Mathematical Analysis 1

Lecture #4

The limit superior and limit inferior. Series (part I)

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- This lecture has two parts:
  1. The limit superior and the limit inferior
  2. Series

1. The limit superior and the limit inferior

Proposition 2.8.1

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then:

a. the following are equivalent:
   - (a.1) $\{a_n\}_{n=1}^{\infty}$ is bounded above,
   - (a.2) all subsequences of $\{a_n\}_{n=1}^{\infty}$ are bounded above,
   - (a.3) for all $k \in \mathbb{N}$, the subsequence $\{a_n\}_{n=k}^{\infty}$ is bounded above,
   - (a.4) there exists some $k \in \mathbb{N}$ s.t. the subsequence $\{a_n\}_{n=k}^{\infty}$ is bounded above;

b. the following are equivalent:
   - (b.1) $\{a_n\}_{n=1}^{\infty}$ is bounded below,
   - (b.2) all subsequences of $\{a_n\}_{n=1}^{\infty}$ are bounded below,
   - (b.3) for all $k \in \mathbb{N}$, the subsequence $\{a_n\}_{n=k}^{\infty}$ is bounded below,
   - (b.4) there exists some $k \in \mathbb{N}$ s.t. the subsequence $\{a_n\}_{n=k}^{\infty}$ is bounded below.

- Proof: Lecture Notes (easy).

Definition

For a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers and a positive integer k:

- we define $\sup_{n \geq k} a_n$ to be the supremum of the subsequence $\{a_n\}_{n=k}^{\infty}$ if this subsequence is bounded above,[a] and we set $\sup_{n \geq k} a_n := +\infty$ otherwise;
- we define $\inf_{n \geq k} a_n$ to be the infimum of the subsequence $\{a_n\}_{n=k}^{\infty}$ if this subsequence is bounded below,[b] and we set $\inf_{n \geq k} a_n := -\infty$ otherwise.

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[a] By Proposition 2.8.1(a), we know that $\{a_n\}_{n=k}^{\infty}$ is bounded above if and only if the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded above. Moreover, since the ordered field $\mathbb{R}$ is complete, we know that if the subsequence $\{a_n\}_{n=k}^{\infty}$ is bounded above, then it does indeed have the least upper bound (supremum).

[b] By Proposition 2.8.1(b), we know that $\{a_n\}_{n=k}^{\infty}$ is bounded below if and only if the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded below. Moreover, since the ordered field $\mathbb{R}$ is complete, we know that if the subsequence $\{a_n\}_{n=k}^{\infty}$ is bounded below, then it does indeed have the greatest lower bound (infimum).

Example 2.8.2

Consider the sequence $\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$, i.e. the sequence

$$-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, \ldots .$$

Then:

- the sequence $\left\{\sup_{n\geq k}\left(\frac{(-1)^n}{n}\right)\right\}_{k=1}^{\infty}$ is the (non-increasing) sequence
$$\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{6}, \frac{1}{6} \ldots ;$$
- the sequence $\left\{\inf_{n\geq k}\left(\frac{(-1)^n}{n}\right)\right\}_{k=1}^{\infty}$ is the (non-decreasing) sequence
$$-1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{5}, -\frac{1}{5}, \ldots .$$

Proposition 2.8.2

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then all the following hold:

(a) for all $k \in \mathbb{N}$, we have that $\inf\limits_{n \geq k} a_n \leq a_k \leq \sup\limits_{n \geq k} a_n$;

(b) the sequence $\{\sup\limits_{n \geq k} a_n\}_{k=1}^{\infty}$ is non-increasing;

(c) the sequence $\{\inf\limits_{n \geq k} a_n\}_{k=1}^{\infty}$ is non-decreasing.

Proof. We first prove (a). Fix $k \in \mathbb{N}$.

Then $\inf\limits_{n \geq k} a_n$ is a lower bound of the subsequence $\{a_n\}_{n=k}^{\infty}$, and in particular, it satisfies $\inf\limits_{n \geq k} a_n \leq a_k$.

