Mathematical Analysis 1

Lecture #6

Limits of functions (part I)

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- **Notation:** When we write "$f : A \subseteq \mathbb{R} \to \mathbb{R}$," we mean that A is a subset of $\mathbb{R}$, and f is a function from A to $\mathbb{R}$. Similarly, "$f : A \subsetneqq \mathbb{R} \to \mathbb{R}$" means that A is a proper subset of $\mathbb{R}$ (i.e. A is a subset of $\mathbb{R}$, but is not equal to $\mathbb{R}$), and f is a function from A to $\mathbb{R}$.

- In what follows, when we define a function f via a formula (without specifying the domain and the codomain), we will assume that the domain of f is the subset of all real numbers x for which the expression $f(x)$ makes sense and is a real number. For example:
  - the domain of $f_1(x) = \frac{x-1}{x+3}$ is $\mathbb{R} \setminus \{-3\}$, whereas the codomain is $\mathbb{R}$;
  - the domain of $f_2(x) = \sqrt{x}$ is $[0, +\infty)$, whereas the codomain is $\mathbb{R}$;
  - both the domain and the codomain of $f_3(x) = \sqrt[3]{x}$ are $\mathbb{R}$.

- As usual, an *interval* in $\mathbb{R}$ is any set of the following form:
    - $[a, b]$, where $a, b \in \mathbb{R}$ are s.t. $a < b$;
    - $[a, b)$, where $a, b \in \mathbb{R}$ are s.t. $a < b$;
    - $(a, b]$, where $a, b \in \mathbb{R}$ are s.t. $a < b$;
    - (a, b), where $a, b \in \mathbb{R}$ are s.t. $a < b$;
    - $(a, +\infty)$, where $a \in \mathbb{R}$;
    - $(-\infty, b)$, where $b \in \mathbb{R}$;
    - $(-\infty, +\infty)$.

  A *closed interval* is an interval of the form $[a, b]$, whereas an *open interval* is an interval of the form (a, b), $(a, +\infty)$, $(-\infty, b)$, or $(-\infty, +\infty)$.

- This lecture has five parts:
  1. The limit of a function at a point: an informal introduction
  2. The Limit of a function at a point: the formal definition
  3. A relationship between limits of functions and limits of sequences
  4. Properties of limits
  5. Continuity

1. The limit of a function at a point: an informal introduction

- Suppose that $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is a function, and that $a \in \mathbb{R}$ (f may or may not be defined at a, i.e. both $a \in A$ and $a \notin A$ are possible).
  - We would like to say that $\lim_{x \to a} f(x) = L$ (which we read: "L is the limit of $f(x)$ as x approaches a") provided that whenever $x \approx a$, we have that $f(x) \approx L$.
  - To make this formal, we need to explain what exactly "$\approx$" means.
  - For now, we take a look at an (informal) example.

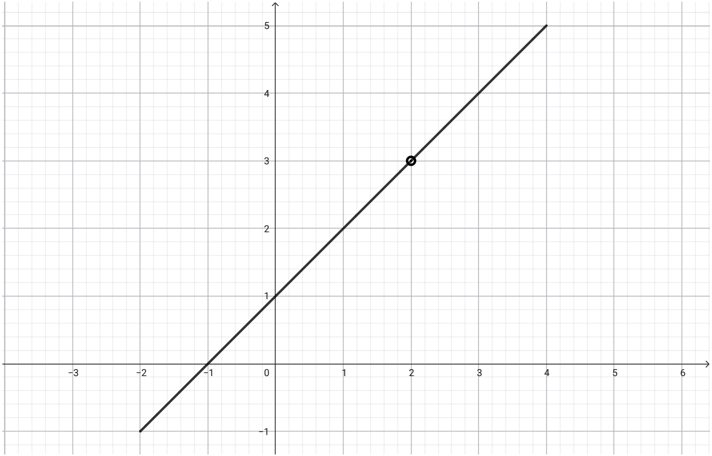
Example 3.1.1

Consider the function $f(x) = \frac{x^2-x-2}{x-2}$. The domain of this function is $\mathbb{R} \setminus \{2\}$ (since we cannot divide by zero). Now, for $x \in \mathbb{R} \setminus \{2\}$, we have that

$$f(x) = \frac{x^2-x-2}{x-2} = \frac{(x+1)(x-2)}{x-2} = x + 1.$$

As pointed out above, $f(2)$ is undefined. However (informally), for $x \approx 2$, we obviously have that $f(x) \approx 2 + 1 = 3$, and so $\lim_{x \to 2} f(x) = 3$.

- The graph of the function $f(x) = \frac{x^2-x-2}{x-2}$ (from Example 3.1.1) is shown below.



- The function is undefined at $x = 2$.
- However, for values of x close to 2, the value of $f(x)$ is close to 3, which is why $\lim_{x \to 2} f(x) = 3$.

2. The Limit of a function at a point: the formal definition

Definition

An *accumulation point* of a set $A \subseteq \mathbb{R}$ is a point $a \in \mathbb{R}$ (note that a may or may not belong to A) s.t. for all real numbers $\varepsilon > 0$, there exists some $a' \in A$ s.t. $0 < |a' - a| < \varepsilon$.[a]

[a] So, a is an accumulation point of A if and only one can find a points of A arbitrarily close to a (but distinct from a).

