Mathematical Analysis 1

Lecture #5

Series (part II)

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March 9, 2026

- This lecture has five parts:
    1. A review of series (from the previous lecture)
    2. The p-series
    3. Absolute convergence
    4. Rearranging the terms of a series
    5. The Ratio Test and the Root Test
    6. Defining functions via series: the exponential function

1. A review of series (from the previous lecture)

- Suppose we are given a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers.
  - We can then form the following infinite sequence of *partial sums*, as follows:
    - $\sum_{n=1}^{1} a_n = a_1;$
    - $\sum_{n=1}^{2} a_n = a_1 + a_2;$
    - $\sum_{n=1}^{3} a_n = a_1 + a_2 + a_3;$
    - $\ldots$

- Suppose we are given a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers.
  - We can also consider the associated *infinite series* (or simply *series*) $\sum_{n=1}^{\infty} a_n$ (also denoted $a_1 + a_2 + a_3 + \dots$).
  - We say that the series $\sum_{n=1}^{\infty} a_n$ *converges* (or is *convergent*) if the sequence $\Big\{ \sum_{n=1}^{k} a_n \Big\}_{k=1}^{\infty}$ of partial sums converges; otherwise, the series $\sum_{n=1}^{\infty} a_n$ *diverges* (or is *divergent*).
  - If the series $\sum_{n=1}^{\infty} a_n$ converges, then its *sum* is the limit $s := \lim_{k\to\infty} \Big(\sum_{n=1}^{k} a_n \Big)$, and we write $\sum_{n=1}^{\infty} a_n = s$.

- Suppose we are given a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers.
  - If $\lim_{k\to\infty}\Big(\sum_{n=1}^{k} a_n\Big) = +\infty$, then we say that the series $\sum_{n=1}^{\infty} a_n$ diverges to $+\infty$, and we write $\sum_{n=1}^{\infty} a_n = +\infty$.
  - Similarly, if $\lim_{k\to\infty}\Big(\sum_{n=1}^{k} a_n\Big) = -\infty$, then we say that the series $\sum_{n=1}^{\infty} a_n$ diverges to $-\infty$, and we write $\sum_{n=1}^{\infty} a_n = -\infty$.

- **Remark:** Sometimes, it is convenient to start our series at an index other than $n = 1$.
  - So, we may get series of the form $\sum_{n=0}^{\infty} a_n$, or $\sum_{n=5}^{\infty} a_n$, or even $\sum_{n=-10}^{\infty} a_n$.
  - Convergence and divergence of such series, as well as their sums, are defined in a natural way.
- Let us now recall Proposition 2.9.1 and Theorem 2.9.2.

Proposition 2.9.1

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers, and let $c \in \mathbb{R}$. Then:

(a) if $\sum_{n=1}^{\infty} a_n$ converges, then so does $\sum_{n=1}^{\infty} (ca_n)$, and in that case, we have that $\sum_{n=1}^{\infty} (ca_n) = c(\sum_{n=1}^{\infty} a_n)$;

(b) if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge, then so does $\sum_{n=1}^{\infty} (a_n + b_n)$, and in that case, we have that $\sum_{n=1}^{\infty} (a_n + b_n) = (\sum_{n=1}^{\infty} a_n) + (\sum_{n=1}^{\infty} b_n)$;

(c) if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge, then so does $\sum_{n=1}^{\infty} (a_n - b_n)$, and in that case, we have that $\sum_{n=1}^{\infty} (a_n - b_n) = (\sum_{n=1}^{\infty} a_n) - (\sum_{n=1}^{\infty} b_n)$;

Theorem 2.9.2

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

- **Remark:** The converse of Theorem 2.9.2 is **false**: it is possible that $\lim_{n\to\infty} a_n = 0$, but that the series $\sum_{n=1}^{\infty} a_n$ still diverges.

- A *geometric series* is a series of the form

$$\sum_{n=0}^{\infty} bq^n \;=\; b + bq + bq^2 + bq^3 + bq^4 + \dots,$$

  where $b, q \in \mathbb{R}$, with $b \neq 0$.
- We proved that this series converges iff $|q| < 1$, and in this case, the sum of the series is

$$\sum_{n=0}^{\infty} bq^n \;=\; \tfrac{b}{1-q}.$$

The Comparison Test

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers s.t. $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$. Then:

(a) if the series $\sum_{n=1}^{\infty} b_n$ converges, then so does the series $\sum_{n=1}^{\infty} a_n$, and moreover, $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$;

(b) if the series $\sum_{n=1}^{\infty} a_n$ diverges, then so does the series $\sum_{n=1}^{\infty} b_n$.

- Using the Comparison Test, we showed that the *harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n} \;=\; 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \ldots$$

diverges.