Similarly, $\sup\limits_{n \geq k} a_n$ is an upper bound of the subsequence $\{a_n\}_{n=k}^{\infty}$, and in particular, it satisfies $a_k \leq \sup\limits_{n \geq k} a_n$.

This proves (a).

Proposition 2.8.2

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then all the following hold:

- (a) for all $k \in \mathbb{N}$, we have that $\inf\limits_{n \geq k} a_n \leq a_k \leq \sup\limits_{n \geq k} a_n$;
- (b) the sequence $\{\sup\limits_{n \geq k} a_n\}_{k=1}^{\infty}$ is non-increasing;
- (c) the sequence $\{\inf\limits_{n \geq k} a_n\}_{k=1}^{\infty}$ is non-decreasing.

Proof (continued). For (b), we simply observe that for all $k \in \mathbb{N}$, we have that

$$\sup_{n \geq k} a_n \quad = \quad \max\Big\{a_k, \sup_{n \geq k+1} a_n\Big\} \quad \geq \quad \sup_{n \geq k+1} a_n.$$

Similarly, for (c), we observe that for all $k \in \mathbb{N}$, we have that

$$\inf_{n \geq k} a_n \quad = \quad \min\Big\{a_k, \inf_{n \geq k+1} a_n\Big\} \quad \leq \quad \inf_{n \geq k+1} a_n.$$

This completes the argument. $\square$

- Reminder:

Proposition 2.7.10

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then:

(a) if $\{a_n\}_{n=1}^{\infty}$ is non-decreasing, then $\lim\limits_{n\to\infty} a_n = \sup\limits_{n\geq 1} a_n$;

(b) if $\{a_n\}_{n=1}^{\infty}$ is non-increasing, then $\lim\limits_{n\to\infty} a_n = \inf\limits_{n\geq 1} a_n$.

- **Remark:** In other words:
  - part (a) states that if a non-decreasing sequence is bounded above, then it converges to its supremum, and otherwise it diverges to $+\infty$;
  - part (b) states that if a non-increasing sequence is bounded below, then it diverges to its infimum, and otherwise it diverges to $-\infty$.

- **Notation:**
  - As a convention, for the constant sequence $+\infty, +\infty, +\infty, \ldots$, we define $\lim_{n\to\infty}(+\infty) = +\infty$, as well as $\sup_{n\geq k}(+\infty) = +\infty$ and $\inf_{n\geq k}(+\infty) = +\infty$ (for each $k \in \mathbb{N}$).
  - Likewise, for the constant sequence $-\infty, -\infty, -\infty, \ldots$, we define $\lim_{n\to\infty}(-\infty) = -\infty$, as well as $\sup_{n\geq k}(-\infty) = -\infty$ and $\inf_{n\geq k}(-\infty) = -\infty$ (for each $k \in \mathbb{N}$).

Definition

The *limit superior* of a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers is defined to be

$$\limsup_{n\to\infty} a_n \quad := \quad \lim_{k\to\infty} \left(\sup_{n\geq k} a_n\right) \quad = \quad \inf_{k\geq 1} \left(\sup_{n\geq k} a_n\right),$$

whereas the *limit inferior* of $\{a_n\}_{n=1}^{\infty}$ is defined to be

$$\liminf_{n\to\infty} a_n \quad := \quad \lim_{k\to\infty} \left(\inf_{n\geq k} a_n\right) \quad = \quad \sup_{k\geq 1} \left(\inf_{n\geq k} a_n\right).$$

- **Remark:** As Proposition 2.8.4 (in a couple of slides) shows, the limit superior and limit inferior are indeed well defined for any sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers.
  - We emphasize that each of $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ may possibly be a real number, $+\infty$, or $-\infty$. This is in contrast to the fact that $\lim_{n\to\infty} a_n$ need not exist.
- First, an example.

