Mathematical Analysis 1

Lecture #7

Limits of functions (part II)

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- This lecture has three parts:
    1. Continuity
    2. The Intermediate Value Theorem
    3. The Extreme Value Theorem

1. Continuity

Definition

A function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is *continuous* at a point $a \in A$ if the following holds:

for all $\varepsilon > 0$, there exists some $\delta > 0$, s.t. for all $x \in A$, if $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$.

- **Remark:** Suppose that $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is a function, and $a \in A$, as in the definition above.
  - Note that if a is an accumulation point of A, then f is continuous at a iff $\lim_{x \to a} f(x) = f(a)$ (and in particular, $\lim_{x \to a} f(x)$ exists).
  - If a is **not** an accumulation point of A, then f is automatically continuous at a.

Definition

If a function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is continuous at all points $a \in A$, then we simply say that f is *continuous*. If $I \subseteq A$ is an interval, then we say that f is *continuous* on I provided that $f \upharpoonright I$ is continuous at all points in I.

- **Remark:** Suppose that $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is a function and that $I \subseteq A$ is an interval.
    - If f is continuous on all points in I, then f is indeed continuous on I, as we would expect.
    - Somewhat surprisingly, the converse is false in general!
        - For instance, consider the function $f : \mathbb{R} \to \infty$ given by

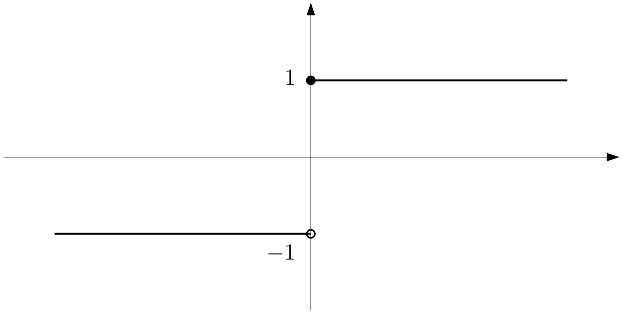
          $$f(x) = \begin{cases} -1 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x \geq 0 \end{cases}$$

          for all $x \in \mathbb{R}$. (The graph of this function is on the next slide.)
        - Then f is **not** continuous at the point $x = 0$, but it **is** continuous at on interval $[0, +\infty)$.
    - This is because whether or not a function is continuous on an interval is determined solely by the behavior of the function on the interval in question, and not by its behavior elsewhere.

- The graph of the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} -1 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x \geq 0 \end{cases}$$

for all $x \in \mathbb{R}$.



Definition

A function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is *continuous* at a point $a \in A$ if the following holds:

for all $\varepsilon > 0$, there exists some $\delta > 0$, s.t. for all $x \in A$, if $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$.

Definition

If a function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is continuous at all points $a \in A$, then we simply say that f is *continuous*. If $I \subseteq A$ is an interval, then we say that f is *continuous* on I provided that $f \upharpoonright I$ is continuous at all points in I.

- **Remark:** The intuition behind the concept of continuity is that a function is supposed to be continuous on an interval if its graph (restricted to the interval in question) can be drawn on a piece of paper without lifting the pen.
    - Of course, pens and paper are not mathematical objects!
    - Our definition attempts to formalize our intuition, although some functions that are continuous according to our formal definition are somewhat strange, and drawing their graphs (with our without lifting the pen) would be quite difficult.
    - We do, however, have the following fact, stated without proof.

Fact 3.5.1

The following functions are all continuous:

- polynomial functions,
- rational functions,
- root functions,
- exponential and logarithmic functions,
- trigonometric and inverse trigonometric functions.