- **Remark:** By definition, for a set $A \subseteq \mathbb{R}$ and a real number a (which may or may not belong to A), we have that a is **not** an accumulation point if and only if there exists some $\varepsilon > 0$ s.t. the set A contain **no points** from interval $(a - \varepsilon, a + \varepsilon)$ other than possibly a itself.
  - So, a is an accumulation point of A if no such $\varepsilon > 0$ exists.

Example 3.2.1

Every real number is an accumulation point of $\mathbb{Q}$ in $\mathbb{R}$. Indeed, consider any $a \in \mathbb{R}$, and fix any real number $\varepsilon > 0$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we know that $(a - \varepsilon, a + \varepsilon) \setminus \{a\} = (a - \varepsilon, a) \cup (a, a + \varepsilon)$ contains a rational number q.[a] Then $q \in \mathbb{Q} \setminus \{a\}$, and moreover, $0 < |q - a| < \varepsilon$. So, all real numbers are indeed accumulation points of $\mathbb{R}$.

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[a] Indeed, by the density of $\mathbb{Q}$ in $\mathbb{R}$ we know that each of the intervals $(a - \varepsilon, a)$ and $(a, a + \varepsilon)$ contains a rational number.

Lemma 3.2.2

Let $A \subseteq \mathbb{R}$ and $a \in \mathbb{R}$. If there exist $p, q \in \mathbb{R}$ such that $p < a < q$ and $(p, q) \setminus \{a\} \subseteq A$,[a] then a is an accumulation point of A.

[a] Equivalently: $(p, a) \cup (a, q) \subseteq A$.

Proof. Assume that there exist some $p, q \in \mathbb{R}$ s.t. $p < a < q$ and $(p, q) \setminus \{a\} \subseteq A$.

Fix $\varepsilon > 0$. Set $p' = \max\{p, a - \varepsilon\}$ and $q' = \min\{q, a + \varepsilon\}$. Then $p' < a < q'$ and $(p', q') \setminus \{a\} \subseteq A$.

Now fix any $a' \in (p', q') \setminus \{a\}$. Then $a' \in A$ and $0 < |a' - a| < \varepsilon$. This proves that a is an accumulation point of A. □

Definition

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, let $a \in \mathbb{R}$ be an accumulation point of A,[a] and let $L \in \mathbb{R}$. We say that L *is the limit of* $f(x)$ *as* x *approaches* a, or that $f(x)$ *tends to* L *as* x *approaches* a, provided that the following holds:

> *for every* $\varepsilon > 0$, *there exists some* $\delta > 0$, *s.t. for all* $x \in A$, *if* $0 < |x - a| < \delta$, *then* $|f(x) - L| < \varepsilon$.

Under such circumstances, we write

$$L = \lim_{x \to a} f(x) \qquad \text{or} \qquad f(x) \to L \quad \text{as} \quad x \to a.$$

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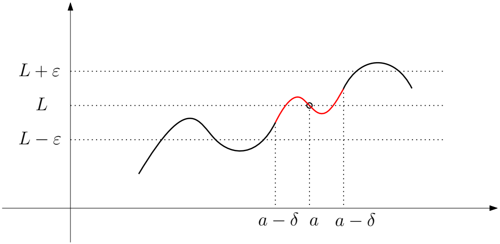
[a] In practice, we will most often encounter the situation from the statement of Lemma 3.2.2: the function f will be defined on some (possibly very small) open interval containing a, except possibly at a itself. Occasionally, we will see slightly more complicated situations.

- **Remark:** Suppose $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is a function, $a \in \mathbb{R}$ is an accumulation point of A, and L is a real number.
    - Then $L = \lim_{x \to a} f(x)$ means that for every $\varepsilon > 0$, we can choose some $\delta > 0$, s.t.

      $$\{f(x) \mid x \in A \setminus \{a\}, x \in (a - \delta, a + \delta)\} \quad \subseteq \quad (L - \varepsilon, L + \varepsilon),$$

      as in the picture below.



    - In other words, all elements of $A \setminus \{a\}$ inside the interval $(a - \delta, a + \delta)$ get mapped by f to the interval $(L - \varepsilon, L + \varepsilon)$.
    - The existence and value of $f(a)$ is irrelevant for the existence and value of $L = \lim_{x \to a} f(x)$.

- **Remark:** We emphasize that the existence and value of $\lim_{x \to a} f(x)$ is determined by the behavior of $f(x)$ for values of x **close to** a, but is unaffected by the existence and value $f(a)$ itself.
  - Indeed, each of the following in possible:
    - $f(a)$ and $\lim_{x \to a} f(x)$ are both defined, and $f(a) = \lim_{x \to a} f(x)$;
    - $f(a)$ and $\lim_{x \to a} f(x)$ are both defined, but $f(a) \neq \lim_{x \to a} f(x)$;
    - $f(a)$ is defined, but $\lim_{x \to a} f(x)$ is undefined;
    - $f(a)$ is undefined, but $\lim_{x \to a} f(x)$ is defined;
    - $f(a)$ and $\lim_{x \to a} f(x)$ are both undefined.