Example 2.8.5

Consider the sequence $\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$, i.e. the sequence $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, \ldots$. Then:

- the sequence $\left\{\sup_{n\geq k}\left(\frac{(-1)^n}{n}\right)\right\}_{k=1}^{\infty}$ is the sequence $\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{6}, \frac{1}{6} \ldots$, and we see that $\limsup_{n\to\infty} a_n = \lim_{k\to\infty}\left(\sup_{n\geq k}\left(\frac{(-1)^n}{n}\right)\right) = 0$;
- the sequence $\left\{\sup_{n\geq k}\left(\frac{(-1)^n}{n}\right)\right\}_{k=1}^{\infty}$ is the sequence $-1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{5}, -\frac{1}{5}, \ldots$, and we see that $\liminf_{n\to\infty} a_n = \lim_{k\to\infty}\left(\inf_{n\geq k}\left(\frac{(-1)^n}{n}\right)\right) = 0$.

Note also that $\lim_{n\to\infty}\left(\frac{(-1)^n}{n}\right) = 0$, so that the limit, the limit superior, and the limit inferior of the sequence $\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$ are all equal.

- **Remark:** As we shall see (see Theorem 2.8.8), for a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers, $\lim_{n\to\infty} a_n$ exists (as a real number, as $+\infty$, or as $-\infty$) if and only if the limit superior and the limit inferior of the sequence are equal, and in this case, $\lim_{n\to\infty} a_n$ is equal to both the limit superior and the limit inferior.
- But first, we must prove that the limit superior and limit inferior are actually well defined!

Proposition 2.8.4

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then $\{a_n\}_{n=1}^{\infty}$ has both the limit superior and the limit inferior, that is, both $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ are well defined. Moreover, the following hold:

- $\limsup_{n\to\infty} a_n = +\infty$ if and only if $\{a_n\}_{n=1}^{\infty}$ is not bounded above;
- $\liminf_{n\to\infty} a_n = -\infty$ if and only if $\{a_n\}_{n=1}^{\infty}$ is not bounded below.

Proof. We prove the statement for the limit superior. The proof for the limit inferior is in the Lecture Notes.

First of all, if $\{a_n\}_{n=1}^{\infty}$ is not bounded above, then for all $k \in \mathbb{N}$, the subsequence $\{a_n\}_{n=k}^{\infty}$ is not bounded above, either, and consequently, $\sup_{n\geq k} a_n = +\infty$. Thus, if $\{a_n\}_{n=1}^{\infty}$ is not bounded above, then we simply have that

$$\limsup_{n\to\infty} a_n = \lim_{k\to\infty}\left(\sup_{n\geq k} a_n\right) = \inf_{k\geq 1}\left(\sup_{n\geq k} a_n\right) = +\infty.$$

Proof (continued). Suppose now that $\{a_n\}_{n=1}^{\infty}$ is bounded above. Then for all $k \in \mathbb{N}$, the subsequence $\{a_n\}_{n=k}^{\infty}$ is also bounded above, and so since the ordered field $\mathbb{R}$ is complete, $\sup\limits_{n \geq k} a_n$ is defined and is a real number.

By Proposition 2.8.3(b), the sequence $\{\sup\limits_{n \geq k} a_n\}_{k=1}^{\infty}$ is non-increasing, and therefore, by Proposition 2.7.10(b), we have that

$$\lim_{k \to \infty} \big(\sup_{n \geq k} a_n\big) \;=\; \inf_{k \geq 1} \big(\sup_{n \geq k} a_n\big),$$

which in particular means that

$$\limsup_{n \to \infty} a_n \;=\; \lim_{k \to \infty} \big(\sup_{n \geq k} a_n\big) \;=\; \inf_{k \geq 1} \big(\sup_{n \geq k} a_n\big),$$

is well defined and is either a real number or $-\infty$. $\square$

Proposition 2.8.4

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then $\{a_n\}_{n=1}^{\infty}$ has both the limit superior and the limit inferior, that is, both $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ are well defined. Moreover, the following hold:

- $\limsup_{n\to\infty} a_n = +\infty$ if and only if $\{a_n\}_{n=1}^{\infty}$ is not bounded above;
- $\liminf_{n\to\infty} a_n = -\infty$ if and only if $\{a_n\}_{n=1}^{\infty}$ is not bounded below.