- Reminder:

Theorem 3.4.4

Let $f_1 : A \subseteq \mathbb{R} \to \mathbb{R}$ and $f_2 : A \subseteq \mathbb{R} \to \mathbb{R}$ be functions, and let $a \in \mathbb{R}$ be an accumulation point of A. Assume that both $\lim_{x\to a} f_1(x)$ and $\lim_{x\to a} f_2(x)$ exist. Then all the following hold:

(a) for all $c \in \mathbb{R}$, $\lim_{x\to a}(cf_1)(x) = c \lim_{x\to a} f_1(x)$;

(b) $\lim_{x\to a}(f_1 + f_2)(x) = (\lim_{x\to a} f_1(x)) + (\lim_{x\to a} f_2(x))$;

(c) $\lim_{x\to a}(f_1 - f_2)(x) = (\lim_{x\to a} f_1(x)) - (\lim_{x\to a} f_2(x))$;

(d) $\lim_{x\to a}(f_1 f_2)(x) = (\lim_{x\to a} f_1(x))(\lim_{x\to a} f_2(x))$;

(e) if $\lim_{x\to a} f_2(x) \neq 0$, then a is an accumulation point of the set $A' := \{x \in A \mid f_2(x) \neq 0\}$, and we have that $\lim_{x\to a}(\frac{f_1}{f_2})(x) = \frac{\lim_{x\to a} f_1(x)}{\lim_{x\to a} f_2(x)}$, where we consider the domain of the function $\frac{f_1}{f_2}$ to be A'.

Theorem 3.5.2

Let $f_1 : A \subseteq \mathbb{R} \to \mathbb{R}$ and $f_2 : A \subseteq \mathbb{R} \to \mathbb{R}$ be functions that are continuous at a point $a \in A$. Then all the following hold:

(a) for all $c \in \mathbb{R}$, the function cf_1 is continuous at a;

(b) functions $f_1 + f_2$, $f_1 - f_2$, and $f_1 f_2$ are continuous at a;

(c) if $f_2(a) \neq 0$, then the function $\frac{f_1}{f_2}$ is continuous at a.

Proof. If a is an accumulation point of A, then this readily follows from Theorem 3.4.4 and from the definition of continuity.

On the other hand, if a is **not** accumulation point of A, then any real-valued function whose domain is A (or any subset of A that contains the point a) is continuous at a. □

Theorem 3.5.3

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and $g : B \subseteq \mathbb{R} \to \mathbb{R}$ (with $f[A] \subseteq B$, so that the function $g \circ f : A \subseteq \mathbb{R} \to \mathbb{R}$ is defined). Let $a \in \mathbb{R}$, and assume that g is continuous at $b := \lim_{x \to a} f(x)$ (in particular, this limit exists and belongs to B). Then:

(a) $\lim_{x \to a} (g \circ f)(x) = g(\lim_{x \to a} f(x)) = g(b)$;

(b) if f is continuous at a, then $g \circ f$ is continuous at a.

- First an example, then a proof.

Example 3.5.4

Compute $\lim_{x \to 2} e^{\frac{x^2-3x+2}{x-2}}$.

Solution. Since the function e^x is continuous, we may compute:

$$\lim_{x \to 2} e^{\frac{x^2-3x+2}{x-2}} = e^{\lim_{x \to 2} \frac{x^2-3x+2}{x-2}} = e^{\lim_{x \to 2} \frac{(x-2)(x-1)}{x-2}} = e^{\lim_{x \to 2} (x-1)} = e,$$

and we are done. □

Theorem 3.5.3

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and $g : B \subseteq \mathbb{R} \to \mathbb{R}$ (with $f[A] \subseteq B$, so that the function $g \circ f : A \subseteq \mathbb{R} \to \mathbb{R}$ is defined). Let $a \in \mathbb{R}$, and assume that g is continuous at $b := \lim_{x \to a} f(x)$ (in particular, this limit exists and belongs to B). Then:

- (a) $\lim_{x \to a} (g \circ f)(x) = g(\lim_{x \to a} f(x)) = g(b)$;
- (b) if f is continuous at a, then $g \circ f$ is continuous at a.

Proof. We begin by observing that a is an accumulation point of A, since $\lim_{x \to a} f(x)$ exists.