The Alternating Series Test

Let $\{a_n\}_{n=1}^{\infty}$ be the a non-increasing sequence of non-negative real numbers s.t. $\lim_{n\to\infty} a_n = 0$. Then the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n \quad = \quad a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \dots$$

converges.

- **Remark:** Here, we are assuming that the following are satisfied:
  - $a_n \geq 0$ for all $n \in \mathbb{N}$;
  - $a_1 \geq a_2 \geq a_3 \geq a_4 \geq \dots;$
  - $\lim_{n\to\infty} a_n = 0.$

  The Alternating Series Test states that, under these circumstances, the "alternating series" $\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \dots$ converges.

The Alternating Series Test

Let $\{a_n\}_{n=1}^{\infty}$ be the a non-increasing sequence of non-negative real numbers s.t. $\lim_{n\to\infty} a_n = 0$. Then the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n \quad = \quad a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \dots$$

converges.

- Using the Alternating Series Test, we proved that the *alternating harmonic series*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \quad = \quad 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \dots$$

converges.

2. The p-series

- A *p-series* is a series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p},$$

where p is some (fixed) real number.

- Note that for $p = 1$, we simply obtain the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ (which diverges).
- We will prove the following theorem.

Theorem 2.9.3

Let $p \in \mathbb{R}$.

(a) If $p > 1$, then the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges;

(b) If $p \leq 1$, then the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

Theorem 2.9.3

Let $p \in \mathbb{R}$.

(a) If $p > 1$, then the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges;

(b) If $p \leq 1$, then the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

Proof. We first prove (b). Assume that $p \leq 1$. Then for all $n \in \mathbb{N}$, we have that $0 < \frac{1}{n} \leq \frac{1}{n^p}$. Since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the Comparison Test now guarantees that the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ also diverges. This proves (b).

Theorem 2.9.3

Let $p \in \mathbb{R}$.

(a) If $p > 1$, then the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges;

(b) If $p \leq 1$, then the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

Proof (continued). It remains to prove (a). Assume that $p > 1$. WTS $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges. For all $n \in \mathbb{N}$, set

$$a_n \ := \ 2^{-pk},$$

where k is the (unique) non-negative integer s.t. $2^k \leq n \leq 2^{k+1} - 1$.

Our goal is to show that $0 < \frac{1}{n^p} \leq a_n$ for all $n \in \mathbb{N}$, and that the series $\sum_{n=1}^{\infty} a_n$ converges; the Comparison Test will then guarantee that the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ also converges, which is what we need.

Proof (continued). Reminder: $p > 1$; $\forall n \in \mathbb{N}$: $a_n = 2^{-pk}$, where k is the (unique) non-negative integer s.t. $2^k \leq n \leq 2^{k+1} - 1$.

Claim 1. *For all $n \in \mathbb{N}$, we have that $0 < \frac{1}{n^p} \leq a_n$.*

Proof of Claim 1. Fix $n \in \mathbb{N}$. Obviously, we have $0 < \frac{1}{n^p}$, and we just need to show that $\frac{1}{n^p} \leq a_n$. Let k be the unique non-negative integer s.t. $2^k \leq n \leq 2^{k+1} - 1$, so that $a_n = 2^{-pk}$. We now have the following sequence of equivalent statements:

$$\begin{array}{rcll} \frac{1}{n^p} \leq a_n & \iff & \frac{1}{n^p} \leq 2^{-pk} & \text{because } a_n = 2^{-pk} \\ & \iff & (\frac{1}{n})^p \leq (\frac{1}{2^k})^p & \\ & \iff & \frac{1}{n} \leq \frac{1}{2^k} & \text{because } p > 0 \\ & \iff & 2^k \leq n. & \end{array}$$

The last statement ("$2^k \leq n$") is true by the choice of k. So, the first statement ("$\frac{1}{n^p} \leq a_n$") is true as well. ♦

Proof (continued). Reminder: $p > 1$; $\forall n \in \mathbb{N}$: $a_n = 2^{-pk}$, where k is the (unique) non-negative integer s.t. $2^k \leq n \leq 2^{k+1} - 1$.

Claim 2. *The series* $\sum_{n=1}^{\infty} a_n$ *converges.*

Proof of Claim 2. We compute:

$$\sum_{n=1}^{\infty} a_n = \sum_{k=0}^{\infty}\Big(\sum_{n=2^k}^{2^{k+1}-1} a_n\Big) = \sum_{k=0}^{\infty}\Big(\sum_{n=2^k}^{2^{k+1}-1} 2^{-pk}\Big)$$

$$= \sum_{k=0}^{\infty}\Big((2^{k+1} - 2^k)2^{-pk}\Big) = \sum_{k=0}^{\infty}(2^k \cdot 2^{-pk})$$

$$= \sum_{k=0}^{\infty} 2^{k-pk} = \sum_{k=0}^{\infty}(2^{1-p})^k$$

Proof (continued). Reminder: $p > 1$; $\forall n \in \mathbb{N}$: $a_n = 2^{-pk}$, where k is the (unique) non-negative integer s.t. $2^k \leq n \leq 2^{k+1} - 1$.