- **Remark:** In what follows, we will occasionally write something like "$\lim_{x \to a} f(x)$ exists (as a real number)."
  - Of course, so far, we have defined $\lim_{x \to a} f(x)$ to be a real number (if it exists).
  - However, we will later introduce the notion of infinite limits, and will introduce the notation "$\lim_{x \to a} f(x) = +\infty$" and "$\lim_{x \to a} f(x) = -\infty$."
  - Hence, we sometimes emphasize that we wish $\lim_{x \to a} f(x)$ to be a real number.

Definition

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, let $a \in \mathbb{R}$ be an accumulation point of A, and let $L \in \mathbb{R}$. We say that L *is the limit of* $f(x)$ *as* x *approaches* a, or that $f(x)$ *tends to* L *as* x *approaches* a, provided that the following holds:

> *for every* $\varepsilon > 0$*, there exists some* $\delta > 0$*, s.t. for all* $x \in A$*,*
> *if* $0 < |x - a| < \delta$*, then* $|f(x) - L| < \varepsilon$*.*

Under such circumstances, we write

$$L = \lim_{x \to a} f(x) \qquad \text{or} \qquad f(x) \to L \quad \text{as} \quad x \to a.$$

Theorem 3.2.3

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and let a be an accumulation point of A. Then $f(x)$ has at most one limit as x approaches a.

- Proof: later!
- First, we take a look at a few examples.

Example 3.2.4

Let $f(x) = \frac{x^2-x-2}{x-2}$. Compute $\lim\limits_{x \to 2} f(x)$ (and give an "ε-δ proof" of the correctness of your answer).

- Here, we are revisiting Example 3.1.1, but this time, we give a formal proof of the fact that $\lim\limits_{x \to 2} f(x) = 3$.

Solution. Clearly, f is defined on $A := \mathbb{R} \setminus \{2\}$. Our goal is to show that $\lim\limits_{x \to 2} f(x) = 3$.

Fix $\varepsilon > 0$. Let $\delta := \varepsilon$. Fix $x \in A$ s.t. $0 < |x-2| < \delta$. (In particular, $x \neq 2$.) Then

$$|f(x) - 3| \quad = \quad |\tfrac{x^2-x-2}{x-2} - 3| \quad = \quad |(x+1) - 3| \quad = \quad |x-2| \quad < \quad \delta \quad = \quad \varepsilon.$$

This proves that $\lim\limits_{x \to 2} f(x) = 3$, as we had claimed. $\square$

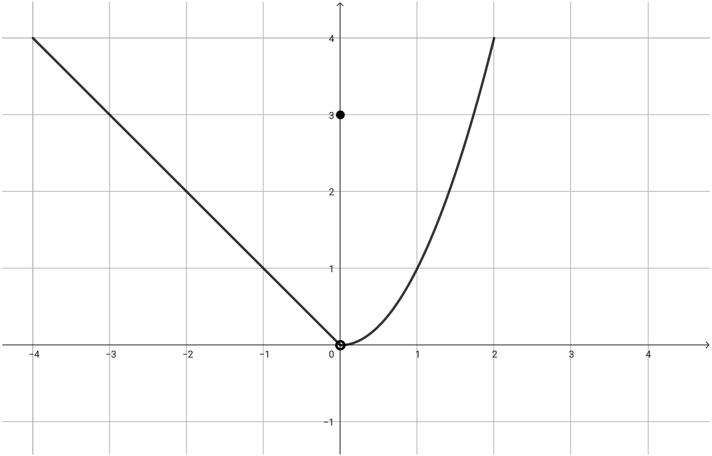
Example 3.2.5

Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} -x & \text{if} \quad x < 0 \\ 3 & \text{if} \quad x = 0 \\ x^2 & \text{if} \quad x > 0 \end{cases}$$

for all $x \in \mathbb{R}$. Either find $\lim_{x \to 0} f(x)$ (and give an "ε-δ proof" of the correctness of your answer), or explain why the limit does not exist (as a real number).

- **Remark:** The graph of this function is shown below.



- By definition, $f(0) = 3$.
- However, when x is very close to 0 (but not actually equal to 0), we have that $f(x)$ is very close to 0.
- So, $\lim_{x \to 0} f(x) = 0$. Our solution gives a formal proof of this fact.

- Reminder:

$$f(x) = \begin{cases} -x & \text{if} \quad x < 0 \\ 3 & \text{if} \quad x = 0 \\ x^2 & \text{if} \quad x > 0 \end{cases}$$

Solution. Our goal is to show that $\lim_{x \to 0} f(x) = 0$.

Fix $\varepsilon > 0$. Let $\delta = \min\{\varepsilon, \sqrt{\varepsilon}\}$. Fix $x \in \mathbb{R}$ s.t. $0 < |x - 0| < \delta$. WTS $|f(x) - 0| < \varepsilon$.

Since $0 < |x - 0| < \delta$, i.e. $0 < |x| < \delta$, we have that either $-\delta < x < 0$ or $0 < x < \delta$. We consider both cases.

Case 1: $-\delta < x < 0$. In this case, we have that $f(x) = -x$. We now compute:

$$|f(x) - 0| = |-x - 0| = |x| < \delta = \min\{\varepsilon, \sqrt{\varepsilon}\} \leq \varepsilon,$$

which is what we needed.