Theorem 3.5.3

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and $g : B \subseteq \mathbb{R} \to \mathbb{R}$ (with $f[A] \subseteq B$, so that the function $g \circ f : A \subseteq \mathbb{R} \to \mathbb{R}$ is defined). Let $a \in \mathbb{R}$, and assume that g is continuous at $b := \lim\limits_{x \to a} f(x)$ (in particular, this limit exists and belongs to B). Then:

(a) $\lim\limits_{x \to a} (g \circ f)(x) = g(\lim\limits_{x \to a} f(x)) = g(b)$;

(b) if f is continuous at a, then $g \circ f$ is continuous at a.

Proof (continued). We first prove (a). Fix $\varepsilon > 0$. Using the fact that g is continuous at b, we fix some $\varepsilon' > 0$ s.t. for all $x \in B$, if $|x - b| < \varepsilon'$, then $|g(x) - g(b)| < \varepsilon$.

Next, using the fact that $b = \lim\limits_{x \to a} f(a)$, we fix some $\delta > 0$ s.t. for all $x \in A$, if $0 < |x - a| < \delta$, then $|f(x) - b| < \varepsilon'$.

Now, fix $x \in A$ s.t. $0 < |x - a| < \delta$. Then $|f(x) - b| < \varepsilon'$. Since $f[A] \subseteq B$, we see that $f(x) \in B$. Now by the choice of ε', we have that $|g(f(x)) - g(b)| < \varepsilon$, i.e. $|(g \circ f)(x) - g(b)| < \varepsilon$. This proves (a).

Theorem 3.5.3

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and $g : B \subseteq \mathbb{R} \to \mathbb{R}$ (with $f[A] \subseteq B$, so that the function $g \circ f : A \subseteq \mathbb{R} \to \mathbb{R}$ is defined). Let $a \in \mathbb{R}$, and assume that g is continuous at $b := \lim_{x \to a} f(x)$ (in particular, this limit exists and belongs to B). Then:

- (a) $\lim_{x \to a} (g \circ f)(x) = g(\lim_{x \to a} f(x)) = g(b)$;
- (b) if f is continuous at a, then $g \circ f$ is continuous at a.

Proof (continued). For (b), we simply observe that, if f is continuous at a, then

$$\begin{aligned} \lim_{x \to a} (g \circ f)(x) &= g(\lim_{x \to a} f(x)) && \text{by (a)} \\ &= g(f(a)) && \text{because } f \text{ is continuous at } a \\ &= (g \circ f)(a), \end{aligned}$$

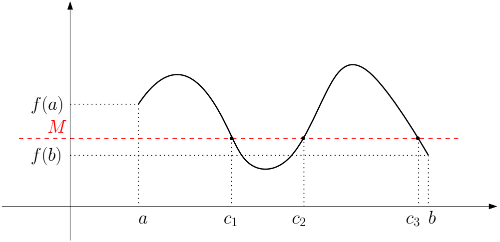
which by definition means that $g \circ f$ is continuous at a. □

2. The Intermediate Value Theorem

The Intermediate Value Theorem

Let a and b be real numbers s.t. $a < b$, let $f : [a, b] \to \mathbb{R}$ be a continuous function, and let $M \in \mathbb{R}$ be s.t. $\min\{f(a), f(b)\} < M < \max\{f(a), f(b)\}$.[a] Then $\exists c \in (a, b)$ s.t. $f(c) = M$.

[a] So, we are assuming that $f(a) \neq f(b)$, and that either $f(a) < M < f(b)$ or $f(b) < M < f(a)$.



- First an example, then a proof.

The Intermediate Value Theorem

Let a and b be real numbers s.t. $a < b$, let $f : [a, b] \to \mathbb{R}$ be a continuous function, and let $M \in \mathbb{R}$ be s.t. $\min\{f(a), f(b)\} < M < \max\{f(a), f(b)\}$. Then $\exists c \in (a, b)$ s.t. $f(c) = M$.

Example 3.5.5

Prove that the equation $\sin x = \ln x$ has at least one solution.

Solution. Let $a = e^{-2}$ and $b = e^2$, and let $f : [a, b] \to \mathbb{R}$ be defined by $f(x) = \sin x - \ln x$. Clearly, f is continuous on $[a, b]$. Furthermore, we have that:

- $f(a) = \sin(e^{-2}) - \ln(e^{-2}) = \sin(e^{-2}) + 2 \geq 1$;
- $f(b) = \sin(e^2) - \ln(e^2) = \sin(e^2) - 2 \leq -1$.

