Mathematical Analysis 1

Lecture #3

The Bolzano-Weierstrass Theorem. Accumulation points. Divergence to infinity

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- This lecture has three parts:
  1. Subsequences and the Bolzano-Weierstrass Theorem
  2. Accumulation points
  3. Divergence to infinity

1. Subsequences and the Bolzano-Weierstrass Theorem

Definition

A *subsequence* of a sequence $\{a_n\}_{n=1}^{\infty}$ is a sequence of the form $\{a_{n_j}\}_{j=1}^{\infty}$, where $\{n_j\}_{j=1}^{\infty}$ is a strictly increasing sequence of positive integers.

- **Remark:** Informally, a subsequence of a sequence $\{a_n\}_{n=1}^{\infty}$ is any sequence that can be obtained from $\{a_n\}_{n=1}^{\infty}$ by possibly deleting some terms, but so that infinitely many terms still remain.
    - In particular, every sequence is a subsequence of itself.

Proposition 2.5.1

For all sequences $\{a_n\}_{n=1}^{\infty}$ of real numbers, the following hold:

(a) $\{a_n\}_{n=1}^{\infty}$ converges iff all subsequences of $\{a_n\}_{n=1}^{\infty}$ converge;

(b) if $\{a_n\}_{n=1}^{\infty}$ converges, then all subsequences of $\{a_n\}_{n=1}^{\infty}$ converge to $\lim_{n\to\infty} a_n$.

Proof. By definition, $\{a_n\}_{n=1}^{\infty}$ is a subsequence of itself. Consequently, if all subsequences of $\{a_n\}_{n=1}^{\infty}$ converge, then in particular, $\{a_n\}_{n=1}^{\infty}$ also converges.

- This proves the "$\Longleftarrow$" part of (a).

Suppose now that $\{a_n\}_{n=1}^{\infty}$ converges, and set $L := \lim_{n\to\infty} a_n$. We will show that all subsequences of $\{a_n\}_{n=1}^{\infty}$ converge to L.

- This will prove the "$\Longrightarrow$" part of (a), as well as all of part (b).

Proposition 2.5.1

For all sequences $\{a_n\}_{n=1}^{\infty}$ of real numbers, the following hold:

- (a) $\{a_n\}_{n=1}^{\infty}$ converges iff all subsequences of $\{a_n\}_{n=1}^{\infty}$ converge;
- (b) if $\{a_n\}_{n=1}^{\infty}$ converges, then all subsequences of $\{a_n\}_{n=1}^{\infty}$ converge to $\lim\limits_{n\to\infty} a_n$.

Proof (continued). Reminder: $L := \lim\limits_{n\to\infty} a_n$; WTS all subsequences of $\{a_n\}_{n=1}^{\infty}$ converge to L.

Fix any subsequence $\{a_{n_j}\}_{j=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$, where $\{n_j\}_{j=1}^{\infty}$ is a strictly increasing sequence of positive integers. WTS $\lim\limits_{j\to\infty} a_{n_j} = L$.

Fix any $\varepsilon > 0$, and using the fact that $L := \lim\limits_{n\to\infty} a_n$, fix $N \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$, if $n \geq N$, then $|a_n - L| < \varepsilon$.

Since $\{n_j\}_{j=1}^{\infty}$ is a strictly increasing sequence of positive integers, we see that $\forall j \in \mathbb{N}$ s.t. $j \geq N$, we have that $n_j \geq n_N \geq N$, and consequently, $|a_{n_j} - L| < \varepsilon$. This proves that $L = \lim\limits_{j\to\infty} a_{n_j}$. $\square$

Lemma 2.5.2

Every sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers has a monotone subsequence.

- Proof: In a minute!

The Bolzano-Weierstrass Theorem

Every bounded sequence of real numbers has a convergent subsequence.

Proof (assuming Lemma 2.5.2) Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers. By Lemma 2.5.2, $\{a_n\}_{n=1}^{\infty}$ has a monotone subsequence, say $\{a_{n_j}\}_{j=1}^{\infty}$, where $\{n_j\}_{j=1}^{\infty}$ is an increasing sequence of positive integers. Since $\{a_n\}_{n=1}^{\infty}$ is bounded, so is its subsequence $\{a_{n_j}\}_{j=1}^{\infty}$. Now $\{a_{n_j}\}_{j=1}^{\infty}$ is a monotone and bounded sequence of real numbers, and so by the Monotone Sequence Theorem, it converges. $\square$

Lemma 2.5.2

Every sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers has a monotone subsequence.

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Let us say that a positive integer m is a *peak* of the sequence $\{a_n\}_{n=1}^{\infty}$ if $\forall n \in \mathbb{N}$ s.t. $n > m$, we have that $a_m > a_n$ (i.e. a_m is strictly greater than every subsequent term of the sequence).

Suppose first that $\{a_n\}_{n=1}^{\infty}$ has infinitely many peaks, and let $\{n_j\}_{j=1}^{\infty}$ be the sequence of all the peaks, arranged in increasing order (i.e. $n_1 < n_2 < n_3 < \dots$). But now $a_{n_1} > a_{n_2} > a_{n_3} > \dots$, i.e. $\{a_{n_j}\}_{j=1}^{\infty}$ is a strictly decreasing (and therefore monotone) subsequence of $\{a_n\}_{n=1}^{\infty}$.

