_1$, then $|f(x) - L| < \varepsilon$, i.e. $L - \varepsilon < f(x) < L + \varepsilon$.

Similarly, using the fact that $\lim_{x \to a} h(x) = L$, we fix $\delta_2 > 0$ s.t. $\forall x \in A$, if $0 < |x - a| < \delta_2$, then $|h(x) - L| < \varepsilon$, i.e. $L - \varepsilon < h(x) < L + \varepsilon$.

Set $\delta = \min\{\delta_1, \delta_2\}$. Then $\forall x \in A$ s.t. $0 < |x - a| < \delta$, we have that $L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon$, and it follows that $|g(x) - L| < \varepsilon$. This proves that $\lim_{x \to a} g(x) = L$, as we had claimed. $\square$

Example 3.7.1

Compute $\lim\limits_{x\to 0}\left(x\sin(\frac{1}{x})\right)$.

Solution. Clearly,

$$-|x| \quad \leq \quad x\sin(\tfrac{1}{x}) \quad \leq \quad |x| \qquad \forall x \in \mathbb{R}.$$

Obviously, $\lim\limits_{x\to 0}(-|x|) = \lim\limits_{x\to 0}|x| = 0$. So, by the Squeeze Theorem, we have that $\lim\limits_{x\to 0}\left(x\sin(\frac{1}{x})\right) = 0$. □

- **Remark:** To apply the Squeeze Theorem, we do not really need the inequality $f(x) \leq g(x) \leq h(x)$ to hold for all x in the domain. It is enough to have the inequality for values of x close to a (but not necessarily at a itself), as follows.

The Squeeze Theorem for functions (stronger version)

Let $f, g, h : A \subseteq \mathbb{R} \to \mathbb{R}$ be functions, let $a \in \mathbb{R}$ be an accumulation point of A, and let $L \in \mathbb{R}$. Assume that for some $\delta > 0$, the following holds:

$$f(x) \quad \leq \quad g(x) \quad \leq \quad h(x) \qquad \forall x \in A \cap \Big((a-\delta, a) \cup (a, a+\delta) \Big)$$

If $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} h(x) = L$, then $\lim\limits_{x \to a} g(x) = L$.

- The proof is very similar to that of the Squeeze Theorem above, and it is left as an exercise.

- **Remark:** Similar “Squeeze Theorems” can obviously be formulated for one-sided limits (“$x \to a^-$” or “$x \to a^+$”), as well as for limits at infinity (“$x \to +\infty$” or “$x \to -\infty$”).
  - The exact formulations and the proofs of these theorems are left as an exercise.

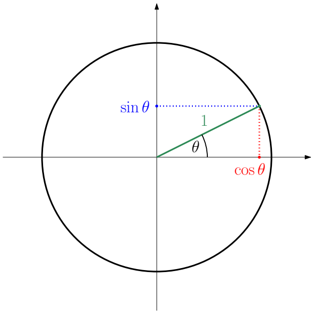
8. The limit $\lim_{x \to 0} \frac{\sin x}{x}$

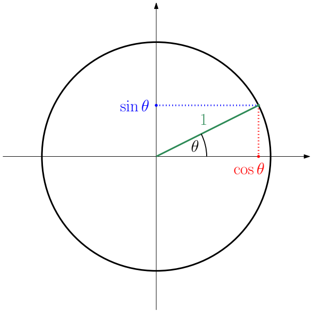
Theorem 3.7.2

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

- The proof of Theorem 3.7.2 relies on trigonometry, geometry, and the Squeeze Theorem.
- Let us first review the trigonometry and geometry that we need.

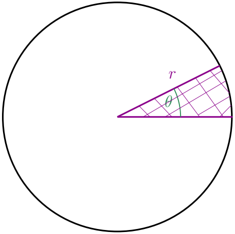
- Given a real number θ (which we consider to be an angle in radians), we obtain $\sin\theta$ and $\cos\theta$ as follows.
- First, we consider the unit circle (i.e. the circle in the coordinate plane, centered at the origin and of radius 1).
- We consider the point on the unit circle determined by the angle θ (starting at the x-axis, we move θ radians counterclockwise if $\theta \geq 0$, and we move $-\theta$ radians clockwise if $\theta < 0$).
- The coordinates of this point are $(\cos\theta, \sin\theta)$, as in the picture below.