Case 2: $0 < x < \delta$. In this case, we have that $f(x) = x^2$ and $x^2 < \delta^2$. We now compute:

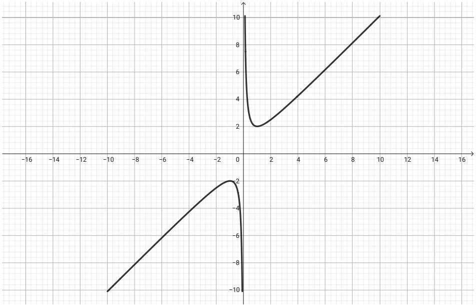
$$|f(x) - 0| = |x^2 - 0| = x^2 < \delta^2 = \min\{\varepsilon^2, \varepsilon\} \leq \varepsilon,$$

which is what we needed. □

Example 3.2.6

Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a given by $f(x) = x + \frac{1}{x}$. Either compute $\lim_{x \to 0} f(x)$, or prove that the limit does not exist (as a real number).

- **Remark:** The graph of the function f is shown below. As we can see, $\lim_{x \to 0} f(x)$ does not exist, which we will prove formally in our solution.



Example 3.2.6

Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a given by $f(x) = x + \frac{1}{x}$. Either compute $\lim_{x \to 0} f(x)$, or prove that the limit does not exist (as a real number).

Solution. We will prove that $\lim_{x \to 0} f(x)$ does not exist (as a real number). For this, we must prove the following:

> *For all $L \in \mathbb{R}$, there exists some $\varepsilon > 0$, s.t. for all $\delta > 0$, there exists some $x \in \mathbb{R} \setminus \{0\}$, s.t. $0 < |x - 0| < \delta$ and $|f(x) - L| \geq \varepsilon$.*

Fix $L \in \mathbb{R}$, and fix an arbitrary $\varepsilon > 0$.

- In this particular case, any $\varepsilon > 0$ will do. In other examples, ε may need to be carefully chosen.

Fix $\delta > 0$. We need to exhibit some $x \in \mathbb{R} \setminus \{0\}$ s.t. $0 < |x - 0| < \delta$ and $|f(x) - L| \geq \varepsilon$.

Example 3.2.6

Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a given by $f(x) = x + \frac{1}{x}$. Either compute $\lim_{x \to 0} f(x)$, or prove that the limit does not exist (as a real number).

Solution (continued).

Claim. *$\exists x \in \mathbb{R} \setminus \{0\}$ s.t. $0 < x < \delta$ and $\frac{1}{x} > L + \varepsilon$.*

Proof of the Claim. If $L + \varepsilon \leq 0$, then we simply choose any $x \in (0, \delta)$, and we observe that $0 < x < \delta$ and $\frac{1}{x} > 0 \geq L + \varepsilon$.

Suppose now that $L + \varepsilon > 0$. We then choose an arbitrary $x \in (0, \min\{\delta, \frac{1}{L+\varepsilon}\})$. Clearly, $0 < x < \delta$. On the other hand, we have that $0 < x < \frac{1}{L+\varepsilon}$, and consequently, $\frac{1}{x} > L + \varepsilon$. ♦

Example 3.2.6

Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a given by $f(x) = x + \frac{1}{x}$. Either compute $\lim_{x \to 0} f(x)$, or prove that the limit does not exist (as a real number).

Solution (continued). We have proven the following:

> ***Claim.*** *$\exists x \in \mathbb{R} \setminus \{0\}$ s.t. $0 < x < \delta$ and $\frac{1}{x} > L + \varepsilon$.*

Now, fix x as in the Claim above. Since $0 < x < \delta$, we have that $0 < |x - 0| < \delta$. Further, we compute:

$$|f(x) - L| \quad = \quad |x + \tfrac{1}{x} - L| \quad \geq \quad x + \tfrac{1}{x} - L \quad \overset{(*)}{>} \quad 0 + (L + \varepsilon) - L \quad = \quad \varepsilon,$$

where (*) follows from the fact that $x > 0$ and $\frac{1}{x} > L + \varepsilon$. This completes the argument. $\square$

Theorem 3.2.3

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and let a be an accumulation point of A. Then $f(x)$ has at most one limit as a approaches x.

Proof. Suppose otherwise, and let L_1 and L_2 be two distinct limits of $f(x)$ as x approaches a. Set $\varepsilon := \frac{|L_1 - L_2|}{2}$. (Obviously, $\varepsilon > 0$.)

Using the fact that L_1 is a limit of $f(x)$ as x approaches a, we fix $\delta_1 > 0$ s.t. for all $x \in A$, if $0 < |x - a| < \delta_1$, then $|f(x) - L_1| < \varepsilon$.

Using the fact that L_2 is a limit of $f(x)$ as x approaches a, we fix $\delta_2 > 0$ s.t. for all $x \in A$, if $0 < |x - a| < \delta_2$, then $|f(x) - L_2| < \varepsilon$.

Set $\delta := \min\{\delta_1, \delta_2\}$. (Obviously, $\delta > 0$.) Fix $x \in A$ s.t. $0 < |x - a| < \delta$. Then $0 < |x - a| < \delta_1$ and $0 < |x - a| < \delta_2$; the former implies that $|f(x) - L_1| < \varepsilon$, whereas the latter implies that $|f(x) - L_2| < \varepsilon$. We now have the following (next slide):

Theorem 3.2.3

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a function, and let a be an accumulation point of A. Then $f(x)$ has at most one limit as a approaches x.