Claim 2. *The series* $\sum_{n=1}^{\infty} a_n$ *converges.*

Proof of Claim 2 (continued). Reminder: $\sum_{n=1}^{\infty} a_n = \sum_{k=0}^{\infty} (2^{1-p})^k$.

But note that $\sum_{k=0}^{\infty} (2^{1-p})^k$ is a geometric series.

- To see this, simply set $b := 1$ and $q := 2^{1-p}$.

Since $p > 1$, we have that $1 - p < 0$, and we deduce that $0 < 2^{1-p} < 1$.

- Thus, for our $q := 2^{1-p}$, we have that $|q| < 1$.

Therefore, the geometric series $\sum_{k=0}^{\infty} (2^{1-p})^k$ converges, and consequently, the series $\sum_{n=1}^{\infty} a_n$ converges as well. ♦

Theorem 2.9.3

Let $p \in \mathbb{R}$.

- (a) If $p > 1$, then the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges;
- (b) If $p \leq 1$, then the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

Proof (continued). Reminder: $p > 1$; $\forall n \in \mathbb{N}$: $a_n = 2^{-pk}$, where k is the (unique) non-negative integer s.t. $2^k \leq n \leq 2^{k+1} - 1$.

Claim 1. *For all $n \in \mathbb{N}$, we have that $0 < \frac{1}{n^p} \leq a_n$.*

Claim 2. *The series $\sum_{n=1}^{\infty} a_n$ converges.*

Claims 1 and 2 and the Comparison Test now guarantee that the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges. This proves (a). $\square$

3. Absolute convergence

Definition

For a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers, the series $\sum_{n=1}^{\infty} a_n$ is said to *converge absolutely* (or to be *absolutely convergent*) if the series $\sum_{n=1}^{\infty} |a_n|$ converges. A series *converges conditionally* (or is *conditionally convergent*) if it is convergent, but not absolutely convergent.

Theorem 2.9.4

Every absolutely convergent series is convergent. In other words, for every sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers, if the series $\sum_{n=1}^{\infty} |a_n|$ converges, then so does the series $\sum_{n=1}^{\infty} a_n$.

- We will prove Theorem 2.9.4. But first, a couple of remarks.

Definition

For a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers, the series $\sum_{n=1}^{\infty} a_n$ is said to *converge absolutely* (or to be *absolutely convergent*) if the series $\sum_{n=1}^{\infty} |a_n|$ converges. A series *converges conditionally* (or is *conditionally convergent*) if it is convergent, but not absolutely convergent.

- **Remark:** Obviously, if $\{a_n\}_{n=1}^{\infty}$ is a sequence of **non-negative** real numbers, so that $|a_n| = a_n$ for all $n \in \mathbb{N}$, then the series $\sum_{n=1}^{\infty} a_n$ cannot be conditionally convergent, and instead, it is convergent iff it is absolutely convergent.

Definition

For a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers, the series $\sum_{n=1}^{\infty} a_n$ is said to *converge absolutely* (or to be *absolutely convergent*) if the series $\sum_{n=1}^{\infty} |a_n|$ converges. A series *converges conditionally* (or is *conditionally convergent*) if it is convergent, but not absolutely convergent.

- **Remark:** Not all convergent series are absolutely convergent.
  - For example, the alternating harmonic series

    $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

    converges, but the series

    $$\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots$$

    is simply the harmonic series, which diverges.
  - This means that the alternating harmonic series is conditionally convergent.

Theorem 2.9.4

Every absolutely convergent series is convergent. In other words, for every sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers, if the series $\sum\limits_{n=1}^{\infty} |a_n|$ converges, then so does the series $\sum\limits_{n=1}^{\infty} a_n$.

Proof. Fix a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers, and assume that the series $\sum\limits_{n=1}^{\infty} |a_n|$ converges. Consequently (by Proposition 2.9.1(a)), the series $\sum\limits_{n=1}^{\infty} 2|a_n|$ also converges. Now, note that for all $n \in \mathbb{N}$, we have that either $a_n = |a_n|$ or $a_n = -|a_n|$, and consequently, $0 \leq a_n + |a_n| \leq 2|a_n|$. Since the series $\sum\limits_{n=1}^{\infty} 2|a_n|$ converges, the Comparison Test guarantees that the series $\sum\limits_{n=1}^{\infty} \left(a_n + |a_n|\right)$ also converges.