- Reminder:

Definition

An *accumulation point* of a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers is a real number A s.t. for all real numbers $\varepsilon > 0$ and all $N \in \mathbb{N}$, there exists some $n \in \mathbb{N}$ s.t. $n \geq N$ and $|a_n - A| < \varepsilon$.

Theorem 2.6.2

Let $\{a_n\}_{n=1}^{\infty}$ be a **convergent** sequence of real numbers. Then $\{a_n\}_{n=1}^{\infty}$ has exactly one accumulation point, namely $L := \lim_{n\to\infty} a_n$.

- We will prove:

Proposition 2.8.6

Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers. Then both $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ are both real numbers, and moreover, they are both accumulation points of the sequence $\{a_n\}_{n=1}^{\infty}$.

Proof. We prove the statement for the limit superior; the proof of the limit inferior is analogous.

Let $m \in \mathbb{R}$ be a lower bound of the sequence $\{a_n\}_{n=1}^{\infty}$. Further, to simplify notation, set $S := \limsup_{n\to\infty} a_n$, and for all $k \in \mathbb{N}$, set $S_k := \sup_{n\geq k} a_n$.

Since $\{a_n\}_{n=1}^{\infty}$ is bounded, we know that S_k is a real number for all $k \in \mathbb{N}$, and moreover, that

$$S = \limsup_{n\to\infty} a_n = \lim_{k\to\infty} \left(\sup_{n\geq k} a_n \right) = \lim_{k\to\infty} S_k.$$

Let us first explain why S is a real number. Clearly, for all $k \in \mathbb{N}$, we have that $m \leq a_k \leq \sup_{n\geq k} a_n = S_k$. Thus, the sequence $\{S_k\}_{k=1}^{\infty}$ is bounded below (by m). Since the sequence $\{S_k\}_{k=1}^{\infty}$ is also non-increasing (by Proposition 2.8.3(b)), Lemma 2.3.2 guarantees that it converges (to its infimum). Thus, S is indeed a real number.

Proof (continued). It remains to show that S is an accumulation point of the sequence $\{a_n\}_{n=1}^{\infty}$. Fix $\varepsilon > 0$ and $N \in \mathbb{N}$. WTS $\exists n \in \mathbb{N}$ s.t. $n \geq N$ and $|a_n - S| < \varepsilon$.

Using the fact that $S = \lim_{k\to\infty} S_k$, we fix some $K \in \mathbb{N}$ s.t. for all $k \in \mathbb{N}$, if $k \geq K$, then $|S_k - S| < \frac{\varepsilon}{2}$.

Now, set $k := \max\{N, K\}$. Then $k \geq K$, and consequently, $|S_k - S| < \frac{\varepsilon}{2}$. Further, by definition, we have that $S_k = \sup_{n\geq k} a_n$, and consequently, $\exists n \in \mathbb{N}$ s.t. $n \geq k$ and $S_k - \frac{\varepsilon}{2} < a_n \leq S_k$.

- The existence of such an n follows from the fact that S_k is an upper bound of $\{a_n\}_{n=k}^{\infty}$, whereas $S_k - \frac{\varepsilon}{2}$ is not.

Note that this implies that $|a_n - S_k| < \frac{\varepsilon}{2}$. We now have that $n \geq N$, and that

$$\begin{aligned} |a_n - S| &= |(a_n - S_k) + (S_k - S)| \\ &\leq |a_n - S_k| + |S_k - S| \qquad \text{by the Triangle Inequality} \\ &< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} \ = \ \varepsilon, \end{aligned}$$

which proves that S is indeed an accumulation point of $\{a_n\}_{n=1}^{\infty}$. $\square$

Proposition 2.8.6

Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers. Then both $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ are both real numbers, and moreover, they are both accumulation points of the sequence $\{a_n\}_{n=1}^{\infty}$.

Proposition 2.8.7

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$.

- Proof: Lecture Notes.

Theorem 2.8.8

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then $\lim_{n\to\infty} a_n$ exists (as a real number, as $+\infty$, or as $-\infty$) if and only if $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$, and in this case, we have that $\lim_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$.

Proof. We give the proof for the case when the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded. The general case is in the Lecture Notes.