It follows that $f(b) < 0 < f(a)$, and so the Intermediate Value Theorem implies that there exists some $x \in (a, b)$ s.t. $f(x) = 0$, i.e. s.t. $\sin x = \ln x$. $\square$

The Intermediate Value Theorem

Let a and b be real numbers s.t. $a < b$, let $f : [a, b] \to \mathbb{R}$ be a continuous function, and let $M \in \mathbb{R}$ be s.t. $\min\{f(a), f(b)\} < M < \max\{f(a), f(b)\}$. Then $\exists c \in (a, b)$ s.t. $f(c) = M$.

Proof. By hypothesis, we have that either $f(a) < M < f(b)$ or $f(b) < M < f(a)$. We will prove the theorem for the case when $f(a) < M < f(b)$; the other case is similar.

Let

$$S \quad := \quad \{x \in [a, b] \mid f(x) < M\}.$$

Then $a \in S$ (because $f(a) < M$), and in particular, $S \neq \emptyset$. On the other hand, the set S is bounded above (by b). So, by the completeness of the ordered field $\mathbb{R}$, S has a supremum, call it c.

Our goal is to show that $c \in (a, b)$ and $f(c) = M$.

Proof (continued). Reminder: $f : [a, b] \to \mathbb{R}$ is continuous; $f(a) < M < f(b)$; $S = \{x \in [a, b] \mid f(x) < M\}$; $c = \sup(S)$; WTS $c \in (a, b)$ and $f(c) = M$.

We will prove the following three claims:

> ***Claim 1.*** *$c \in [a, b]$, and in particular, $f(c)$ is defined.*
>
> ***Claim 2.*** *$c < b$.*
>
> ***Claim 3.*** *$f(c) = M$.*

Before proving these three claims, let us explain why they are enough to complete our proof.

By Claims 1 and 2, we have that $c \in [a, b)$, and by Claim 3, we have that $f(c) = M$. It remains to show that $c \neq a$. But this follows from the fact that $f(a) < M = f(c)$, and in particular, $f(a) \neq f(c)$.

Let us now prove the three claims!

Proof (continued). Reminder: $f : [a, b] \to \mathbb{R}$ is continuous; $f(a) < M < f(b)$; $S = \{x \in [a, b] \mid f(x) < M\}$; $c = \sup(S)$.

Claim 1. *$c \in [a, b]$, and in particular, $f(c)$ is defined.*

Proof of Claim 1. Since c is an upper bound of S, and $a \in S$, we have that $a \leq c$. On the other hand, since b is an upper bound of S, and c is the supremum (i.e. the least upper bound) of S, we have that $c \leq b$. So, $c \in [a, b]$. This proves Claim 1. ♦

Claim 2. *$c < b$.*

Proof of Claim 2. By Claim 1, we have that $c \in [a, b]$, and consequently, $c \leq b$. So, we need only show that $c \neq b$. Suppose otherwise, i.e. suppose that $c = b$, so that b is the supremum of S.

Recall that $M < f(b)$, and set $\varepsilon := \frac{f(b)-M}{2}$; then $\varepsilon > 0$.

Using the continuity of f at b, we fix some $\delta > 0$ s.t. $\forall x \in [a, b]$, if $|x - b| < \delta$, then $|f(x) - f(b)| < \varepsilon$.

Proof (continued). Reminder: $f : [a,b] \to \mathbb{R}$ is continuous; $f(a) < M < f(b)$; $S = \{x \in [a,b] \mid f(x) < M\}$; $c = \sup(S)$.