Theorem 2.9.4

Every absolutely convergent series is convergent. In other words, for every sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers, if the series $\sum_{n=1}^{\infty} |a_n|$ converges, then so does the series $\sum_{n=1}^{\infty} a_n$.

Proof. Reminder: series $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} (a_n + |a_n|)$ converge.

But now

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \Big((a_n + |a_n|) - |a_n| \Big)$$

$$\overset{\text{Prop. 2.9.1(c)}}{=} \underbrace{\left(\sum_{n=1}^{\infty} (a_n + |a_n|) \right)}_{\text{converges}} - \underbrace{\left(\sum_{n=1}^{\infty} |a_n| \right)}_{\text{converges}},$$

and it follows that $\sum_{n=1}^{\infty} a_n$ converges. $\square$

4. Rearranging the terms of a series

- We note that rearranging the terms of a convergent series may alter the sum of the series, or even produce a divergent series.
    - Let us take a look at an example.
    - Recall that the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \dots$$

    converges.
- Let us show that simply rearranging the terms of this series can produce a divergent series.

Set

$$s \;:=\; \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \;=\; 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \dots.$$

Note that $1 > \frac{1}{2}$, $\frac{1}{3} > \frac{1}{4}$, $\frac{1}{5} > \frac{1}{6}$, and so on; therefore, $s > 0$. Now, by multiplying both sides of the equality by $\frac{1}{2}$, we obtain

$$\frac{s}{2} \;=\; \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \frac{1}{16} + \dots.$$

By inserting some zeros, we obtain

$$\tfrac{s}{2} \;=\; 0 + \tfrac{1}{2} + 0 - \tfrac{1}{4} + 0 + \tfrac{1}{6} + 0 - \tfrac{1}{8} + 0 + \tfrac{1}{10} + 0 - \tfrac{1}{12} + 0 + \tfrac{1}{14} + 0 - \tfrac{1}{16} + \dots$$

Now we sum up the following two equalities:

$$s \;=\; 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \tfrac{1}{7} - \tfrac{1}{8} + \tfrac{1}{9} - \tfrac{1}{10} + \tfrac{1}{11} - \tfrac{1}{12} + \tfrac{1}{13} - \tfrac{1}{14} + \tfrac{1}{15} - \tfrac{1}{16} + \dots$$

$$\tfrac{s}{2} \;=\; 0 + \tfrac{1}{2} + 0 - \tfrac{1}{4} + 0 + \tfrac{1}{6} + 0 - \tfrac{1}{8} + 0 + \tfrac{1}{10} + 0 - \tfrac{1}{12} + 0 + \tfrac{1}{14} + 0 - \tfrac{1}{16} + \dots$$

to obtain

$$\begin{aligned} \tfrac{3s}{2} &= 1 + 0 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + 0 + \tfrac{1}{7} - \tfrac{1}{4} + \tfrac{1}{9} + 0 + \tfrac{1}{11} - \tfrac{1}{6} + \tfrac{1}{13} + 0 + \tfrac{1}{15} - \tfrac{1}{8} + \dots \\ &= 1 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{11} - \tfrac{1}{6} + \tfrac{1}{13} + \tfrac{1}{15} - \tfrac{1}{8} + \dots \end{aligned}$$

We have now obtained:

$$\begin{aligned} s &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \frac{1}{13} - \frac{1}{14} + \frac{1}{15} - \frac{1}{16} + \dots \\ \frac{3s}{2} &= 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} + \frac{1}{15} - \frac{1}{8} + \dots \end{aligned}$$

Note that the last series that we obtained has exactly the same terms as the alternating harmonic series, but in a different order. However, the sum of this last series is $\frac{3s}{2}$, rather than s. (Since $s \neq 0$, we see that $\frac{3s}{2} \neq s$, i.e. the sum has indeed changed.)

- The phenomenon that we observed above may only happen with conditionally convergent series.
  - In the case of absolutely convergent series, rearranging the terms will never alter the sum of the series (see Theorem 2.9.6 in a couple of slides).
  - This is one of the reasons why we care about absolute convergence.

- Recall that a *permutation* of a set X is simply a bijection from X to X.
- For a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers, a *rearrangement* of the series $\sum_{n=1}^{\infty} a_n$ is any series of the form

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = a_{\sigma(1)} + a_{\sigma(2)} + a_{\sigma(3)} + a_{\sigma(4)} + \dots,$$

where σ is a permutation of $\mathbb{N}$.