From now on, we assume that $\{a_n\}_{n=1}^{\infty}$ has only finitely many (if any) peaks. Fix some $N \in \mathbb{N}$ s.t. all the peaks of $\{a_n\}_{n=1}^{\infty}$ are strictly smaller than N.

- Thus, $\forall m \in \mathbb{N}$ s.t. $m \geq N$, the integer m is **not** a peak, i.e. $\exists n \in \mathbb{N}$ s.t. $m < n$ and $a_m \leq a_n$.

Lemma 2.5.2

Every sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers has a monotone subsequence.

Proof (continued). Reminder: $\forall m \in \mathbb{N}$ s.t. $m \geq N$, the integer m is **not** a peak, i.e. $\exists n \in \mathbb{N}$ s.t. $m < n$ and $a_m \leq a_n$.

Our goal is now to (recursively) form an increasing sequence $\{n_j\}_{j=1}^{\infty}$ of positive integers s.t. $\forall j \in \mathbb{N}$: $a_{n_j} \leq a_{n_{j+1}}$. Then $\{a_{n_j}\}_{j=1}^{\infty}$ will be a non-decreasing (and therefore monotone) subsequence of $\{a_n\}_{n=1}^{\infty}$, which is what we need.

First, fix $n_1 = N$. Next, fix $j \in \mathbb{N}$, and suppose we have constructed positive integers $n_1, \ldots, n_j$ so that $n_1 < \cdots < n_j$ and $a_{n_1} \leq \cdots \leq a_{n_j}$. Since n_j is not a peak (because $n_j \geq N$), $\exists n \in \mathbb{N}$ s.t. $n_j < n$ and $a_{n_j} \leq a_n$; choose n_{j+1} to be the smallest such n.

By construction, $\{n_j\}_{j=1}^{\infty}$ is a strictly increasing sequence of positive integers, and $\{a_{n_j}\}_{j=1}^{\infty}$ is a non-decreasing (and therefore monotone) subsequence of $\{a_n\}_{n=1}^{\infty}$. $\square$

- We have proven:

Lemma 2.5.2

Every sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers has a monotone subsequence.

The Bolzano-Weierstrass Theorem

Every bounded sequence of real numbers has a convergent subsequence.

2. Accumulation points

Definition

An *accumulation point* of a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers is a real number A s.t. for all real numbers $\varepsilon > 0$ and all $N \in \mathbb{N}$, there exists some $n \in \mathbb{N}$ s.t. $n \geq N$ and $|a_n - A| < \varepsilon$.

- How is this different from a limit?
  - Next slide!

Definition

We say that a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers *converges* to a real number L provided that the following holds:

For all real numbers $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ s.t. for all $n \in \mathbb{N}$: if $n \geq N$, then $|a_n - L| < \varepsilon$.

Under such circumstances, we say that L is the *limit* of the sequence $\{a_n\}_{n=1}^{\infty}$, and we write

$$L = \lim_{n\to\infty} a_n \qquad \text{or} \qquad a_n \to L \quad \text{as} \quad n \to \infty.$$

A sequence is *convergent* (or *converges*) if it has a limit. Otherwise, it is *divergent* (or *diverges*).

Definition

An *accumulation point* of a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers is a real number A s.t. for all real numbers $\varepsilon > 0$ and all $N \in \mathbb{N}$, there exists some $n \in \mathbb{N}$ s.t. $n \geq N$ and $|a_n - A| < \varepsilon$.

Definition

An *accumulation point* of a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers is a real number A s.t. for all real numbers $\varepsilon > 0$ and all $N \in \mathbb{N}$, there exists some $n \in \mathbb{N}$ s.t. $n \geq N$ and $|a_n - A| < \varepsilon$.

Proposition 2.6.1

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers, and let $A \in \mathbb{R}$. Then the following are equivalent:

1. (i) A is an accumulation point of $\{a_n\}_{n=1}^{\infty}$;
2. (ii) for all real numbers $\varepsilon > 0$, the interval $(A - \varepsilon, A + \varepsilon)$ contains infinitely many terms of the sequence $\{a_n\}_{n=1}^{\infty}$;
3. (iii) some subsequence of $\{a_n\}_{n=1}^{\infty}$ converges to A.

- **Terminology:** For a real number $\varepsilon > 0$, the open interval $(A - \varepsilon, A + \varepsilon)$ is called the *ε-neighborhood of A*.

Proof. It suffices to prove the following sequence of implications: "(i) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i)."

Proposition 2.6.1

- (i) A is an accumulation point of $\{a_n\}_{n=1}^{\infty}$;
- (iii) some subsequence of $\{a_n\}_{n=1}^{\infty}$ converges to A.

Proof (continued). We first assume (i) and prove (iii). We define a strictly increasing sequence $\{n_j\}_{j=1}^{\infty}$ recursively as follows. First, using the fact that A is an accumulation point of $\{a_n\}_{n=1}^{\infty}$, we fix $n_1 \in \mathbb{N}$ s.t. $|a_{n_1} - A| < 1$.