Claim 2. *$c < b$.*

Proof of Claim 2 (continued). Reminder: $c = b$ (toward a contradiction); $\varepsilon = \frac{f(b)-M}{2}$; $\delta > 0$ is s.t. $\forall x \in [a,b]$, if $|x-b| < \delta$, then $|f(x) - f(b)| < \varepsilon$.
Since b is the supremum of S, we see that there exists some $x \in S$ s.t. $b - \delta < x \leq b$. We now have that $x \in [a,b]$ and $|x-b| < \delta$, and consequently $|f(x) - f(b)| < \varepsilon$. Therefore:

$$\begin{array}{rcll} f(x) & > & f(b) - \varepsilon & \text{because } |f(x) - f(b)| < \varepsilon \\ & = & f(b) - \frac{f(b)-M}{2} & \text{because } \varepsilon = \frac{f(b)-M}{2} \\ & = & \frac{f(b)+M}{2} \;\; > \;\; M & \text{because } M < f(b). \end{array}$$

We have now shown that $f(x) > M$, contrary to the fact that $x \in S$. This proves Claim 2. ♦

Proof (continued). Reminder: $f : [a, b] \to \mathbb{R}$ is continuous; $f(a) < M < f(b)$; $S = \{x \in [a, b] \mid f(x) < M\}$; $c = \sup(S)$.

So far, we have proven the following two claims:

> ***Claim 1.*** *$c \in [a, b]$, and in particular, $f(c)$ is defined.*
>
> ***Claim 2.*** *$c < b$.*

We will now prove:

> ***Claim 3.*** *$f(c) = M$.*

Proof of Claim 3. Our goal is to show that, for all $\varepsilon > 0$, we have that $M - \varepsilon < f(c) < M + \varepsilon$. Indeed, if this holds for every single $\varepsilon > 0$, then it is clear that $f(c) = M$, which is what we need to show.

So, fix $\varepsilon > 0$. Using the continuity of f and the fact that $c \in [a, b]$ (by Claim 1), we fix some $\delta > 0$ s.t. $\forall x \in [a, b]$, if $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$.

Proof (continued). Reminder: $f : [a, b] \to \mathbb{R}$ is continuous; $f(a) < M < f(b)$; $S = \{x \in [a, b] \mid f(x) < M\}$; $c = \sup(S)$.

Claim 3. $f(c) = M$.

Proof of Claim 3 (continued). Reminder: $\varepsilon > 0$; $\delta > 0$ is s.t. $\forall x \in [a, b]$, if $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$; WTS $M - \varepsilon < f(c) < M + \varepsilon$.

We first show that $f(c) < M + \varepsilon$. Since c is the supremum of S, we see that $\exists a^* \in S$ s.t. $c - \delta < a^* \leq c$. By the definition of S, we have that $f(a^*) \leq M$.

Now, since $a^* \in (c - \delta, c]$, we see that $|a^* - c| < \delta$. Also, $a^* \in [a, b]$ (because $a^* \in S \subseteq [a, b]$). So, by the choice of δ, we have that $|f(a^*) - f(c)| < \varepsilon$. It follows that $f(c) < f(a^*) + \varepsilon \leq M + \varepsilon$.

Proof (continued). Reminder: $f : [a, b] \to \mathbb{R}$ is continuous; $f(a) < M < f(b)$; $S = \{x \in [a, b] \mid f(x) < M\}$; $c = \sup(S)$.

Claim 3. *$f(c) = M$.*

Proof of Claim 3 (continued). Reminder: $\varepsilon > 0$; $\delta > 0$ is s.t. $\forall x \in [a, b]$, if $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$; WTS $M - \varepsilon < f(c) < M + \varepsilon$.

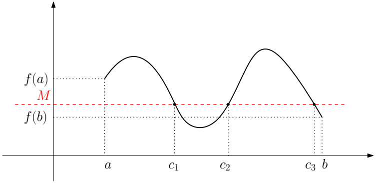
It remains to show that $M - \varepsilon < f(c)$. By Claim 2, we have that $c < b$. So, fix some $b^* \in (c, \min\{c + \delta, b\})$. Then $b^* \in [a, b]$ and $|b^* - c| < \delta$, and it follows that $|f(b^*) - f(c)| < \varepsilon$. Consequently, $f(b^*) - \varepsilon < f(c)$.