Proof (continued). Reminder: $|f(x) - L_1| < \varepsilon$; $|f(x) - L_2| < \varepsilon$.

$$\begin{aligned}
|L_1 - L_2| &= \big|\big(L_1 - f(x)\big) + \big(f(x) - L_2\big)\big| \\
&\leq |L_1 - f(x)| + |f(x) - L_2| && \text{by the Triangle Inequality} \\
&= |f(x) - L_1| + |f(x) - L_2| \\
&< \varepsilon + \varepsilon \\
&= 2\varepsilon \\
&= |L_1 - L_2| && \text{by the choice of } \varepsilon,
\end{aligned}$$

a contradiction. □

3. A relationship between limits of functions and limits of sequences

- Reminder:

Definition

We say that a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers *converges* to a real number L provided that the following holds:

> *For all real numbers $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ s.t. for all $n \in \mathbb{N}$: if $n \geq N$, then $|a_n - L| < \varepsilon$.*

Under such circumstances, we say that L is the *limit* of the sequence $\{a_n\}_{n=1}^{\infty}$, and we write

$$L = \lim_{n \to \infty} a_n \qquad \text{or} \qquad a_n \to L \quad \text{as} \quad n \to \infty.$$

A sequence is *convergent* (or *converges*) if it has a limit. Otherwise, it is *divergent* (or *diverges*).

Definition

An *accumulation point* of a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers is a real number A s.t. for all real numbers $\varepsilon > 0$ and all $N \in \mathbb{N}$, there exists some $n \in \mathbb{N}$ s.t. $n \geq N$ and $|a_n - A| < \varepsilon$.

Proposition 2.6.1

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers, and let $A \in \mathbb{R}$. Then the following are equivalent:

(i) A is an accumulation point of $\{a_n\}_{n=1}^{\infty}$;

(ii) for all real numbers $\varepsilon > 0$, the interval $(A - \varepsilon, A + \varepsilon)$ contains infinitely many terms of the sequence $\{a_n\}_{n=1}^{\infty}$;

(iii) some subsequence of $\{a_n\}_{n=1}^{\infty}$ converges to A.

Theorem 2.6.2

Let $\{a_n\}_{n=1}^{\infty}$ be a **convergent** sequence of real numbers. Then $\{a_n\}_{n=1}^{\infty}$ has exactly one accumulation point, namely $L := \lim\limits_{n\to\infty} a_n$.

Theorem 3.3.1

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$, let $a \in \mathbb{R}$ be an accumulation point of the set A,and let $L \in \mathbb{R}$. Then the following are equivalent:

(i) $\lim\limits_{x \to a} f(x) = L$;

(ii) for all sequences $\{a_n\}_{n=1}^{\infty}$ of real numbers that all belong to the set $A \setminus \{a\}$, if $\lim\limits_{n\to\infty} a_n = a$, then $\lim\limits_{n\to\infty} f(a_n) = L$.

Proof. First, we assume (i) and prove (ii). Fix a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers in $A \setminus \{a\}$, and assume that $\lim\limits_{n\to\infty} a_n = a$. WTS $\lim\limits_{n\to\infty} f(a_n) = L$. Fix $\varepsilon > 0$. By (i), $\lim\limits_{x\to a} f(x) = L$. So, fix $\delta > 0$ s.t. $\forall x \in A$, if $0 < |x - a| < \delta$, then$|f(x) - L| < \varepsilon$. Now, using the fact that $\lim\limits_{n\to\infty} a_n = a$, we fix $N \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$, if $n \geq N$, then $|a_n - a| < \delta$. But recall that $a_n \in A \setminus \{a\}$ (so, $a_n \neq a$) $\forall n \in \mathbb{N}$. We conclude that $\forall n \in \mathbb{N}$, if $n \geq N$, then $0 < |a_n - a| < \delta$ (with $a_n \in A$), and therefore, by the choice of δ, we have that $|f(a_n) - L| < \varepsilon$. Thus, $\lim\limits_{n\to\infty} f(a_n) = L$, which proves (ii).

Theorem 3.3.1

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$, let $a \in \mathbb{R}$ be an accumulation point of the set A,and let $L \in \mathbb{R}$. Then the following are equivalent:

(i) $\lim\limits_{x \to a} f(x) = L$;

(ii) for all sequences $\{a_n\}_{n=1}^{\infty}$ of real numbers that all belong to the set $A \setminus \{a\}$, if $\lim\limits_{n \to \infty} a_n = a$, then $\lim\limits_{n \to \infty} f(a_n) = L$.

Proof (continued). We now assume that (i) is false, and we show that (ii) is false.

Using the fact that (i) is false, we fix $\varepsilon > 0$ s.t. for all $\delta > 0$, there exists some $x \in A$ s.t. $0 < |x - a| < \delta$ and $|f(x) - L| \geq \varepsilon$. Now, for all $n \in \mathbb{N}$, we choose some $a_n \in A$ s.t. $0 < |a_n - a| < \frac{1}{n}$ and $|f(a_n) - L| \geq \varepsilon$.

- Let us explain why such an a_n exists.
  - Here, we are setting $\delta := \frac{1}{n}$.
  - By our assumption, we know $\exists x \in A$ s.t. $0 < |x - a| < \delta$ and $|f(x) - L| \geq \varepsilon$; we let a_n be any such x.