- Here, we are using the def. of an acc. pt. for $\varepsilon = 1$ and $N = 1$.

Next, assume that for some $j \in \mathbb{N}$, we have defined the positive integer n_j; we then define $n_{j+1} \in \mathbb{N}$ as follows. Since A is an accumulation point of $\{a_n\}_{n=1}^{\infty}$, we let n_{j+1} be the smallest positive integer satisfying $n_{j+1} \geq n_j + 1$ and $|A - a_{n_{j+1}}| < \frac{1}{j+1}$.

- Here, we are using the def. of an acc. pt. for $\varepsilon = \frac{1}{j+1}$ and $N = n_j + 1$.

We have now constructed a strictly increasing sequence $\{n_j\}_{j=1}^{\infty}$ of positive integers s.t. for all $j \in \mathbb{N}$, we have that $|a_{n_j} - A| < \frac{1}{j}$.

Proposition 2.6.1

1. (i) A is an accumulation point of $\{a_n\}_{n=1}^{\infty}$;
2. (iii) some subsequence of $\{a_n\}_{n=1}^{\infty}$ converges to A.

Proof (continued). It is now easy to verify that $\lim_{j\to\infty} a_{n_j} = A$.

Indeed, fix $\varepsilon > 0$. Let $J \in \mathbb{N}$ be s.t. $J > \frac{1}{\varepsilon}$. Fix $j \in \mathbb{N} \in \mathbb{N}$ s.t. $j \geq J$. Then

$$\begin{aligned} |a_{n_j} - A| &< \frac{1}{j} && \text{by the construction of } n_j \\ &\leq \frac{1}{J} && \text{because } j \geq J \\ &< \varepsilon && \text{because } J > \tfrac{1}{\varepsilon}. \end{aligned}$$

This proves that $\lim_{j\to\infty} a_{n_j} = A$, i.e. (iii) holds.

Proposition 2.6.1

- (ii) for all real numbers $\varepsilon > 0$, the interval $(A-\varepsilon, A+\varepsilon)$ contains infinitely many terms of the sequence $\{a_n\}_{n=1}^{\infty}$;
- (iii) some subsequence of $\{a_n\}_{n=1}^{\infty}$ converges to A.

Proof (continued). Next, we assume (iii) and prove (ii).

Using (iii), we fix a strictly increasing sequence $\{n_j\}_{n=1}^{\infty}$ of positive integers s.t. $\lim\limits_{j\to\infty} a_{n_j} = A$.

Now fix $\varepsilon > 0$. Using the definition of a limit, we now fix $J \in \mathbb{N}$ s.t. for all $j \in \mathbb{N}$, if $j \geq J$, then $|a_{n_j} - A| < \varepsilon$, i.e. $a_{n_j} \in (A-\varepsilon, A+\varepsilon)$.

But now $a_{n_J}, a_{n_{J+1}}, a_{n_{J+2}}, \dots$ all belong to $(A-\varepsilon, A+\varepsilon)$. This proves (ii).

Proposition 2.6.1

1. (i) A is an accumulation point of $\{a_n\}_{n=1}^{\infty}$;
2. (ii) for all real numbers $\varepsilon > 0$, the interval $(A - \varepsilon, A + \varepsilon)$ contains infinitely many terms of the sequence $\{a_n\}_{n=1}^{\infty}$;

Proof (continued). Finally, we assume (ii) and prove (i).

Fix $\varepsilon > 0$ and $N \in \mathbb{N}$. Now, using (ii), we know that there exist infinitely many positive integers n s.t. $a_n \in (A - \varepsilon, A + \varepsilon)$, i.e. $|a_n - A| < \varepsilon$. Since there are infinitely many such n's, one of them (in fact, infinitely many of them) must satisfy $n \geq N$. This proves (i). $\square$

Definition

An *accumulation point* of a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers is a real number A s.t. for all real numbers $\varepsilon > 0$ and all $N \in \mathbb{N}$, there exists some $n \in \mathbb{N}$ s.t. $n \geq N$ and $|a_n - A| < \varepsilon$.

Proposition 2.6.1

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers, and let $A \in \mathbb{R}$. Then the following are equivalent:

(i) A is an accumulation point of $\{a_n\}_{n=1}^{\infty}$;

(ii) for all real numbers $\varepsilon > 0$, the interval $(A - \varepsilon, A + \varepsilon)$ contains infinitely many terms of the sequence $\{a_n\}_{n=1}^{\infty}$;

(iii) some subsequence of $\{a_n\}_{n=1}^{\infty}$ converges to A.

- **Terminology:** For a real number $\varepsilon > 0$, the open interval $(A - \varepsilon, A + \varepsilon)$ is called the *ε-neighborhood of A*.

Proposition 2.6.1 (abbreviated)

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers, and let $A \in \mathbb{R}$. Then the following are equivalent:

1. A is an accumulation point of $\{a_n\}_{n=1}^{\infty}$;
2. some subsequence of $\{a_n\}_{n=1}^{\infty}$ converges to A.