On the other hand, since $c < b^*$, and c is the supremum (and an in particular, an upper bound) of S, we deduce that $b^* \notin S$. Since $b^* \in [a, b]$ and $b^* \notin S$, we see that $M \leq f(b^*)$. So, $M - \varepsilon \leq f(b^*) - \varepsilon < f(c)$. This proves Claim 3. ♦

□

The Intermediate Value Theorem

Let a and b be real numbers s.t. $a < b$, let $f : [a, b] \to \mathbb{R}$ be a continuous function, and let $M \in \mathbb{R}$ be s.t. $\min\{f(a), f(b)\} < M < \max\{f(a), f(b)\}$. Then $\exists c \in (a, b)$ s.t. $f(c) = M$.



- **Remark:** In practice, we may have a function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ that is continuous on some closed interval $[a, b] \subseteq A$, though not necessarily on all of A.
  - We can still apply the Intermediate Value Theorem to f and $[a, b]$.
  - Technically, we are applying the theorem to the function $f \restriction [a, b]$, i.e. the restriction of f to the interval $[a, b]$.

3. The Extreme Value Theorem

Definition

A function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is

- *bounded above* if there exists some $U \in \mathbb{R}$, called an *upper bound* for f, s.t. for all $a \in A$, we have that $f(a) \leq U$;
- *bounded below* if there exists some $L \in \mathbb{R}$, called a *lower bound* for f, s.t. for all $a \in A$, we have that $L \leq f(a)$;
- *bounded* on A if it is bounded above and bounded below.

For $B \subseteq A$, we say that

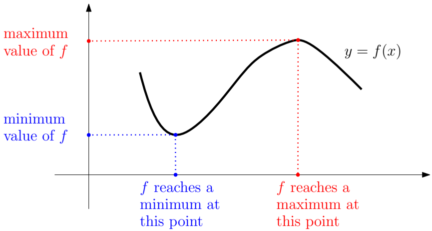
- f is *bounded above* on B if $f \restriction B$ is bounded above;[a]
- f is *bounded below* on B if $f \restriction B$ is bounded below;
- f is *bounded* on B if $f \restriction B$ is bounded.

[a] $f \restriction B$ is the restriction of f to B

Definition

Given a function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and $a \in A$, we say that

- f reaches a *global maximum* (or simply *maximum*) at a, and that $f(a)$ is the *maximum value* of f, if for all $x \in A$, we have that $f(x) \leq f(a)$;
- f reaches a *global minimum* (or simply *minimum*) at a, and that $f(a)$ is the *minimum value* of f, if for all $x \in A$, we have that $f(x) \geq f(a)$.



- Our goal is to prove the following two theorems:

Theorem 3.5.6

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f is bounded on $[a, b]$.

The Extreme Value Theorem

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f reaches both a maximum and a minimum on $[a, b]$.

- **Remark:** Both theorems above **fail** for open intervals.
  - For instance, $\tan x$ is continuous on the open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, but it is bounded neither above nor below, and consequently, it reaches neither a maximum nor a minimum on $(-\frac{\pi}{2}, \frac{\pi}{2})$.
- **Notation:** For $a \in \mathbb{R}$, we set $[a, a] = \{a\}$.
  - So, we introduce a slight generalization of closed intervals, where the set consisting of a single real number is a closed interval.

Theorem 3.5.6

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f is bounded on $[a, b]$.

Proof. Let $B := \{x \in [a, b] \mid f \text{ is bounded on } [a, x]\}$. WTS $b \in B$; this will immediately imply that f is bounded on $[a, b]$, which is what we need to show.

Clearly, $a \in B$, and in particular, $B \neq \emptyset$.

- Indeed, f is bounded both above and below on $[a, a] = \{a\}$ by $f(a)$.

Furthermore, B is bounded above by b. The completeness of the ordered field $\mathbb{R}$ now guarantees that B has a supremum, call it s. Clearly, $a \leq s \leq b$, that is, $s \in [a, b]$.

- Indeed, since $a \in B$ and s is an upper bound of B, we have that $a \leq s$. On the other hand, since s is the least upper bound of B, and b is some upper bound of B, we have that $s \leq b$.