Theorem 3.3.1

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$, let $a \in \mathbb{R}$ be an accumulation point of the set A,and let $L \in \mathbb{R}$. Then the following are equivalent:

1. (i) $\lim_{x \to a} f(x) = L$;
2. (ii) for all sequences $\{a_n\}_{n=1}^{\infty}$ of real numbers that all belong to the set $A \setminus \{a\}$, if $\lim_{n \to \infty} a_n = a$, then $\lim_{n \to \infty} f(a_n) = L$.

Proof (continued). Reminder: We assumed that (i) is false; WTS (ii) is false; we have chosen a suitable $\varepsilon > 0$, and we have constructed a sequence $\{a_n\}_{n=1}^{\infty}$ in $A \setminus \{a\}$ s.t. $\forall n \in \mathbb{N}$:
$0 < |a_n - a| < \frac{1}{n}$ and $|f(a_n) - L| \geq \varepsilon$.

Our goal is to show that $\lim_{n \to \infty} a_n = a$, but that the sequence $\{f(a_n)\}_{n=1}^{\infty}$ fails to converge to L; this will prove that (ii) is false.

Proof (continued). Reminder: $\{a_n\}_{n=1}^{\infty}$ is a sequence in $A \setminus \{a\}$ s.t. $\forall n \in \mathbb{N}$: $0 < |a_n - a| < \frac{1}{n}$ and $|f(a_n) - L| \geq \varepsilon$; WTS $\lim_{n\to\infty} a_n = a$, but $\{f(a_n)\}_{n=1}^{\infty}$ does not converge to L.

We first show that $\lim_{n\to\infty} a_n = a$. Fix $\varepsilon' > 0$. Let $N \in \mathbb{N}$ be s.t. $N > \frac{1}{\varepsilon'}$. Fix any $n \in \mathbb{N}$ s.t. $n \geq N$. Then

$$
\begin{aligned}
|a_n - a| &< \frac{1}{n} && \text{by the choice of } a_n \\
&\leq \frac{1}{N} && \text{because } n \geq N \\
&< \varepsilon' && \text{because } N > \frac{1}{\varepsilon'}.
\end{aligned}
$$

This proves that $\lim_{n\to\infty} a_n = a$.

Proof (continued). Reminder: $\{a_n\}_{n=1}^{\infty}$ is a sequence in $A \setminus \{a\}$ s.t. $\forall n \in \mathbb{N}$: $0 < |a_n - a| < \frac{1}{n}$ and $|f(a_n) - L| \geq \varepsilon$; WTS $\lim_{n\to\infty} a_n = a$, but $\{f(a_n)\}_{n=1}^{\infty}$ does not converge to L.

It remains to show that the sequence $\{f(a_n)\}_{n=1}^{\infty}$ does not converge to L. Reminder:

Proposition 2.6.1 (abbreviated)

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers, and let $A \in \mathbb{R}$. Then the following are equivalent:

- (i) A is an accumulation point of $\{a_n\}_{n=1}^{\infty}$;
- (ii) for all real numbers $\varepsilon > 0$, the interval $(A - \varepsilon, A + \varepsilon)$ contains infinitely many terms of the sequence $\{a_n\}_{n=1}^{\infty}$.

Theorem 2.6.2

Let $\{a_n\}_{n=1}^{\infty}$ be a **convergent** sequence of real numbers. Then $\{a_n\}_{n=1}^{\infty}$ has exactly one accumulation point, namely $L := \lim_{n\to\infty} a_n$.

Theorem 3.3.1

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$, let $a \in \mathbb{R}$ be an accumulation point of the set A,and let $L \in \mathbb{R}$. Then the following are equivalent:

1. $\lim\limits_{x \to a} f(x) = L$;
2. for all sequences $\{a_n\}_{n=1}^{\infty}$ of real numbers that all belong to the set $A \setminus \{a\}$, if $\lim\limits_{n \to \infty} a_n = a$, then $\lim\limits_{n \to \infty} f(a_n) = L$.

Proof (continued). Reminder: $\{a_n\}_{n=1}^{\infty}$ is a sequence in $A \setminus \{a\}$ s.t. $\forall n \in \mathbb{N}$: $0 < |a_n - a| < \frac{1}{n}$ and $|f(a_n) - L| \geq \varepsilon$; WTS $\{f(a_n)\}_{n=1}^{\infty}$ does not converge to L.

By the construction of our sequence $\{a_n\}_{n=1}^{\infty}$, we know that $\forall n \in \mathbb{N}$, we have that $|f(a_n) - L| \geq \varepsilon$, that is, $f(a_n) \notin (L - \varepsilon, L + \varepsilon)$. By Proposition 2.6.1, this implies that L is not an accumulation point of the sequence $\{f(a_n)\}_{n=1}^{\infty}$. Therefore, by Theorem 2.6.2, the sequence $\{f(a_n)\}_{n=1}^{\infty}$ does not converge to L. This completes the argument. $\square$