Theorem 2.6.2

Let $\{a_n\}_{n=1}^{\infty}$ be a **convergent** sequence of real numbers. Then $\{a_n\}_{n=1}^{\infty}$ has exactly one accumulation point, namely $L := \lim_{n\to\infty} a_n$.

Proof. First of all, $\{a_n\}_{n=1}^{\infty}$ is a subsequence of itself, and it converges to L; so, by Proposition 2.6.1, L is indeed an accumulation point of $\{a_n\}_{n=1}^{\infty}$.

On the other hand, by Proposition 2.5.1, all subsequences of $\{a_n\}_{n=1}^{\infty}$ converge to L, and so by Proposition 2.6.2, $\{a_n\}_{n=1}^{\infty}$ has no accumulation points other than L. $\square$

Theorem 2.6.2

Let $\{a_n\}_{n=1}^{\infty}$ be a **convergent** sequence of real numbers. Then $\{a_n\}_{n=1}^{\infty}$ has exactly one accumulation point, namely $L := \lim_{n\to\infty} a_n$.

- **Remark:**
  - By Theorem 2.6.2, if a sequence has more than one accumulation point, then it diverges.
    - For instance, the sequence $\{(-1)^n\}_{n=1}^{\infty}$ has two accumulation points, namely 1 and -1, and so it diverges.
  - However, the converse of Theorem 2.6.2 is false in general, i.e. some sequences that only have one accumulation point nevertheless diverge.
    - One example is the sequence $\{a_n\}_{n=1}^{\infty}$ given by

$$a_n \quad := \quad \begin{cases} \frac{1}{n} & \text{if} \quad n \text{ is even} \\ n & \text{if} \quad n \text{ is odd} \end{cases}$$

      for all $n \in \mathbb{N}$. (This is the sequence $1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, 7, \frac{1}{8}, \ldots$.)
    - Clearly, the only accumulation point of the sequence $\{a_n\}_{n=1}^{\infty}$ is 0, and yet the sequence diverges.

3. Divergence to infinity

Definition

A sequence $\{a_n\}_{n=1}^{\infty}$ *diverges to infinity*, and we write

$$\lim_{n\to\infty} a_n = \infty \qquad \text{or} \qquad \lim_{n\to\infty} a_n = +\infty,$$

or alternatively,

$$a_n \to \infty \quad \text{as} \quad n \to \infty \qquad \text{or} \qquad a_n \to +\infty \quad \text{as} \quad n \to \infty,$$

if for all $M \in \mathbb{R}$, there exists some $N \in \mathbb{N}$ s.t. for all $n \in \mathbb{N}$, if $n \geq N$, then $a_n > M$.

Definition

A sequence $\{a_n\}_{n=1}^{\infty}$ *diverges to infinity*, and we write

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if for all $M \in \mathbb{R}$, there exists some $N \in \mathbb{N}$ s.t. for all $n \in \mathbb{N}$, if $n \geq N$, then $a_n > M$.

Example 2.7.1

Using the definition, show that $\lim_{n\to\infty} n^2 = +\infty$.

Solution. Fix $M \in \mathbb{R}$. Let $N \in \mathbb{N}$ be s.t. $N > M$. Then $\forall n \in \mathbb{N}$ s.t. $n \geq N$, we have that $n^2 \geq N^2 \geq N > M$. This proves that $\lim_{n\to\infty} n^2 = +\infty$. $\square$

Definition

A sequence $\{a_n\}_{n=1}^{\infty}$ *diverges to negative infinity*, and we write

$$\lim_{n\to\infty} a_n = -\infty,$$

or alternatively,

$$a_n \to -\infty \quad \text{as} \quad n \to \infty,$$

if for all $M \in \mathbb{R}$, there exists some $N \in \mathbb{N}$ s.t. for all $n \in \mathbb{N}$, if $n \geq N$, then $a_n < M$.

Example 2.7.2

Using the definition, show that $\lim_{n\to\infty} \left(-\sqrt{n}\right) = -\infty$.

Solution. Fix $M \in \mathbb{R}$. Fix $N \in \mathbb{N}$ s.t. $N > M^2$. Then $\sqrt{N} > |M| \geq -M$, and consequently, $-\sqrt{N} < M$. It follows that $\forall n \in \mathbb{N}$ s.t. $n \geq N$, we have that $-\sqrt{n} \leq -\sqrt{N} < M$. This proves that $\lim_{n\to\infty} \left(-\sqrt{n}\right) = -\infty$. $\square$

Proposition 2.7.3

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then:

(a) if $\lim\limits_{n\to\infty} a_n = +\infty$, then $\{a_n\}_{n=1}^{\infty}$ is bounded below, but is not bounded above;

(b) if $\lim\limits_{n\to\infty} a_n = -\infty$, then $\{a_n\}_{n=1}^{\infty}$ is bounded above, but is not bounded below.

- Proof: exercise (easy!).