Theorem 3.5.6

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f is bounded on $[a, b]$.

Proof (continued). Reminder: $B := \{x \in [a, b] \mid f \text{ is bounded on } [a, x]\}$; $B \neq \emptyset$; $s = \sup(B)$; $s \in [a, b]$.

Our goal is to prove the following two claims:

Claim 1. *$s \in B$.*

Claim 2. *$s = b$.*

Clearly, these two Claims together imply that $b \in B$, and consequently, f is bounded on $[a, b]$, which is what we need to show.

Let's prove the two claims!

Theorem 3.5.6

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f is bounded on $[a, b]$.

Proof (continued). Reminder: $B := \{x \in [a, b] \mid f \text{ is bounded on } [a, x]\}$; $B \neq \emptyset$; $s = \sup(B)$; $s \in [a, b]$.

Claim 1. *$s \in B$.*

Proof of Claim 1. If $s = a$, then this follows from the fact that $a \in B$. So, assume that $s > a$. WTS f is bounded on $[a, s]$. (Since $s \in [a, b]$, this will immediately imply that $s \in B$.)

Fix $\varepsilon > 0$. Using the continuity of f, we fix some $\delta > 0$ s.t. $\forall x \in [a, b]$, if $|x - s| < \delta$, then $|f(x) - f(s)| < \varepsilon$.

Proof (continued). Reminder: $B := \{x \in [a,b] \mid f \text{ is bounded on } [a,x]\}$; $B \neq \emptyset$; $s = \sup(B)$; $s \in [a,b]$.

Claim 1. *$s \in B$.*

Proof of Claim 1. Reminder: $s > a$; $\varepsilon > 0$; $\delta > 0$ is s.t. $\forall x \in [a,b]$, if $|x - s| < \delta$, then $|f(x) - f(s)| < \varepsilon$; WTS f is bounded on $[a,s]$.

Using the fact that $s = \sup(B)$, we fix some $b' \in B$ s.t. $s - \delta < b' \leq s$. Since $b' \in B$, we have that $b' \in [a,b]$ and that f is bounded on $[a,b']$; fix $L', U' \in \mathbb{R}$ s.t. for all $x \in [a,b']$, we have that $L' \leq f(x) \leq U'$.

Set $L := \min\{L', f(s) - \varepsilon\}$ and $U := \max\{U', f(s) + \varepsilon\}$. WTS $\forall x \in [a,s]$: $L \leq f(x) \leq U$. This will immediately imply that f is bounded on $[a,s]$, which is what we need. Fix any $x \in [a,s]$.

If $x > s - \delta$, then $|s - x| < \delta$, and consequently, $|f(x) - f(s)| < \varepsilon$, which implies that $L \leq f(s) - \varepsilon < f(x) < f(s) + \varepsilon \leq U$.

Suppose now that $x \leq s - \delta$. Then $a \leq x \leq s - \delta < b'$, and in particular, $x \in [a,b']$. But now $L \leq L' \leq f(x) \leq U' \leq U$. ♦

Proof (continued). Reminder: $B := \{x \in [a,b] \mid f \text{ is bounded on } [a,x]\}$; $B \neq \emptyset$; $s = \sup(B)$; $s \in [a,b]$.

> ***Claim 1.*** *$s \in B$.*
>
> ***Claim 2.*** *$s = b$.*

Proof of Claim 2. By Claim 1, we have that $s \in B$, and in particular, $s \in [a,b]$. So, WMA $a \leq s < b$, for otherwise we are done.

Fix $\varepsilon > 0$. Using the continuity of f on $[a,b]$, we fix some $\delta > 0$ s.t. for all $x \in [a,b]$, if $|x - s| < \delta$, then $|f(x) - f(s)| < \varepsilon$.

Set $s' := \min\{b, s + \frac{\delta}{2}\}$. Then $\forall x \in [s, s']$, we have that $x \in [a,b]$ and $|x - s| < \delta$, and consequently, $|f(x) - f(s)| < \varepsilon$, that is, $f(s) - \varepsilon < f(x) < f(s) + \varepsilon$. Thus, f is bounded on $[s, s']$.

