# Mathematical Analysis 1

## Lecture #3

## The Bolzano-Weierstrass Theorem. Accumulation points. Divergence to infinity

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- This lecture has three parts:
  1. Subsequences and the Bolzano-Weierstrass Theorem
  2. Accumulation points
  3. Divergence to infinity

1. Subsequences and the Bolzano-Weierstrass Theorem

### Definition

A *subsequence* of a sequence $\{a_n\}_{n=1}^{\infty}$ is a sequence of the form $\{a_{n_j}\}_{j=1}^{\infty}$, where $\{n_j\}_{j=1}^{\infty}$ is a strictly increasing sequence of positive integers.

- **Remark:** Informally, a subsequence of a sequence $\{a_n\}_{n=1}^{\infty}$ is any sequence that can be obtained from $\{a_n\}_{n=1}^{\infty}$ by possibly deleting some terms, but so that infinitely many terms still remain.
  - In particular, every sequence is a subsequence of itself.

#### Proposition 2.5.1

For all sequences $\{a_n\}_{n=1}^{\infty}$ of real numbers, the following hold:

- **(a)** $\{a_n\}_{n=1}^{\infty}$ converges iff all subsequences of $\{a_n\}_{n=1}^{\infty}$ converge;
- **(b)** if $\{a_n\}_{n=1}^{\infty}$ converges, then all subsequences of $\{a_n\}_{n=1}^{\infty}$ converge to $\lim\limits_{n\to\infty} a_n$.

*Proof.* By definition, $\{a_n\}_{n=1}^{\infty}$ is a subsequence of itself. Consequently, if all subsequences of $\{a_n\}_{n=1}^{\infty}$ converge, then in particular, $\{a_n\}_{n=1}^{\infty}$ also converges.

- This proves the "$\Longleftarrow$" part of (a).

Suppose now that $\{a_n\}_{n=1}^{\infty}$ converges, and set $L := \lim\limits_{n\to\infty} a_n$. We will show that all subsequences of $\{a_n\}_{n=1}^{\infty}$ converge to $L$.

- This will prove the "$\Longrightarrow$" part of (a), as well as all of part (b).

### Proposition 2.5.1

For all sequences $\{a_n\}_{n=1}^\infty$ of real numbers, the following hold:

- $\{a_n\}_{n=1}^\infty$ converges iff all subsequences of $\{a_n\}_{n=1}^\infty$ converge;
- if $\{a_n\}_{n=1}^\infty$ converges, then all subsequences of $\{a_n\}_{n=1}^\infty$ converge to $\lim\limits_{n\to\infty} a_n$.

*Proof (continued).* Reminder: $L := \lim\limits_{n\to\infty} a_n$; WTS all subsequences of $\{a_n\}_{n=1}^\infty$ converge to $L$.

Fix any subsequence $\{a_{n_j}\}_{j=1}^\infty$ of $\{a_n\}_{n=1}^\infty$, where $\{n_j\}_{j=1}^\infty$ is a strictly increasing sequence of positive integers. WTS $\lim\limits_{j\to\infty} a_{n_j} = L$.

Fix any $\varepsilon > 0$, and using the fact that $L := \lim\limits_{n\to\infty} a_n$, fix $N \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$, if $n \geq N$, then $|a_n - L| < \varepsilon$.

Since $\{n_j\}_{j=1}^\infty$ is a strictly increasing sequence of positive integers, we see that $\forall j \in \mathbb{N}$ s.t. $j \geq N$, we have that $n_j \geq n_N \geq N$, and consequently, $|a_{n_j} - L| < \varepsilon$. This proves that $L = \lim\limits_{j\to\infty} a_{n_j}$. $\square$

### Lemma 2.5.2

Every sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers has a monotone subsequence.

- Proof: In a minute!

### The Bolzano-Weierstrass Theorem

Every bounded sequence of real numbers has a convergent subsequence.

*Proof (assuming Lemma 2.5.2)* Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers. By Lemma 2.5.2, $\{a_n\}_{n=1}^{\infty}$ has a monotone subsequence, say $\{a_{n_j}\}_{j=1}^{\infty}$, where $\{n_j\}_{j=1}^{\infty}$ is an increasing sequence of positive integers. Since $\{a_n\}_{n=1}^{\infty}$ is bounded, so is its subsequence $\{a_{n_j}\}_{j=1}^{\infty}$. Now $\{a_{n_j}\}_{j=1}^{\infty}$ is a monotone and bounded sequence of real numbers, and so by the Monotone Sequence Theorem, it converges. $\square$

### Lemma 2.5.2

Every sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers has a monotone subsequence.