- Reminder:

Theorem 2.2.5

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be convergent sequences of real numbers, and let $c \in \mathbb{R}$. Then all the following hold:

a. $\lim_{n\to\infty}(ca_n) = c\lim_{n\to\infty}(a_n)$;
b. $\lim_{n\to\infty}(a_n + b_n) = (\lim_{n\to\infty} a_n) + (\lim_{n\to\infty} b_n)$;
c. $\lim_{n\to\infty}(a_n - b_n) = (\lim_{n\to\infty} a_n) - (\lim_{n\to\infty} b_n)$;
d. $\lim_{n\to\infty}(a_n b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n)$;
e. if $b_n \neq 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} b_n \neq 0$, then
$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}.$$

- **Remark:** Properties of limits from Theorem 2.2.5 (from the previous slide) readily generalize to divergence to (positive or negative) infinity.
    - We do not state an analogous theorem formally.
    - Instead, we focus on the "danger zones."
    - In particular, the following forms are "indeterminate," i.e. they can in principle be anything:

$$\frac{0}{0} \qquad \frac{\infty}{\infty} \qquad 0 \cdot \infty \qquad 1^{\infty} \qquad \infty - \infty \qquad 0^0 \qquad \infty^0$$

- Let us take a look at a few examples!

Example 2.7.4

Consider the behavior of the following "$\frac{\infty}{\infty}$" forms:

(a) $$\lim_{n\to\infty} \frac{\overbrace{2n}^{+\infty}}{\underbrace{7n}_{+\infty}} = \lim_{n\to\infty} \frac{2}{7} = \frac{2}{7};$$

(b) $$\lim_{n\to\infty} \frac{\overbrace{-2\sqrt{n}}^{-\infty}}{\underbrace{-3\sqrt{n}}_{-\infty}} = \lim_{n\to\infty} \frac{2}{3} = \frac{2}{3}$$

Example 2.7.4 (continued)

Consider the behavior of the following "$\frac{\infty}{\infty}$" forms:

(c) $$\lim_{n\to\infty} \frac{\overset{+\infty}{\overbrace{3n^2}}}{\underset{+\infty}{\underbrace{2n}}} = \lim_{n\to\infty} \frac{3n}{2} = +\infty;$$

(d) $$\lim_{n\to\infty} \frac{\overset{-\infty}{\overbrace{-2n^2}}}{\underset{+\infty}{\underbrace{5n^3}}} = \lim_{n\to\infty} \left(-\frac{2}{5n}\right) = 0.$$

Exaple 2.7.5

Compute the following limits:

(a) $\displaystyle\lim_{n\to\infty} \frac{2n^2-3n+7}{-n^2+2}$;

(b) $\displaystyle\lim_{n\to\infty} \frac{-n^3+5n-1}{2n-1}$;

(c) $\displaystyle\lim_{n\to\infty} \frac{n^2-1}{n^5+n^4+n^2}$.

(Note that each of the above is of the form "$\frac{\infty}{\infty}$.")

Solution. In each part, we start by factoring out the largest degree term from both the numerator and the denominator, and then we evaluate.

$$\text{(a)}\ \lim_{n\to\infty} \frac{2n^2-3n+7}{-n^2+2} = \lim_{n\to\infty} \frac{\cancel{n^2}\left(2-\overset{\to 0}{\frac{3}{n}}+\overset{\to 0}{\frac{7}{n^2}}\right)}{\cancel{n^2}\left(-1+\underset{\to 0}{\frac{2}{n^2}}\right)} = -2$$

Solution (continued). (b)

$$\lim_{n\to\infty} \frac{-n^3+5n-1}{2n-1} = \lim_{n\to\infty} \frac{n^3\left(-1+\frac{5}{n^2}-\frac{1}{n^3}\right)}{n\left(2-\frac{1}{n}\right)}$$

$$= \lim_{n\to\infty} \left(\underbrace{n^2}_{\to +\infty} \cdot \underbrace{\frac{-1+\frac{5}{n^2}-\frac{1}{n^3}}{2-\frac{1}{n}}}_{\to -\frac{1}{2}} \right)$$

$$= -\infty$$

Solution (continued). (c)

$$
\begin{aligned}
\lim_{n\to\infty} \frac{n^2-1}{n^5+n^4+n^2} &= \lim_{n\to\infty} \frac{n^2\left(1-\frac{1}{n^2}\right)}{n^5\left(1+\frac{1}{n}+\frac{1}{n^3}\right)} \\
&= \lim_{n\to\infty} \left(\underbrace{\frac{1}{n^3}}_{\to 0} \cdot \underbrace{\frac{1-\frac{1}{n^2}}{1+\frac{1}{n}+\frac{1}{n^3}}}_{\to 1} \right) \\
&= 0
\end{aligned}
$$

Example 2.7.6

Compute the following limits:

(a) $\lim_{n\to\infty} \left(\sqrt{7n^2+n} - \sqrt{2n^2-2n}\right)$;
(b) $\lim_{n\to\infty} \left(\sqrt{n^2+n} - \sqrt{n^3-1}\right)$;
(c) $\lim_{n\to\infty} \left(\sqrt{n^2+n} - \sqrt{n^2-2n}\right)$.

(Note that each of the above is of the form "$\infty - \infty$.")