So, we assume that $\{a_n\}_{n=1}^{\infty}$ is bounded, and to simplify notation, we set $S := \limsup_{n\to\infty} a_n$ and $I := \liminf_{n\to\infty} a_n$.

Since the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded, we know that the subsequence $\{a_n\}_{n=k}^{\infty}$ is bounded for all $k \in \mathbb{N}$, and consequently, $\inf_{n\geq k} a_n$ and $\sup_{n\geq k} a_n$ are real numbers for all $k \in \mathbb{N}$. Moreover, by Proposition 2.8.6, both I and S are real numbers and accumulation points of $\{a_n\}_{n=1}^{\infty}$

Theorem 2.8.8

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then $\lim_{n\to\infty} a_n$ exists (as a real number, as $+\infty$, or as $-\infty$) if and only if $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$, and in this case, we have that $\lim_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$.

Proof (continued). Suppose first that $L := \lim_{n\to\infty} a_n$ exists.

Since $\{a_n\}_{n=1}^{\infty}$ is bounded, Proposition 2.7.3 guarantees that $L \neq \pm\infty$; consequently, L is a real number, i.e. the sequence $\{a_n\}_{n=1}^{\infty}$ converges to L.

Consequently, by Theorem 2.6.2, L is the only accumulation point of the sequence $\{a_n\}_{n=1}^{\infty}$. Since S and I are accumulation points of $\{a_n\}_{n=1}^{\infty}$, it follows that $L = S = I$.

Theorem 2.8.8

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then $\lim_{n\to\infty} a_n$ exists (as a real number, as $+\infty$, or as $-\infty$) if and only if $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$, and in this case, we have that $\lim_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$.

Proof (continued). Suppose now that $S = I$. Now, note that

$$\inf_{n\geq k} a_n \quad \leq \quad a_k \quad \leq \quad \sup_{n\geq k} a_n$$

for all $k \in \mathbb{N}$. Since $\lim_{k\to\infty} \inf_{n\geq k} a_n = I = S = \lim_{k\to\infty} \sup_{n\geq k} a_n$, the Squeeze Theorem guarantees that $\lim_{k\to\infty} a_k = I = S$. $\square$

2. Series

- Suppose we are given a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers.
  - We can then form the following infinite sequence of *partial sums*, as follows:
    - $\sum_{n=1}^{1} a_n = a_1;$
    - $\sum_{n=1}^{2} a_n = a_1 + a_2;$
    - $\sum_{n=1}^{3} a_n = a_1 + a_2 + a_3;$
    - $\dots$

- Suppose we are given a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers.
    - We can also consider the associated *infinite series* (or simply *series*) $\sum_{n=1}^{\infty} a_n$ (also denoted $a_1 + a_2 + a_3 + \dots$).
    - We say that the series $\sum_{n=1}^{\infty} a_n$ *converges* (or is *convergent*) if the sequence $\Big\{ \sum_{n=1}^{k} a_n \Big\}_{k=1}^{\infty}$ of partial sums converges; otherwise, the series $\sum_{n=1}^{\infty} a_n$ *diverges* (or is *divergent*).
    - If the series $\sum_{n=1}^{\infty} a_n$ converges, then its *sum* is the limit $s := \lim_{k\to\infty} \Big(\sum_{n=1}^{k} a_n \Big)$, and we write $\sum_{n=1}^{\infty} a_n = s$.

- Suppose we are given a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers.
  - If $\lim_{k\to\infty}\left(\sum_{n=1}^{k} a_n\right) = +\infty$, then we say that the series $\sum_{n=1}^{\infty} a_n$ diverges to $+\infty$, and we write $\sum_{n=1}^{\infty} a_n = +\infty$.
  - Similarly, if $\lim_{k\to\infty}\left(\sum_{n=1}^{k} a_n\right) = -\infty$, then we say that the series $\sum_{n=1}^{\infty} a_n$ diverges to $-\infty$, and we write $\sum_{n=1}^{\infty} a_n = -\infty$.