*Proof.* Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Let us say that a positive integer $m$ is a *peak* of the sequence $\{a_n\}_{n=1}^{\infty}$ if $\forall n \in \mathbb{N}$ s.t. $n > m$, we have that $a_m > a_n$ (i.e. $a_m$ is strictly greater than every subsequent term of the sequence).

Suppose first that $\{a_n\}_{n=1}^{\infty}$ has infinitely many peaks, and let $\{n_j\}_{j=1}^{\infty}$ be the sequence of all the peaks, arranged in increasing order (i.e. $n_1 < n_2 < n_3 < \dots$). But now $a_{n_1} > a_{n_2} > a_{n_3} > \dots$, i.e. $\{a_{n_j}\}_{j=1}^{\infty}$ is a strictly decreasing (and therefore monotone) subsequence of $\{a_n\}_{n=1}^{\infty}$.

From now on, we assume that $\{a_n\}_{n=1}^{\infty}$ has only finitely many (if any) peaks. Fix some $N \in \mathbb{N}$ s.t. all the peaks of $\{a_n\}_{n=1}^{\infty}$ are strictly smaller than $N$.

- Thus, $\forall m \in \mathbb{N}$ s.t. $m \geq N$, the integer $m$ is **not** a peak, i.e. $\exists n \in \mathbb{N}$ s.t. $m < n$ and $a_m \leq a_n$.

### Lemma 2.5.2

Every sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers has a monotone subsequence.

*Proof (continued).* Reminder: $\forall m \in \mathbb{N}$ s.t. $m \geq N$, the integer $m$ is **not** a peak, i.e. $\exists n \in \mathbb{N}$ s.t. $m < n$ and $a_m \leq a_n$.

Our goal is now to (recursively) form an increasing sequence $\{n_j\}_{j=1}^{\infty}$ of positive integers s.t. $\forall j \in \mathbb{N}$: $a_{n_j} \leq a_{n_{j+1}}$. Then $\{a_{n_j}\}_{j=1}^{\infty}$ will be a non-decreasing (and therefore monotone) subsequence of $\{a_n\}_{n=1}^{\infty}$, which is what we need.

First, fix $n_1 = N$. Next, fix $j \in \mathbb{N}$, and suppose we have constructed positive integers $n_1, \ldots, n_j$ so that $n_1 < \cdots < n_j$ and $a_{n_1} \leq \cdots \leq a_{n_j}$. Since $n_j$ is not a peak (because $n_j \geq N$), $\exists n \in \mathbb{N}$ s.t. $n_j < n$ and $a_{n_j} \leq a_n$; choose $n_{j+1}$ to be the smallest such $n$.

By construction, $\{n_j\}_{j=1}^{\infty}$ is a strictly increasing sequence of positive integers, and $\{a_{n_j}\}_{j=1}^{\infty}$ is a non-decreasing (and therefore monotone) subsequence of $\{a_n\}_{n=1}^{\infty}$. $\square$

- We have proven:

### Lemma 2.5.2

Every sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers has a monotone subsequence.

### The Bolzano-Weierstrass Theorem

Every bounded sequence of real numbers has a convergent subsequence.

② Accumulation points

### Definition

An *accumulation point* of a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers is a real number $A$ s.t. for all real numbers $\varepsilon > 0$ and all $N \in \mathbb{N}$, there exists some $n \in \mathbb{N}$ s.t. $n \geq N$ and $|a_n - A| < \varepsilon$.

- How is this different from a limit?
  - Next slide!

### Definition

We say that a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers *converges* to a real number $L$ provided that the following holds:

> *For all real numbers $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ s.t. for all $n \in \mathbb{N}$: if $n \geq N$, then $|a_n - L| < \varepsilon$.*

Under such circumstances, we say that $L$ is the *limit* of the sequence $\{a_n\}_{n=1}^{\infty}$, and we write

$$L = \lim_{n \to \infty} a_n \qquad \text{or} \qquad a_n \to L \quad \text{as} \quad n \to \infty.$$

A sequence is *convergent* (or *converges*) if it has a limit. Otherwise, it is *divergent* (or *diverges*).

### Definition

An *accumulation point* of a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers is a real number $A$ s.t. for all real numbers $\varepsilon > 0$ and all $N \in \mathbb{N}$, there exists some $n \in \mathbb{N}$ s.t. $n \geq N$ and $|a_n - A| < \varepsilon$.