Solution. **Informally**, the idea is as follows:

(a) $\sqrt{7n^2+n} - \sqrt{2n^2-2n} \approx \sqrt{7n^2} - \sqrt{2n^2} = n(\sqrt{7}-\sqrt{2}) \to \infty$;
(b) $\sqrt{n^2+n} - \sqrt{n^3-1} \approx \sqrt{n^2} - \sqrt{n^3} = n - n\sqrt{n} \to -\infty$ (because $n\sqrt{n}$ increases much faster than n);
(c) $\sqrt{n^2+n} - \sqrt{n^2-2n} \approx \sqrt{n^2} - \sqrt{n^2}$ (pure "$\infty - \infty$").
    - For $\sqrt{n^2+n} \approx \sqrt{n^2}$ and $\sqrt{n^2-2n} \approx \sqrt{n^2}$, the problem is that the error is only small relative to $\sqrt{n^2}$, and it **cannot** necessarily be made smaller than an arbitrarily small $\varepsilon > 0$.
    - Thus, $\lim_{n\to\infty} \left(\sqrt{n^2+n} - \sqrt{n^2-2n}\right) \not\asymp 0$, and we need to compute the limit more intelligently.

Example 2.7.6

Compute the following limits:

(a) $\lim_{n\to\infty}\left(\sqrt{7n^2+n}-\sqrt{2n^2-2n}\right)$;
(b) $\lim_{n\to\infty}\left(\sqrt{n^2+n}-\sqrt{n^3-1}\right)$;
(c) $\lim_{n\to\infty}\left(\sqrt{n^2+n}-\sqrt{n^2-2n}\right)$.

(Note that each of the above is of the form "$\infty-\infty$.")

Solution (continued). The above only gives the intuition, and it does **not** count as a proper proof! Let us try to formalize this, i.e. give a proper solution.

Solution (continued). (a)

$$\lim_{n\to\infty}\left(\sqrt{7n^2+n}-\sqrt{2n^2-2n}\right)$$

$$= \lim_{n\to\infty}\Big(\underset{\nearrow +\infty}{\boxed{n}}\;\underbrace{\left(\sqrt{7+\frac{1}{n}}-\sqrt{2-\frac{2}{n}}\right)}_{\longrightarrow\sqrt{7}-\sqrt{2}} = \infty$$

Solution (continued). (b)

$$\lim_{n\to\infty}\left(\sqrt{n^2+n}-\sqrt{n^3-1}\right)$$

$$= \lim_{n\to\infty}\Bigg(\underbrace{n\sqrt{n}}_{\to+\infty}\Bigg(\underbrace{\sqrt{\frac{1}{n}+\frac{1}{n^2}}}_{\longrightarrow 0}-\underbrace{\sqrt{1-\frac{1}{n^3}}}_{\longrightarrow 1}\Bigg) = -\infty$$

Solution. (c) Here, the trick is to multiply and divide by $\sqrt{n^2+n}+\sqrt{n^2-2n}$, and then make use of the familiar formula $(x-y)(x+y)=x^2-y^2$. This way, we will eliminate square roots in the numerator, while obtaining the sum (rather than difference) of square roots in the denominator. Formally, we have the following.

$$
\begin{aligned}
\lim_{n\to\infty}\left(\sqrt{n^2+n}-\sqrt{n^2-2n}\right) &= \lim_{n\to\infty}\frac{\left(\sqrt{n^2+n}-\sqrt{n^2-2n}\right)\left(\sqrt{n^2+n}+\sqrt{n^2-2n}\right)}{\sqrt{n^2+n}+\sqrt{n^2-2n}} \\
&= \lim_{n\to\infty}\frac{(n^2+n)-(n^2-2n)}{\sqrt{n^2+n}+\sqrt{n^2-2n}} \\
&= \lim_{n\to\infty}\frac{3n}{n\left(\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{2}{n}}\right)} \\
&= \lim_{n\to\infty}\frac{3}{\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{2}{n}}} \\
&= \frac{3}{2}
\end{aligned}
$$

Theorem 2.7.7

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers s.t. $a_n \leq b_n$ for all $n \in \mathbb{N}$. Then

(a) if $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ both converge, then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$;

(b) if $\lim_{n\to\infty} a_n = +\infty$, then $\lim_{n\to\infty} b_n = +\infty$;

(c) if $\lim_{n\to\infty} b_n = -\infty$, then $\lim_{n\to\infty} a_n = -\infty$.

- First an example, then a proof.

Theorem 2.7.7

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers s.t. $a_n \le b_n$ for all $n \in \mathbb{N}$. Then

(a) if $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ both converge, then $\lim\limits_{n\to\infty} a_n \le \lim\limits_{n\to\infty} b_n$;

(b) if $\lim\limits_{n\to\infty} a_n = +\infty$, then $\lim\limits_{n\to\infty} b_n = +\infty$;

(c) if $\lim\limits_{n\to\infty} b_n = -\infty$, then $\lim\limits_{n\to\infty} a_n = -\infty$.

Example 2.7.8

Compute $\lim\limits_{n\to\infty} (n + (-1)^n\sqrt{n})$.