  - For example, if $\sigma : \mathbb{N} \to \mathbb{N}$ is given by

$$\sigma(n) = \begin{cases} n+1 & \text{if} \quad n \text{ is odd} \\ n-1 & \text{if} \quad n \text{ is even} \end{cases}$$

for all $n \in \mathbb{N}$ (so that σ swaps 1 and 2, swaps 3 and 4, swaps 5 and 6, and so on), then we have that

$$\begin{aligned} \sum_{n=1}^{\infty} a_{\sigma(n)} &= a_{\sigma(1)} + a_{\sigma(2)} + a_{\sigma(3)} + a_{\sigma(4)} + a_{\sigma(5)} + a_{\sigma(6)} + \dots \\ &= a_2 + a_1 + a_4 + a_3 + a_6 + a_5 + \dots \end{aligned}$$

Proposition 2.9.5

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non-negative real numbers, and let σ be a permutation of $\mathbb{N}$. Then the series $\sum_{n=1}^{\infty} a_n$ converges iff the series $\sum_{n=1}^{\infty} a_{\sigma(n)}$ converges, and in this case, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\sigma(n)}$.

- Proof: Lecture Notes.
- The proof relies on Lemma 2.3.1 (below).

Lemma 2.3.1

Let $\{a_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of real numbers bounded above. Then $\{a_n\}_{n=1}^{\infty}$ converges, and its limit is precisely the supremum of the sequence.

Proposition 2.9.5

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non-negative real numbers, and let σ be a permutation of $\mathbb{N}$. Then the series $\sum_{n=1}^{\infty} a_n$ converges iff the series $\sum_{n=1}^{\infty} a_{\sigma(n)}$ converges, and in this case, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\sigma(n)}$.

- Informally, the idea of the proof is that the sequences of partial sums of the two series are both non-decreasing (because $a_n \geq 0 \, \forall n \in \mathbb{N}$), and if one is bounded above, then so is the other.
- So, if one of them (i.e. of the two sequences of partial sums) converges, then it converges to its least upper bound s (by Lemma 2.3.1), and it can be shown that s is also the least upper bound of the other one.
- In this case, s is the limit of this other sequence as well, by Lemma 2.3.1.

Theorem 2.9.6

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers s.t. the series $\sum\limits_{n=1}^{\infty} a_n$ converges **absolutely**. Then for all permutations σ of $\mathbb{N}$, the series $\sum\limits_{n=1}^{\infty} a_{\sigma(n)}$ converges absolutely, and moreover, $\sum\limits_{n=1}^{\infty} a_{\sigma(n)} = \sum\limits_{n=1}^{\infty} a_n$.

Proof. Fix a permutation σ of $\mathbb{N}$.

By assumption, $\sum\limits_{n=1}^{\infty} |a_n|$ converges; since all terms of the sequence $\{|a_n|\}_{n=1}^{\infty}$ are non-negative, Proposition 2.9.5 guarantees that $\sum\limits_{n=1}^{\infty} |a_{\sigma(n)}|$ converges, i.e. that the series $\sum\limits_{n=1}^{\infty} a_{\sigma(n)}$ converges absolutely.

Proof (continued). We define sequences $\{a_n^+\}_{n=1}^\infty$ and $\{a_n^+\}_{n=1}^\infty$ by setting

$$a_n^+ \quad := \quad \begin{cases} |a_n| & \text{if} \quad a_n \geq 0 \\ 0 & \text{if} \quad a_n < 0 \end{cases} \qquad \text{and} \qquad a_n^- \quad := \quad \begin{cases} 0 & \text{if} \quad a_n \geq 0 \\ |a_n| & \text{if} \quad a_n < 0 \end{cases}$$

for all $n \in \mathbb{N}$. We observe that $a_n = a_n^+ - a_n^-$ for all $n \in \mathbb{N}$.

Obviously, $0 \leq a_n^+, a_n^- \leq |a_n|$ for all $n \in \mathbb{N}$; since $\sum_{n=1}^{\infty} |a_n|$ converges (by assumption), the Comparison Test guarantees that $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ both converge. Since all terms of the sequences $\{a_n^+\}_{n=1}^\infty$ and $\{a_n^-\}_{n=1}^\infty$ are non-negative, Proposition 2.9.5 guarantees that

$$\sum_{n=1}^{\infty} a_{\sigma(n)}^+ \quad = \quad \sum_{n=1}^{\infty} a_n^+ \qquad \text{and} \qquad \sum_{n=1}^{\infty} a_{\sigma(n)}^- \quad = \quad \sum_{n=1}^{\infty} a_n^- .$$

Proof (continued). Reminder: $a_n = a_n^+ - a_n^-$ for all $n \in \mathbb{N}$; $\sum_{n=1}^{\infty} a_{\sigma(n)}^+ = \sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_{\sigma(n)}^- = \sum_{n=1}^{\infty} a_n^-$.