4. Properties of limits

Theorem 3.4.4

Let $f_1 : A \subseteq \mathbb{R} \to \mathbb{R}$ and $f_2 : A \subseteq \mathbb{R} \to \mathbb{R}$ be functions, and let $a \in \mathbb{R}$ be an accumulation point of A. Assume that both $\lim\limits_{x \to a} f_1(x)$ and $\lim\limits_{x \to a} f_2(x)$ exist. Then all the following hold:

a. for all $c \in \mathbb{R}$, $\lim\limits_{x \to a} (cf_1)(x) = c \lim\limits_{x \to a} f_1(x)$;
b. $\lim\limits_{x \to a} (f_1 + f_2)(x) = (\lim\limits_{x \to a} f_1(x)) + (\lim\limits_{x \to a} f_2(x))$;
c. $\lim\limits_{x \to a} (f_1 - f_2)(x) = (\lim\limits_{x \to a} f_1(x)) - (\lim\limits_{x \to a} f_2(x))$;
d. $\lim\limits_{x \to a} (f_1 f_2)(x) = (\lim\limits_{x \to a} f_1(x))(\lim\limits_{x \to a} f_2(x))$;
e. if $\lim\limits_{x \to a} f_2(x) \neq 0$, then a is an accumulation point of the set $A' := \{x \in A \mid f_2(x) \neq 0\}$, and we have that $\lim\limits_{x \to a} (\frac{f_1}{f_2})(x) = \frac{\lim\limits_{x \to a} f_1(x)}{\lim\limits_{x \to a} f_2(x)}$, where we consider the domain of the function $\frac{f_1}{f_2}$ to be A'.

- The proof of Theorem 3.4.4 is given in the Lecture Notes.
  - The proof of Theorem 3.4.4 is similar to the proof of the corresponding theorem for sequences (Theorem 2.2.5 from the Lecture Notes).
- Parts (a), (b), and (c) are straightforward, whereas parts (d) and (e) are somewhat technical.
  - The proofs of (d) and (e) rely on certain technical results from section 3.4 of the Lecture Notes, which we omit here.

5. Continuity

Definition

A function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is *continuous* at a point $a \in A$ if the following holds:

for all $\varepsilon > 0$, there exists some $\delta > 0$, s.t. for all $x \in A$, if $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$.

- **Remark:** Suppose that $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is a function, and $a \in A$, as in the definition above.
  - Note that if a is an accumulation point of A, then f is continuous at a if and only if $\lim_{x \to a} f(x) = f(a)$ (and in particular, $\lim_{x \to a} f(x)$ exists).
  - If a is **not** an accumulation point of A, then f is automatically continuous at a.

Definition

If a function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is continuous at all points $a \in A$, then we simply say that f is *continuous*. If $I \subseteq A$ is an interval, then we say that f is *continuous* on I provided that $f \restriction I$ is continuous at all points in I.

- **Remark:** Suppose that $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is a function and that $I \subseteq A$ is an interval.
  - If f is continuous on all points in I, then f is indeed continuous on I, as we would expect.
  - Somewhat surprisingly, the converse is false in general!
    - For instance, consider the function $f : \mathbb{R} \to \infty$ given by

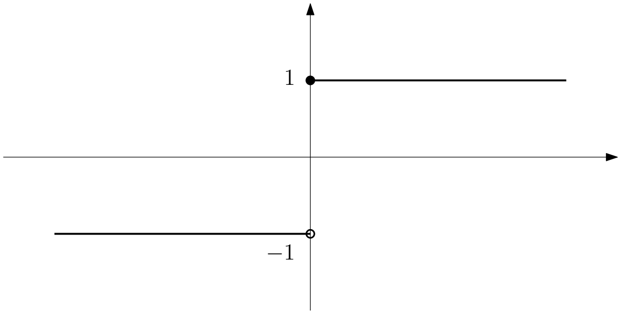
$$f(x) = \begin{cases} -1 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x \geq 0 \end{cases}$$

      for all $x \in \mathbb{R}$. (The graph of this function is on the next slide.)
    - Then f is **not** continuous at the point $x = 0$, but it **is** continuous at on interval $[0, +\infty)$.
  - This is because whether or not a function is continuous on an interval is determined solely by the behavior of the function on the interval in question, and not by its behavior elsewhere.

- The graph of the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} -1 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x \geq 0 \end{cases}$$

for all $x \in \mathbb{R}$.



Definition

A function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is *continuous* at a point $a \in A$ if the following holds:

> *for all $\varepsilon > 0$, there exists some $\delta > 0$, s.t. for all $x \in A$, if $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$.*

Definition

If a function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ is continuous at all points $a \in A$, then we simply say that f is *continuous*. If $I \subseteq A$ is an interval, then we say that f is *continuous* on I provided that $f \restriction I$ is continuous at all points in I.

- **Remark:** The intuition behind the concept of continuity is that a function is supposed to be continuous on an interval if its graph (restricted to the interval in question) can be drawn on a piece of paper without lifting the pen.
  - Of course, pens and paper are not mathematical objects!
  - Our definition attempts to formalize our intuition, although some functions that are continuous according to our formal definition are somewhat strange, and drawing their graphs (with our without lifting the pen) would be quite difficult.
  - We do, however, have the following fact, stated without proof.

Fact 3.5.1

The following functions are all continuous:

- polynomial functions,
- rational functions,
- root functions,
- exponential and logarithmic functions,
- trigonometric and inverse trigonometric functions.