Solution. Clearly, $n - \sqrt{n} \le n + (-1)^n\sqrt{n} \quad \forall n \in \mathbb{N}$. Since

$$\lim_{n\to\infty} (n - \sqrt{n}) = \lim_{n\to\infty} \Big(\underbrace{n}_{\to +\infty} \big(1 - \underbrace{\tfrac{1}{\sqrt{n}}}_{\to 0}\big)\Big) = +\infty,$$

Theorem 2.7.7(b) implies that $\lim\limits_{n\to\infty} (n + (-1)^n\sqrt{n}) = +\infty$. $\square$

Theorem 2.7.7

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers s.t. $a_n \leq b_n$ for all $n \in \mathbb{N}$. Then

(a) if $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ both converge, then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$;

(b) if $\lim_{n\to\infty} a_n = +\infty$, then $\lim_{n\to\infty} b_n = +\infty$;

(c) if $\lim_{n\to\infty} b_n = -\infty$, then $\lim_{n\to\infty} a_n = -\infty$.

Proof. We prove (a) and (b). The proof of (c) is similar to the proof of (b).

Theorem 2.7.7

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers s.t. $a_n \leq b_n$ for all $n \in \mathbb{N}$. Then

(a) if $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ both converge, then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$;

Proof of (a). Assume that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ both converge, and set $a := \lim_{n\to\infty} a_n$ and $b := \lim_{n\to\infty} b_n$. WTS $a \leq b$. Suppose otherwise, so that $b < a$. Set $\varepsilon := \frac{a-b}{2}$, so that $a - \varepsilon = b + \varepsilon$.

Using the fact that $a = \lim_{n\to\infty} a_n$, we fix some $N_1 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$, if $n \geq N_1$, then $|a_n - a| < \varepsilon$, i.e. $a - \varepsilon < a_n < a + \varepsilon$.

Similarly, using the fact that $b = \lim_{n\to\infty} b_n$, we fix some $N_2 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$, if $n \geq N_2$, then $|b_n - b| < \varepsilon$, i.e. $b - \varepsilon < b_n < b + \varepsilon$.

Set $N := \max\{N_1, N_2\}$. Then

$$b_N < b + \varepsilon = a - \varepsilon < a_N,$$

contrary to the fact that $a_N \leq b_N$. $\square$

Theorem 2.7.7

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers s.t. $a_n \leq b_n$ for all $n \in \mathbb{N}$. Then

(b) if $\lim\limits_{n\to\infty} a_n = +\infty$, then $\lim\limits_{n\to\infty} b_n = +\infty$;

Proof of (b). Assume that $\lim\limits_{n\to\infty} a_n = +\infty$. WTS $\lim\limits_{n\to\infty} b_n = +\infty$.

Fix $M \in \mathbb{R}$.

Using the fact that $\lim\limits_{n\to\infty} a_n = +\infty$, we fix some $N \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$, if $n \geq N$, then $a_n > M$.

But now $\forall n \in \mathbb{N}$ s.t. $n \geq N$, we have that $b_n \geq a_n > M$. This proves that $\lim\limits_{n\to\infty} b_n = +\infty$. $\square$

Theorem 2.7.7

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers s.t. $a_n \leq b_n$ for all $n \in \mathbb{N}$. Then

(a) if $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ both converge, then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$;

(b) if $\lim_{n\to\infty} a_n = +\infty$, then $\lim_{n\to\infty} b_n = +\infty$;

(c) if $\lim_{n\to\infty} b_n = -\infty$, then $\lim_{n\to\infty} a_n = -\infty$.

Proposition 2.7.9

Let $q \in \mathbb{R}$. Then:

(a) if $|q| < 1$, then $\lim_{n\to\infty} q^n = 0$;

(b) if $q = 1$, $\lim_{n\to\infty} q^n = 1$;

(c) if $q > 1$, then $\lim_{n\to\infty} q^n = +\infty$;

(d) if $q \leq -1$, then $\lim_{n\to\infty} q^n$ does not exist, i.e. the sequence $\{q^n\}_{n=1}^{\infty}$ diverges, but neither to $+\infty$ nor to $-\infty$.

- Part (b) is obvious. We will prove (a) and (c). The proof of (d) is in the Lecture Notes.

Proposition 2.7.9

Let $q \in \mathbb{R}$. Then:

(a) if $|q| < 1$, then $\lim\limits_{n\to\infty} q^n = 0$;

(b) if $q = 1$, $\lim\limits_{n\to\infty} q^n = 1$;

(c) if $q > 1$, then $\lim\limits_{n\to\infty} q^n = +\infty$;

(d) if $q \leq -1$, then $\lim\limits_{n\to\infty} q^n$ does not exist, i.e. the sequence $\{q^n\}_{n=1}^{\infty}$ diverges, but neither to $+\infty$ nor to $-\infty$.

Proof of (c). Assume that $q > 1$. Set $x := q - 1$. Then $x > 0$, and so by Bernoulli's inequality, we have that

$$q^n = (1+x)^n \geq 1 + nx,$$

and clearly, $\lim\limits_{n\to\infty}(1 + nx) = +\infty$. Therefore, by Theorem 2.7.7(b), we have that $\lim\limits_{n\to\infty} q^n = +\infty$. This proves (c).