By Claim 1, f is bounded on $[a, s]$, and by what we just showed, f is bounded on $[s, s']$. Therefore f is bounded on $[a, s'] = [a, s] \cup [s, s']$, and consequently, $s' \in B$. But this is impossible since s is an upper bound of B, and $s < s'$. ♦

□

- We have proven:

Theorem 3.5.6

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f is bounded on $[a, b]$.

- It remains to prove:

The Extreme Value Theorem

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f reaches both a maximum and a minimum on $[a, b]$.

The Extreme Value Theorem

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f reaches both a maximum and a minimum on $[a, b]$.

- The Extreme Value Theorem follows immediately from the two lemmas below.
  - So it suffices to prove the two lemmas.

Lemma 3.5.7

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f reaches a maximum on the interval $[a, b]$.

Lemma 3.5.8

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f reaches a minimum on the interval $[a, b]$.

Lemma 3.5.7

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f reaches a maximum on the interval $[a, b]$.

Proof. By Theorem 3.5.6, f is bounded on $[a, b]$, i.e. the set $Y := \{f(x) \mid x \in [a, b]\}$ is bounded. Clearly, $Y \neq \emptyset$, and so by the completeness of $\mathbb{R}$, Y has a supremum, call it s. WTS $\exists x \in [a, b]$ s.t. $f(x) = s$.

- Indeed, if such an x exists, then f reaches a maximum at x, and we are done.

Suppose otherwise. Then $\forall x \in [a, b]$, we have that $f(x) < s$, and in particular, $s - f(x) \neq 0$. Now, define $g : [a, b] \to \mathbb{R}$ by setting

$$g(x) := \frac{1}{s - f(x)} \qquad \forall x \in [a, b].$$

Since f is continuous on $[a, b]$, so is g (by Theorem 3.5.2). So, by Theorem 3.5.6, g is bounded. Let U be an upper bound for g. We will derive a contradiction by exhibiting an $x \in [a, b]$ s.t. $g(x) > U$.

Lemma 3.5.7

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f reaches a maximum on the interval $[a, b]$.

Proof (continued). Reminder: $Y = \{f(x) \mid x \in [a, b]\}$;
$s = \sup(Y)$; $g(x) = \frac{1}{s - f(x)}$ $\forall x \in [a, b]$; U is an upper bound for g.

Note that $g(x) > 0$ for all $x \in [a, b]$.

- This is because $f(x) < s$ for all $x \in [a, b]$.

So, $g(x) = \frac{1}{s - f(x)} > 0$ for all $x \in [a, b]$. Consequently, $U > 0$.

Since $s = \sup(Y)$, we know that $s - \frac{1}{U}$ is not an upper bound of Y, and consequently, $\exists x \in [a, b]$ s.t. $f(x) > s - \frac{1}{U}$.

But then $\frac{1}{s - f(x)} > U$, i.e. $g(x) > U$, contrary to the fact that U is an upper bound for g. $\square$

- We have proven:

Lemma 3.5.7

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f reaches a maximum on the interval $[a, b]$.

- It remains to prove:

Lemma 3.5.8

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f reaches a minimum on the interval $[a, b]$.

Proof. Since f is continuous on $[a, b]$, so is $-f$. So, by Lemma 3.5.7, $-f$ reaches a maximum at some $x \in [a, b]$. But then f reaches a minimum at x. $\square$

- We have proven:

Lemma 3.5.7

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f reaches a maximum on the interval $[a, b]$.

Lemma 3.5.8

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f reaches a minimum on the interval $[a, b]$.

The Extreme Value Theorem

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f reaches both a maximum and a minimum on $[a, b]$.

Proof. This follows immediately from Lemmas 3.5.7 and 3.5.8. □

Theorem 3.5.6

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f is bounded on $[a, b]$.

The Extreme Value Theorem

Let $a, b \in \mathbb{R}$ be s.t. $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f reaches both a maximum and a minimum on $[a, b]$.