- If θ is not an odd integer multiple of $\frac{\pi}{2}$ (and so $\cos\theta \neq 0$), then we define $\tan\theta := \frac{\sin\theta}{\cos\theta}$. Similarly, if θ is not an integer multiple of π (and so $\sin\theta \neq 0$), then we define $\cot\theta := \frac{\cos\theta}{\sin\theta}$.
- **Remark:** Unless explicitly specified otherwise, when we write $\sin x$, $\cot x$, $\tan x$, or $\cot x$ (in analysis), we always assume that x is an angle in **radians** (not degrees).

- Suppose we are given real numbers $r > 0$ and $\theta \in [0, 2\pi]$, where we think of θ as an angle in radians.
- The area of a disk with radius r is $r^2\pi$.
- Meanwhile, the area of the sector of the disk determined by the angle θ (see the picture below) is $\frac{\theta}{2\pi} r^2\pi = \frac{1}{2} r^2\theta$.



Theorem 3.7.2

$\lim\limits_{x\to 0} \frac{\sin x}{x} = 1.$

Proof. Our goal is to show that $\lim\limits_{x\to 0^+} \frac{\sin x}{x} = 1$. This is enough, because once we have shown this, it will follow that

$$\lim_{x\to 0^-} \frac{\sin x}{x} = \lim_{x\to 0^-} \frac{-\sin x}{-x} = \lim_{x\to 0^-} \frac{\sin(-x)}{-x} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1,$$

and consequently, $\lim\limits_{x\to 0^+} \frac{\sin x}{x} = \lim\limits_{x\to 0^-} \frac{\sin x}{x} = 1$, which in turn implies that $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$ (by Lemma 3.6.3). We will prove that $\lim\limits_{x\to 0^+} \frac{\sin x}{x} = 1$ by showing that

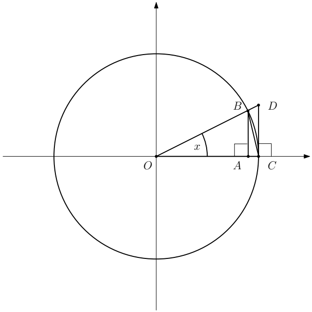
$$\cos x < \frac{\sin x}{x} < 1 \qquad \forall x \in \left(0, \tfrac{\pi}{2}\right).$$

Since $\lim\limits_{x\to 0^+} \cos x = 1$, the Squeeze Theorem will immediately imply that $\lim\limits_{x\to 0^+} \frac{\sin x}{x} = 1$, which is what we need to show.

Theorem 3.7.2

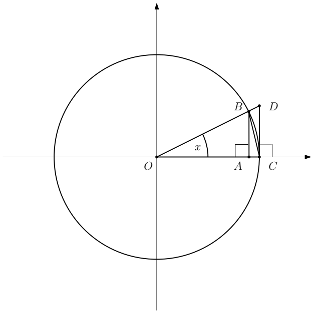
$\lim_{x\to 0} \frac{\sin x}{x} = 1.$

Proof. Reminder: WTS $\cos x < \frac{\sin x}{x} < 1 \ \forall x \in (0, \frac{\pi}{2})$.
Now, fix $x \in (0, \frac{\pi}{2})$, and consider the diagram below (where the circle is the unit circle, i.e. its radius is 1).



As usual, for points X and Y, we will denote by $|XY|$ the length of the line segment XY.

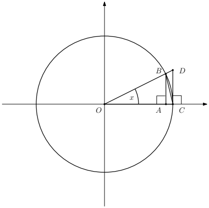
Proof (continued). Reminder: $x \in (0, \frac{\pi}{2})$; WTS $\cos x < \frac{\sin x}{x} < 1$.