- **Remark:** Sometimes, it is convenient to start our series at an index other than $n = 1$.
  - So, we may get series of the form $\sum_{n=0}^{\infty} a_n$, or $\sum_{n=5}^{\infty} a_n$, or even $\sum_{n=-10}^{\infty} a_n$.
  - Convergence and divergence of such series, as well as their sums, are defined in a natural way.
- By applying properties of limits to the sequences of partial sums, we obtain the following proposition (next slide).

Proposition 2.9.1

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers, and let $c \in \mathbb{R}$. Then:

a. if $\sum\limits_{n=1}^{\infty} a_n$ converges, then so does $\sum\limits_{n=1}^{\infty}(ca_n)$, and in that case, we have that $\sum\limits_{n=1}^{\infty}(ca_n) = c\Big(\sum\limits_{n=1}^{\infty} a_n\Big)$;
b. if $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ both converge, then so does $\sum\limits_{n=1}^{\infty}(a_n + b_n)$, and in that case, we have that $\sum\limits_{n=1}^{\infty}(a_n + b_n) = \Big(\sum\limits_{n=1}^{\infty} a_n\Big) + \Big(\sum\limits_{n=1}^{\infty} b_n\Big)$;
c. if $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ both converge, then so does $\sum\limits_{n=1}^{\infty}(a_n - b_n)$, and in that case, we have that $\sum\limits_{n=1}^{\infty}(a_n - b_n) = \Big(\sum\limits_{n=1}^{\infty} a_n\Big) - \Big(\sum\limits_{n=1}^{\infty} b_n\Big)$;

- The *geometric series* is a series of the form

$$\sum_{n=0}^{\infty} bq^n \quad = \quad b + bq + bq^2 + bq^3 + bq^4 + \dots,$$

where $b, q \in \mathbb{R}$, with $b \neq 0$.

- Let us check that this series converges if and only if $|q| < 1$, and that in this case (i.e. if $|q| < 1$), we have that

$$\sum_{n=0}^{\infty} bq^n \quad = \quad \tfrac{b}{1-q},$$

- First, if $q = 1$, then our series is simply of the form $b + b + b + b + \dots$, and since $b \neq 0$, it obviously diverges (to $+\infty$ if $b > 0$, and to $-\infty$ if $b < 0$).
- So, let us assume that $q \neq 1$.
- Then

$$\sum_{n=0}^{k} bq^n \quad = \quad \tfrac{b-bq^{k+1}}{1-q} \qquad \forall k \in \mathbb{N}.$$

- Reminder: $b, q \in \mathbb{R}$; $b \neq 0$; $q \neq 1$; $\sum_{n=0}^{k} bq^n = \frac{b - bq^{k+1}}{1-q}$ $\forall k \in \mathbb{N}$.
- Now, if $|q| < 1$, then $\lim_{n\to\infty} q^n = 0$ (by Proposition 2.7.9), and consequently,

$$\sum_{n=0}^{\infty} bq^n \;=\; \lim_{k\to\infty} \Big(\sum_{n=0}^{k} q^n \Big) \;=\; \lim_{k\to\infty} \frac{b - bq^{k+1}}{1-q} \;=\; \frac{b}{1-q},$$

  and in particular, the series $\sum_{n=0}^{\infty} bq^n$ converges.
- On the other hand, if $|q| \geq 1$, then since $q \neq 1$, we know that $\{q^n\}_{n=1}^{\infty}$ diverges (by Proposition 2.7.9), and consequently, the series $\sum_{n=0}^{\infty} bq^n$ diverges as well.

Theorem 2.9.2

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Proof. Assume that the series $\sum_{n=1}^{\infty} a_n$ converges, and set

$s := \sum_{n=1}^{\infty} a_n$.