Proposition 2.7.9

Let $q \in \mathbb{R}$. Then:

- (a) if $|q| < 1$, then $\lim_{n\to\infty} q^n = 0$;
- (b) if $q = 1$, $\lim_{n\to\infty} q^n = 1$;
- (c) if $q > 1$, then $\lim_{n\to\infty} q^n = +\infty$;
- (d) if $q \leq -1$, then $\lim_{n\to\infty} q^n$ does not exist, i.e. the sequence $\{q^n\}_{n=1}^{\infty}$ diverges, but neither to $+\infty$ nor to $-\infty$.

Proof of (a). Assume that $|q| < 1$, and we prove that $\lim_{n\to\infty} q^n = 0$. If $q = 0$, then this is obviously true. Suppose now that $0 < q < 1$. Then $\frac{1}{q} > 1$, and so by (c), we have that $\lim_{n\to\infty} \left(\frac{1}{q}\right)^n = +\infty$, and consequently,

$$\lim_{n\to\infty} q^n = \lim_{n\to\infty} \frac{1}{\left(\frac{1}{q}\right)^n} = 0.$$

$$\left(\frac{1}{q}\right)^n \to +\infty$$

Proposition 2.7.9

Let $q \in \mathbb{R}$. Then:

- (a) if $|q| < 1$, then $\lim_{n\to\infty} q^n = 0$;
- (b) if $q = 1$, $\lim_{n\to\infty} q^n = 1$;
- (c) if $q > 1$, then $\lim_{n\to\infty} q^n = +\infty$;
- (d) if $q \leq -1$, then $\lim_{n\to\infty} q^n$ does not exist, i.e. the sequence $\{q^n\}_{n=1}^{\infty}$ diverges, but neither to $+\infty$ nor to $-\infty$.

Proof of (a) (continued). Suppose now that $-1 < q < 0$. Then $0 < -q < 1$, and so by what we just showed, we have that $\lim_{n\to\infty} (-q)^n = 0$. Since

$$-(-q)^n \quad \leq \quad q^n \quad \leq \quad (-q)^n$$

for all $n \in \mathbb{N}$, the Squeeze Theorem now implies that $\lim_{n\to\infty} q^n = 0$. This proves (a). $\square$

- Reminder:

Lemma 2.3.1

Let $\{a_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of real numbers bounded above. Then $\{a_n\}_{n=1}^{\infty}$ converges, and its limit is precisely the supremum of the sequence.

Lemma 2.3.2

Let $\{a_n\}_{n=1}^{\infty}$ be a non-increasing sequence of real numbers bounded below. Then $\{a_n\}_{n=1}^{\infty}$ converges, and its limit is precisely the infimum of the sequence.

The Monotone Sequence Theorem

Every monotone and bounded sequence of real numbers is convergent.

- As we shall see (in a couple of slides), Lemmas 2.3.1 and 2.3.2 can be generalized to the non-bounded case, except that in that case, the sequence may diverge to $+\infty$ or $-\infty$.

- For a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers:
  - we define $\sup_{n\geq 1} a_n$ to be the supremum of the subsequence $\{a_n\}_{n=1}^{\infty}$ if this subsequence is bounded above,[1] and we set $\sup_{n\geq 1} a_n := +\infty$ otherwise;
  - we define $\inf_{n\geq 1} a_n$ to be the infimum of the subsequence $\{a_n\}_{n=1}^{\infty}$ if this subsequence is bounded below,[2] and we set $\inf_{n\geq 1} a_n := -\infty$ otherwise.

---

[1] Since the ordered field $\mathbb{R}$ is complete, we know that if the subsequence $\{a_n\}_{n=1}^{\infty}$ is bounded above, then it does indeed have the least upper bound (supremum).

[2] Since the ordered field $\mathbb{R}$ is complete, we know that if the subsequence $\{a_n\}_{n=1}^{\infty}$ is bounded below, then it does indeed have the greatest lower bound (infimum).

Proposition 2.7.10

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then:

(a) if $\{a_n\}_{n=1}^{\infty}$ is non-decreasing, then $\lim\limits_{n\to\infty} a_n = \sup\limits_{n\geq 1} a_n$;

(b) if $\{a_n\}_{n=1}^{\infty}$ is non-increasing, then $\lim\limits_{n\to\infty} a_n = \inf\limits_{n\geq 1} a_n$.

- Proof: Lecture Notes.
  - Easy! Uses Lemmas 2.3.1 and 2.3.2, plus the definition of divergence to $+\infty$ or $-\infty$.

Proposition 2.7.10

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then:

(a) if $\{a_n\}_{n=1}^{\infty}$ is non-decreasing, then $\lim_{n\to\infty} a_n = \sup_{n\geq 1} a_n$;

(b) if $\{a_n\}_{n=1}^{\infty}$ is non-increasing, then $\lim_{n\to\infty} a_n = \inf_{n\geq 1} a_n$.

- **Remark:** In other words:
  - part (a) states that if a non-decreasing sequence is bounded above, then it converges to its supremum, and otherwise it diverges to $+\infty$;
  - part (b) states that if a non-increasing sequence is bounded below, then it diverges to its infimum, and otherwise it diverges to $-\infty$.