Note that:

- $|OB| = |OC| = 1$;
- $|AB| = \sin x$;
- $|OA| = \cos x$;
- $|CD| = \tan x$.
    - The triangle ΔOCD is similar to the triangle ΔOAB, and so $\frac{|CD|}{|AB|} = \frac{|OC|}{|OA|}$. Therefore, $|CD| = \frac{|OC|}{|OA|}|AB| = \frac{1}{\cos x}\sin x = \tan x$.

Proof (continued). Reminder: $x \in (0, \frac{\pi}{2})$; $|OB| = |OC| = 1$; $|AB| = \sin x$, $|OA| = \cos x$, $|CD| = \tan x$; WTS $\cos x < \frac{\sin x}{x} < 1$.



We now observe the following:

- the area of the triangle ΔOCB is $\frac{1}{2}|OC||AB| = \frac{1}{2}\sin x$;
- the area of the sector OCB is $\frac{1}{2}x$;
- the area of the triangle ΔOCD is $\frac{1}{2}|OC||CD| = \frac{1}{2}\tan x$.

Obviously,

$$\text{area}(\Delta OCB) \quad < \quad \text{area}(\text{sector } OCB) \quad < \quad \text{area}(\Delta OCD),$$

and consequently,

$$\frac{1}{2}\sin x \quad < \quad \frac{1}{2}x \quad < \quad \frac{1}{2}\tan x.$$

Theorem 3.7.2

$\lim_{x\to 0} \frac{\sin x}{x} = 1.$

Proof (continued.) Reminder: $x \in (0, \frac{\pi}{2})$; WTS $\cos x < \frac{\sin x}{x} < 1$.

We have obtained:

$$\tfrac{1}{2}\sin x \quad < \quad \tfrac{1}{2}x \quad < \quad \tfrac{1}{2}\tan x.$$

Since $0 < x < \frac{\pi}{2}$, we know that $\sin x > 0$; therefore, we may multiply the inequality above by $\frac{2}{\sin x}$ to obtain

$$1 \quad < \quad \frac{x}{\sin x} \quad < \quad \frac{1}{\cos x}.$$

Since all values in the inequality above are positive (because $0 < x < \frac{\pi}{2}$), we may take reciprocals to obtain

$$1 \quad > \quad \frac{\sin x}{x} \quad > \quad \cos x,$$

and we are done. ☐

⑩ Analogues of the Squeeze Theorem for infinite limits

Theorem 3.7.3

Let $f, g : A \subseteq \mathbb{R} \to \mathbb{R}$ be functions s.t. $f(x) \leq g(x)\ \forall x \in A$. Let $a \in \mathbb{R}$ be an accumulation point of A. Then the following hold:

ⓐ if $\lim_{x \to a} f(x) = +\infty$, then $\lim_{x \to a} g(x) = +\infty$;

ⓑ if $\lim_{x \to a} g(x) = -\infty$, then $\lim_{x \to a} f(x) = -\infty$.

Proof. We prove (a); the proof of (b) is similar. Fix $M \in \mathbb{R}$. Using the fact that $\lim_{x \to a} f(x) = +\infty$, we fix some $\delta > 0$ such that for all $x \in A$, if $0 < |x - a| < \delta$, then $f(x) > M$. But now for all $x \in A$ such that $0 < |x - a| < \delta$, we have that

$$g(x) \quad \geq \quad f(x) \quad > \quad M,$$

which proves that $\lim_{x \to a} f(x) = +\infty$. $\square$

Theorem 3.7.4

Let $f, g : A \subseteq \mathbb{R} \to \mathbb{R}$ be functions, and let $a \in \mathbb{R}$ be an accumulation point of A. Assume that for some $\delta > 0$, we following holds:

$$f(x) \quad \leq \quad g(x) \qquad\qquad \forall x \in A \cap \Big((a-\delta, a) \cup (a, a+\delta)\Big).$$

Then the following hold:

1. if $\lim_{x \to a} f(x) = +\infty$, then $\lim_{x \to a} g(x) = +\infty$;
2. if $\lim_{x \to a} g(x) = -\infty$, then $\lim_{x \to a} f(x) = -\infty$.

- Proof: Exercise.