For each $k \in \mathbb{N}$, set $s_k := \sum_{n=1}^{k} a_n$, so that $s = \lim_{k\to\infty} s_k$. For convenience, we may set $s_0 := 0$.
We then have the following:

$$0 = \underbrace{\left(\lim_{k\to\infty} s_k\right)}_{=s} - \underbrace{\left(\lim_{k\to\infty} s_{k-1}\right)}_{=s} = \lim_{k\to\infty} (s_k - s_{k-1}) = \lim_{k\to\infty} a_k,$$

and we are done. □

Theorem 2.9.2

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

- **Remark:** Theorem 2.9.2 can sometimes be used to show that a series **diverges**: if a sequence $\{a_n\}_{n=1}^{\infty}$ does not converge to 0, then the series $\sum_{n=1}^{\infty} a_n$ diverges.
- However, the converse of Theorem 2.9.2 is **false** in general.
  - For example, we shall see (later!) that the so called "harmonic series"

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \dots$$

    diverges, even though $\lim_{n\to\infty} \frac{1}{n} = 0$.

The Comparison Test

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers s.t. $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$. Then:

(a) if the series $\sum_{n=1}^{\infty} b_n$ converges, then so does the series $\sum_{n=1}^{\infty} a_n$, and moreover, $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$;

(b) if the series $\sum_{n=1}^{\infty} a_n$ diverges, then so does the series $\sum_{n=1}^{\infty} b_n$.

Proof. Note that (b) is simply the contrapositive of (a). So, it is enough to prove (a). Let us therefore assume that the series $\sum_{n=1}^{\infty} b_n$ converges.

Proof. For all $k \in \mathbb{N}$, consider the partial sums

$$A_k := \sum_{n=1}^{k} a_n \qquad \text{and} \qquad B_k := \sum_{n=1}^{k} b_n$$

Since all terms of the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are non-negative, we see that the sequences $\{A_k\}_{k=1}^{\infty}$ and $\{B_k\}_{k=1}^{\infty}$ are both non-decreasing, and moreover, we have that

$$0 \quad \leq \quad A_k \quad \leq \quad B_k \qquad\qquad \forall k \in \mathbb{N}.$$

Moreover, since $\{B_k\}_{k=1}^{\infty}$ converges (by assumption), Lemma 2.2.3 guarantees that it is bounded. Consequently, $\{A_k\}_{k=1}^{\infty}$ is also bounded. Now the sequence $\{A_k\}_{k=1}^{\infty}$ is monotone and bounded, and so by the Monotone Sequence Theorem, it converges, that is, the series $\sum_{n=1}^{\infty} a_n$ converges. Moreover, by Theorem 2.7.7, we have that $\lim_{k\to\infty} A_k \leq \lim_{k\to\infty} B_k$, that is, $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$. $\square$

The Comparison Test

Let $\{a_n\}_{n=1}^{\infty}$ and $\{_n\}_{n=1}^{\infty}$ be sequences of real numbers s.t. $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$. Then:

(a) if the series $\sum_{n=1}^{\infty} b_n$ converges, then so does the series $\sum_{n=1}^{\infty} a_n$, and moreover, $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$;

(b) if the series $\sum_{n=1}^{\infty} a_n$ diverges, then so does the series $\sum_{n=1}^{\infty} b_n$.

- The *harmonic series* is the series

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad = \quad 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \ldots$$

We will use the Comparison Test to show that this sequence **diverges**.

- Define the sequence $\{a_n\}_{n=1}^{\infty}$ by setting

$$a_{2^k+\ell} \quad = \quad \frac{1}{2^{k+1}}$$

for all $k, \ell \in \mathbb{N}_0$ s.t. $0 \leq \ell \leq 2^k - 1$.

- Note that the sequence $\{a_n\}_{n=1}^{\infty}$ is precisely the sequence

$$\underbrace{\frac{1}{2}}_{1}, \underbrace{\frac{1}{4}, \frac{1}{4}}_{2}, \underbrace{\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}}_{4}, \underbrace{\frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}}_{8}, \ldots$$

- Clearly, for all $n \in \mathbb{N}$, we have that $0 < a_n < \frac{1}{n}$.
- Thus, by the Comparison Test, we need only show that $\sum_{n=1}^{\infty} a_n$ diverges.
- But note that

$$\begin{aligned} \sum_{n=1}^{\infty} a_n \quad &= \quad \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \ldots \\ &= \quad \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots \quad = \quad +\infty, \end{aligned}$$

and we are done.