It now follows that:

$$\begin{aligned}
\sum_{n=1}^{\infty} a_{\sigma(n)} &= \sum_{n=1}^{\infty} \left(a_{\sigma(n)}^+ - a_{\sigma(n)}^-\right) \\
&= \left(\sum_{n=1}^{\infty} a_{\sigma(n)}^+\right) - \left(\sum_{n=1}^{\infty} a_{\sigma(n)}^-\right) \\
&= \left(\sum_{n=1}^{\infty} a_n^+\right) - \left(\sum_{n=1}^{\infty} a_n^-\right) \\
&= \sum_{n=1}^{\infty} \left(a_n^+ - a_n^-\right) = \sum_{n=1}^{\infty} a_n,
\end{aligned}$$

and we are done. □

Theorem 2.9.6

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers s.t. the series $\sum_{n=1}^{\infty} a_n$ converges **absolutely**. Then for all permutations σ of $\mathbb{N}$, the series $\sum_{n=1}^{\infty} a_{\sigma(n)}$ converges absolutely, and moreover, $\sum_{n=1}^{\infty} a_{\sigma(n)} = \sum_{n=1}^{\infty} a_n$.

- For conditionally convergent series, we have the following:

The Riemann Rearrangement Theorem

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers s.t. the series $\sum_{n=1}^{\infty} a_n$ converges **conditionally**. Then all the following hold:

(a) for all $s \in \mathbb{R}$, there exists a permutation σ of $\mathbb{N}$ s.t. $\sum_{n=1}^{\infty} a_{\sigma(n)} = s$;

(b) there exists a permutation σ of $\mathbb{N}$ s.t. $\sum_{n=1}^{\infty} a_{\sigma(n)} = +\infty$;

(c) there exists a permutation σ of $\mathbb{N}$ s.t. $\sum_{n=1}^{\infty} a_{\sigma(n)} = -\infty$;

(d) there exists a permutation σ of $\mathbb{N}$ s.t. $\sum_{n=1}^{\infty} a_{\sigma(n)}$ diverges (but neither to $+\infty$ nor to $-\infty$).

- Proof: Omitted.

5. The Ratio Test and the Root Test

- First, we state the Ratio Test, and we give some examples.
- Then, we state the Root Test, and we give some example.
- Finally, we prove the Root Test.
    - The proof of the Ratio Test is in the Lecture Notes.
    - It is similar to the proof of the Root Test, but a bit more complicated.

The Ratio Test

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers, and assume that $L := \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists (either as a real number or as $+\infty$).

(a) If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely;

(b) If $L > 1$,[a] then the series $\sum_{n=1}^{\infty} a_n$ diverges.

[a] Possibly, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = +\infty$.

- **Remark:** If $L = 1$, then the Ratio Test gives us no information about the convergence/divergence of the series $\sum_{n=1}^{\infty} a_n$.

Example 2.9.7

Consider the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^7}{7^n}$. Note that

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} \frac{(n+1)^7}{7^{n+1}}}{(-1)^n \frac{n^7}{7^n}} \right| = \lim_{n\to\infty} \frac{(n+1)^7}{7n^7} = \lim_{n\to\infty} \frac{(1+\frac{1}{n})^7}{7} = \frac{1}{7} < 1,$$

and so by the Ratio Test, the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^7}{7^n}$ converges absolutely.

Example 2.9.8

Consider the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$. Note that

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} \right| = \lim_{n\to\infty} \frac{(n+1)^n}{n^n} = \lim_{n\to\infty} (1 + \frac{1}{n})^n = e > 1,$$

and so by the Ratio Test, the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges.

The Ratio Test

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers, and assume that $L := \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists (either as a real number or as $+\infty$).

(a) If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely;

(b) If $L > 1$,[a] then the series $\sum_{n=1}^{\infty} a_n$ diverges.

[a] Possibly, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = +\infty$.

- **Remark:** As pointed out above, if $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers s.t. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, then the Ratio Test gives us no information about the convergence/divergence of the series $\sum_{n=1}^{\infty} a_n$, and in principle, it is possible that the series converges absolutely, converges conditionally, or diverges.
  - For example, consider the following three series (next slide):

1. the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, and we have that

$$\lim_{n\to\infty} \left| \frac{\frac{1}{n+1}}{\frac{1}{n}} \right| = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1;$$

2. the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges conditionally, and we have that

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^n}{n+1}}{\frac{(-1)^{n-1}}{n}} \right| = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1;$$

3. the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges (absolutely) or diverges depending on the value of the real number p; however, for all real numbers p, we have that

$$\lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)^p}}{\frac{1}{n^p}} \right| = \lim_{n\to\infty} \frac{n^p}{(n+1)^p} = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^p} = 1.$$

The Root Test

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers, and assume that $L := \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists (either as a real number or as $+\infty$).