- Reminder:

Theorem 3.4.4

Let $f_1 : A \subseteq \mathbb{R} \to \mathbb{R}$ and $f_2 : A \subseteq \mathbb{R} \to \mathbb{R}$ be functions, and let $a \in \mathbb{R}$ be an accumulation point of A. Assume that both $\lim_{x \to a} f_1(x)$ and $\lim_{x \to a} f_2(x)$ exist. Then all the following hold:

(a) for all $c \in \mathbb{R}$, $\lim_{x \to a} (cf_1)(x) = c \lim_{x \to a} f_1(x)$;

(b) $\lim_{x \to a} (f_1 + f_2)(x) = (\lim_{x \to a} f_1(x)) + (\lim_{x \to a} f_2(x))$;

(c) $\lim_{x \to a} (f_1 - f_2)(x) = (\lim_{x \to a} f_1(x)) - (\lim_{x \to a} f_2(x))$;

(d) $\lim_{x \to a} (f_1 f_2)(x) = (\lim_{x \to a} f_1(x))(\lim_{x \to a} f_2(x))$;

(e) if $\lim_{x \to a} f_2(x) \neq 0$, then a is an accumulation point of the set $A' := \{x \in A \mid f_2(x) \neq 0\}$, and we have that $\lim_{x \to a} (\frac{f_1}{f_2})(x) = \frac{\lim_{x \to a} f_1(x)}{\lim_{x \to a} f_2(x)}$, where we consider the domain of the function $\frac{f_1}{f_2}$ to be A'.

Theorem 3.5.2

Let $f_1 : A \subseteq \mathbb{R} \to \mathbb{R}$ and $f_2 : A \subseteq \mathbb{R} \to \mathbb{R}$ be functions that are continuous at a point $a \in A$. Then all the following hold:

a. for all $c \in \mathbb{R}$, the function cf_1 is continuous at a;
b. functions $f_1 + f_2$, $f_1 - f_2$, and $f_1 f_2$ are continuous at a;
c. if $f_2(a) \neq 0$, then the function $\frac{f_1}{f_2}$ is continuous at a.

Proof. If a is an accumulation point of A, then this readily follows from Theorem 3.4.4 and from the definition of continuity.

On the other hand, if a is **not** accumulation point of A, then any real-valued function whose domain is A (or any subset of A that contains the point a) is continuous at a. $\square$

Theorem 3.5.3

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and $g : B \subseteq \mathbb{R} \to \mathbb{R}$ (with $f[A] \subseteq B$, so that the function $g \circ f : A \subseteq \mathbb{R} \to \mathbb{R}$ is defined). Let $a \in \mathbb{R}$, and assume that g is continuous at $b := \lim_{x \to a} f(x)$ (in particular, this limit exists and belongs to B). Then:

(a) $\lim_{x \to a} (g \circ f)(x) = g(\lim_{x \to a} f(x)) = g(b)$;

(b) if f is continuous at a, then $g \circ f$ is continuous at a.

Proof. We begin by observing that a is an accumulation point of A, since $\lim_{x \to a} f(x)$ exists.

Theorem 3.5.3

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and $g : B \subseteq \mathbb{R} \to \mathbb{R}$ (with $f[A] \subseteq B$, so that the function $g \circ f : A \subseteq \mathbb{R} \to \mathbb{R}$ is defined). Let $a \in \mathbb{R}$, and assume that g is continuous at $b := \lim\limits_{x \to a} f(x)$ (in particular, this limit exists and belongs to B). Then:

- (a) $\lim\limits_{x \to a} (g \circ f)(x) = g(\lim\limits_{x \to a} f(x)) = g(b)$;
- (b) if f is continuous at a, then $g \circ f$ is continuous at a.

Proof (continued). We first prove (a). Fix $\varepsilon > 0$. Using the fact that g is continuous at b, we fix some $\varepsilon' > 0$ s.t. for all $x \in B$, if $|x - b| < \varepsilon'$, then $|g(x) - g(b)| < \varepsilon$.

Next, using the fact that $b = \lim\limits_{x \to a} f(a)$, we fix some $\delta > 0$ s.t. for all $x \in A$, if $0 < |x - a| < \delta$, then $|f(x) - b| < \varepsilon'$.

Now, fix $x \in A$ s.t. $0 < |x - a| < \delta$. Then $|f(x) - b| < \varepsilon'$. Since $f[A] \subseteq B$, we see that $f(x) \in B$. Now by the choice of ε', we have that $|g(f(x)) - g(b)| < \varepsilon$, i.e. $|(g \circ f)(x) - g(b)| < \varepsilon$. This proves (a).

Theorem 3.5.3

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and $g : B \subseteq \mathbb{R} \to \mathbb{R}$ (with $f[A] \subseteq B$, so that the function $g \circ f : A \subseteq \mathbb{R} \to \mathbb{R}$ is defined). Let $a \in \mathbb{R}$, and assume that g is continuous at $b := \lim_{x \to a} f(x)$ (in particular, this limit exists and belongs to B). Then:

a. $\lim_{x \to a} (g \circ f)(x) = g(\lim_{x \to a} f(x)) = g(b)$;
b. if f is continuous at a, then $g \circ f$ is continuous at a.

Proof (continued). For (b), we simply observe that, if f is continuous at a, then

$$\begin{aligned} \lim_{x \to a} (g \circ f)(x) &= g(\lim_{x \to a} f(x)) && \text{by (a)} \\ &= g(f(a)) && \text{because } f \text{ is continuous at } a \\ &= (g \circ f)(a), \end{aligned}$$

which by definition means that $g \circ f$ is continuous at a. □