- We have shown that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.
- **Remark:** The harmonic series and the Comparison Test can often be used to show that a series diverges.
  - For example, note that for all $n \in \mathbb{N}$, we have that $0 < \frac{1}{n} \leq \frac{1}{\sqrt{n}}$.
  - So, since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so does the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

The Alternating Series Test

Let $\{a_n\}_{n=1}^{\infty}$ be the a non-increasing sequence of non-negative real numbers s.t. $\lim_{n\to\infty} a_n = 0$. Then the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \dots$$

converges.

- **Remark:** Here, we are assuming that the following are satisfied:
  - $a_n \geq 0$ for all $n \in \mathbb{N}$;
  - $a_1 \geq a_2 \geq a_3 \geq a_4 \geq \dots$;
  - $\lim_{n\to\infty} a_n = 0$.

  The Alternating Series Test states that, under these circumstances, the "alternating series" $\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \dots$ converges.

The Alternating Series Test

Let $\{a_n\}_{n=1}^{\infty}$ be the a non-increasing sequence of non-negative real numbers s.t. $\lim_{n\to\infty} a_n = 0$. Then the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n \quad = \quad a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \dots$$

converges.

Proof. For each $k \in \mathbb{N}$, consider the partial sum
$s_k := \sum_{n=1}^{k} (-1)^{n-1} a_n$.

We will show that $\lim_{k\to\infty} s_{2k}$ and $\lim_{k\to\infty} s_{2k+1}$ converge to the same limit; this will imply that $\lim_{k\to\infty} s_k$ also converges to that same limit (check this!).

Proof (continued). First, we note that $\forall k \in \mathbb{N}$:

$$s_{2k+2} \quad = \quad s_{2k} + a_{2k+1} - a_{2k+2} \quad \overset{(*)}{\geq} \quad s_{2k},$$

where (*) follows from the fact that the sequence $\{a_n\}_{n=1}^{\infty}$ is non-increasing, and consequently, $a_{2k+1} - a_{2k+2} \geq 0$. Thus, the subsequence $\{s_{2k}\}_{k=1}^{\infty}$ is non-decreasing.

On the other hand, we note that for all $k \in \mathbb{N}$, we have that

$$s_{2k} \quad = \quad a_1 - (\underbrace{a_2 - a_3}_{\overset{(*)}{\geq} 0}) - (\underbrace{a_4 - a_5}_{\overset{(*)}{\geq} 0}) - \cdots - (\underbrace{a_{2k-2} - a_{2k-1}}_{\overset{(*)}{\geq} 0}) - \underbrace{a_{2k}}_{\overset{(**)}{\geq} 0} \quad \leq \quad a_1,$$

where each instance of (*) follows from the fact that the sequence $\{a_n\}_{n=1}^{\infty}$ is non-increasing, and where (**) follows from the fact that all terms of the sequence $\{a_n\}_{n=1}^{\infty}$ are non-negative. Thus, the subsequence $\{s_{2k}\}_{k=1}^{\infty}$ is bounded above by a_1.

- Since the sequence $\{a_{2k}\}_{k=1}^{\infty}$ is non-decreasing, it is obviously bounded below by its first term, that is, by the term $s_2 = a_1 - a_2$.

Proof (continued). The Monotone Sequence Theorem now implies that $\{s_{2k}\}_{k=1}^{\infty}$ converges.

Now, set $s := \lim_{k\to\infty} s_k$. Then

$$\begin{aligned}
\lim_{k\to\infty} s_{2k+1} &= \lim_{k\to\infty}(s_{2k} + a_{2k+1}) \\
&= (\lim_{k\to\infty} s_{2k}) + (\lim_{k\to\infty} a_{2k+1}) \\
&= s + 0 \\
&= s.
\end{aligned}$$

This completes the argument. □

The Alternating Series Test

Let $\{a_n\}_{n=1}^{\infty}$ be the a non-increasing sequence of non-negative real numbers s.t. $\lim_{n\to\infty} a_n = 0$. Then the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n \quad = \quad a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \dots$$

converges.

- By the Alternating Series Test, the *alternating harmonic series*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \quad = \quad 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \dots$$

converges.