(a) If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(b) If $L > 1$,[a] then the series $\sum_{n=1}^{\infty} a_n$ diverges.

[a] Possibly, $\lim_{n\to\infty} \sqrt[n]{|a_n|} = +\infty$.

- **Remark:** If $L = 1$, then the Root Test gives us no information about the convergence/divergence of the series $\sum_{n=1}^{\infty} a_n$.

- Reminder:

Proposition 2.4.2

$$\lim_{n\to\infty} \sqrt[n]{n} = 1.$$

Example 2.9.9

Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{5^n}$, and note that

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^n n}{5^n}\right|} \;=\; \lim_{n\to\infty} \frac{\sqrt[n]{n}}{5} \;=\; \frac{1}{5} \;<\; 1.$$

So, by the Root Test, the series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{5^n}$ converges absolutely.

Example 2.9.10

Consider the series $\sum_{n=1}^{\infty} \left(\frac{16n+3}{-7n+6}\right)^n$. Note that

$$
\begin{aligned}
\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{16n+3}{-7n+6}\right)^n\right|} &= \lim_{n\to\infty} \left|\frac{16n+3}{-7n+6}\right| \\
&= \lim_{n\to\infty} \frac{16n+3}{7n-6} \\
&= \lim_{n\to\infty} \frac{16+\frac{3}{n}}{7-\frac{6}{n}} \\
&= \frac{16}{7} \\
&> 1,
\end{aligned}
$$

and so by the Root Test, the series $\sum_{n=1}^{\infty} \left(\frac{-7n+6}{16n+3}\right)^n$ diverges.

The Root Test

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers, and assume that $L := \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists (either as a real number or as $+\infty$).

(a) If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(b) If $L > 1$,[a] then the series $\sum_{n=1}^{\infty} a_n$ diverges.

---

[a] Possibly, $\lim_{n\to\infty} \sqrt[n]{|a_n|} = +\infty$.

- **Remark:** As pointed out above, if $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers s.t. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then the Root Test gives us no information about the convergence/divergence of the series $\sum_{n=1}^{\infty} a_n$, and in principle, it is possible that the series converges absolutely, converges conditionally, or diverges.
  - Recall that $\lim_{n\to\infty} \sqrt[n]{n} = 1$ (by Proposition 2.4.2), and consider, for example, the following three series (next slide):

1. the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, and we have that

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{n}\right|} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1;$$

2. the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges conditionally, and we have that

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^{n-1}}{n}\right|} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1;$$

3. the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges (absolutely) or diverges depending on the value of the real number p; however, for all real numbers p, we have that

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{n^p}\right|} = \lim_{n\to\infty} \frac{1}{(\sqrt[n]{n})^p} = \frac{1}{(\lim_{n\to\infty} \sqrt[n]{n})^p} = \frac{1}{1^p} = 1.$$

- Let's prove the Root Test!
- First, a reminder:

Theorem 2.9.2

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

The Root Test

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers, and assume that $L := \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists (either as a real number or as $+\infty$).

- (a) If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.
- (b) If $L > 1$,[a] then the series $\sum_{n=1}^{\infty} a_n$ diverges.

[a] Possibly, $\lim_{n\to\infty} \sqrt[n]{|a_n|} = +\infty$.

Proof. We will prove (a) by using a suitably chosen convergent geometric series and the Comparison Test. We will prove (b) by showing that $\{a_n\}_{n=1}^{\infty}$ does **not** converge to 0, and consequently, the series $\sum_{n=1}^{\infty} a_n$ diverges (by Theorem 2.9.2. We now give the formal details.

Proof (continued). We first prove (a). Suppose that $L < 1$. Since all terms of the sequence $\{\sqrt[n]{|a_n|}\}_{n=1}^{\infty}$ are non-negative, $L \geq 0$.

Set $\varepsilon := \frac{1-L}{2}$; since $0 \leq L < 1$, we see that $0 < \varepsilon \leq \frac{1}{2}$.

Now, using the fact that $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L$, we fix $N \in \mathbb{N}$ s.t. for all $n \in \mathbb{N}$, if $n \geq N$, then $|\sqrt[n]{|a_n|} - L| < \varepsilon$, and consequently, $\sqrt[n]{|a_n|} < L + \varepsilon = \frac{1+L}{2} =: q$. It follows that for all $n \in \mathbb{N}$ s.t. $n \geq N$, we have that $0 \leq |a_n| \leq q^n$.

On the other hand, since $0 \leq L < 1$, we have that $\frac{1}{2} \leq q < 1$, and in particular, $|q| < 1$. So, the geometric series $\sum\limits_{n=N}^{\infty} q^n$ converges.

Thus, by the Comparison Test, the series $\sum\limits_{n=N}^{\infty} |a_n|$ converges.

If $N = 1$, then we are done. On the other hand, if $N \geq 2$, then $\sum\limits_{n=1}^{\infty} |a_n| = \underbrace{\Big(\sum\limits_{n=1}^{N-1} |a_n| \Big)}_{\text{finite sum}} + \underbrace{\Big(\sum\limits_{n=N}^{\infty} |a_n| \Big)}_{\text{convergent}}$, and so $\sum\limits_{n=1}^{\infty} |a_n|$ converges.

This proves (a).

Proof (continued). It remains to prove (b). So, let us assume that $L > 1$.

Claim. *$\exists N \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$, if $n \geq N$, then $|a_n| > 1$.*

Proof of the Claim. Suppose first that $L = +\infty$, i.e. $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = +\infty$. In this case, we know that $\exists N \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$, if $n \geq N$, then $\sqrt[n]{|a_n|} > 1$, and consequently, $|a_n| > 1$.

Let us now suppose that L is a real number. Set $\varepsilon := L - 1$; since $L > 1$, we see that $\varepsilon > 0$.

Now, using the fact that $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L$, we fix some $N \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$, if $n \geq N$, then $|\sqrt[n]{|a_n|} - L| < \varepsilon$, and consequently, $L - \varepsilon < \sqrt[n]{|a_n|} < L + \varepsilon$.

But note that $L - \varepsilon = 1$. Thus, $\forall n \in \mathbb{N}$ s.t. $n \geq N$, we have that $\sqrt[n]{|a_n|} > L - \varepsilon = 1$, and consequently, $|a_n| > 1^n = 1$. ♦

Proof (continued). We have proven:

Claim. $\exists N \in \mathbb{N}$ *s.t.* $\forall n \in \mathbb{N}$, *if* $n \geq N$, *then* $|a_n| > 1$.

Now, suppose toward a contradiction that $\lim_{n\to\infty} a_n = 0$. Fix $N' \in \mathbb{N}$ s.t. for all $n \in \mathbb{N}$, if $n \geq N'$, then $|a_n| < 1$.

- Here are the details. Set $\varepsilon := 1$. The, since $\lim_{n\to\infty} a_n = 0$, $\exists N' \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$, if $n \geq N'$, then $|a_n - 0| < \varepsilon$, that is, $|a_n| < 1$.

Now, let $N \in \mathbb{N}$ be as in the Claim, so that $\forall n \in \mathbb{N}$ s.t. $n \geq N$, we have that $|a_n| > 1$. Then for $n := \max\{N, N'\}$, we have that

$$1 \overset{(*)}{<} |a_n| \overset{(**)}{<} 1,$$

where (*) follows from the fact that $n \geq N$, whereas (**) follows from the fact that $n \geq N'$. We have now obtained $1 < 1$, a contradiction. This proves that $\{a_n\}_{n=1}^{\infty}$ does not converge to 0. So, by Theorem 2.9.2, the series $\sum_{n=1}^{\infty} a_n$ diverges. □

6. Defining functions via series: the exponential function

- Some important functions can be defined as series.
- For instance, we may define the function $\exp : \mathbb{R} \to \mathbb{R}$ by setting

$$\exp(x) \quad := \quad \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad = \quad 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

  for all $x \in \mathbb{R}$.
- Let us explain why this series converges for all $x \in \mathbb{R}$ (and so the function exp is well defined).
  - Obviously, the series above converges for $x = 0$, and we have that $\exp(0) = 1$.
  - On the other hand, for fixed $x \in \mathbb{R} \setminus \{0\}$, we have that

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| \quad = \quad \lim_{n\to\infty} \frac{|x|}{n+1} \quad \overset{(*)}{=} \quad |x| \lim_{n\to\infty} \frac{1}{n+1} \quad = \quad |x| \cdot 0 \quad = \quad 0,$$

    where (*) follows from the fact that x is fixed (and may therefore be treated as a constant). So, by the Ratio Test, the series $\exp(x)$ converges for all $x \in \mathbb{R} \setminus \{0\}$.
  - This proves that the function exp is well defined.

- Reminder: The function $\exp : \mathbb{R} \to \mathbb{R}$ is defined by

$$\exp(x) \quad := \quad \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad = \quad 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

  for all $x \in \mathbb{R}$.

- **Remark:** It can be shown that $\exp(x) = e^x$ for all $x \in \mathbb{R}$.
  - In other words, for all $x \in \mathbb{R}$, we have that

$$e^x \quad = \quad \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

    but we omit the proof.

- We do note that this implies, in particular, that

$$e \quad = \quad \sum_{n=0}^{\infty} \frac{1}{n!} \quad = \quad 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots .$$