## Definition

An *accumulation point* of a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers is a real number $A$ s.t. for all real numbers $\varepsilon > 0$ and all $N \in \mathbb{N}$, there exists some $n \in \mathbb{N}$ s.t. $n \geq N$ and $|a_n - A| < \varepsilon$.

## Proposition 2.6.1

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers, and let $A \in \mathbb{R}$. Then the following are equivalent:

- (i) $A$ is an accumulation point of $\{a_n\}_{n=1}^{\infty}$;
- (ii) for all real numbers $\varepsilon > 0$, the interval $(A - \varepsilon, A + \varepsilon)$ contains infinitely many terms of the sequence $\{a_n\}_{n=1}^{\infty}$;
- (iii) some subsequence of $\{a_n\}_{n=1}^{\infty}$ converges to $A$.

- **Terminology:** For a real number $\varepsilon > 0$, the open interval $(A - \varepsilon, A + \varepsilon)$ is called the $\varepsilon$-*neighborhood of $A$*.

*Proof.* It suffices to prove the following sequence of implications: "(i) $\implies$ (iii) $\implies$ (ii) $\implies$ (i)."

- **(i)** $A$ is an accumulation point of $\{a_n\}_{n=1}^{\infty}$;
- **(iii)** some subsequence of $\{a_n\}_{n=1}^{\infty}$ converges to $A$.

*Proof (continued).* We first assume (i) and prove (iii). We define a strictly increasing sequence $\{n_j\}_{j=1}^{\infty}$ recursively as follows. First, using the fact that $A$ is an accumulation point of $\{a_n\}_{n=1}^{\infty}$, we fix $n_1 \in \mathbb{N}$ s.t. $|a_{n_1} - A| < 1$.

- Here, we are using the def. of an acc. pt. for $\varepsilon = 1$ and $N = 1$.

Next, assume that for some $j \in \mathbb{N}$, we have defined the positive integer $n_j$; we then define $n_{j+1} \in \mathbb{N}$ as follows. Since $A$ is an accumulation point of $\{a_n\}_{n=1}^{\infty}$, we let $n_{j+1}$ be the smallest positive integer satisfying $n_{j+1} \geq n_j + 1$ and $|A - a_{n_{j+1}}| < \frac{1}{j+1}$.

- Here, we are using the def. of an acc. pt. for $\varepsilon = \frac{1}{j+1}$ and $N = n_j + 1$.

We have now constructed a strictly increasing sequence $\{n_j\}_{j=1}^{\infty}$ of positive integers s.t. for all $j \in \mathbb{N}$, we have that $|a_{n_j} - A| < \frac{1}{j}$.

- $A$ is an accumulation point of $\{a_n\}_{n=1}^{\infty}$;
- some subsequence of $\{a_n\}_{n=1}^{\infty}$ converges to $A$.

*Proof (continued).* It is now easy to verify that $\lim\limits_{j \to \infty} a_{n_j} = A$.

Indeed, fix $\varepsilon > 0$. Let $J \in \mathbb{N}$ be s.t. $J > \frac{1}{\varepsilon}$. Fix $j \in \mathbb{N} \in \mathbb{N}$ s.t. $j \geq J$. Then

$$
\begin{aligned}
|a_{n_j} - A| \quad &< \quad \tfrac{1}{j} \qquad \text{by the construction of } n_j \\[1mm]
&\leq \quad \tfrac{1}{J} \qquad \text{because } j \geq J \\[1mm]
&< \quad \varepsilon \qquad \text{because } J > \tfrac{1}{\varepsilon}.
\end{aligned}
$$

This proves that $\lim\limits_{j \to \infty} a_{n_j} = A$, i.e. (iii) holds.

## Proposition 2.6.1

- (ii) for all real numbers $\varepsilon > 0$, the interval $(A - \varepsilon, A + \varepsilon)$ contains infinitely many terms of the sequence $\{a_n\}_{n=1}^{\infty}$;
- (iii) some subsequence of $\{a_n\}_{n=1}^{\infty}$ converges to $A$.

*Proof (continued).* Next, we assume (iii) and prove (ii).

Using (iii), we fix a strictly increasing sequence $\{n_j\}_{j=1}^{\infty}$ of positive integers s.t. $\lim_{j \to \infty} a_{n_j} = A$.

Now fix $\varepsilon > 0$. Using the definition of a limit, we now fix $J \in \mathbb{N}$ s.t. for all $j \in \mathbb{N}$, if $j \geq J$, then $|a_{n_j} - A| < \varepsilon$, i.e. $a_{n_j} \in (A - \varepsilon, A + \varepsilon)$.

But now $a_{n_J}, a_{n_{J+1}}, a_{n_{J+2}}, \ldots$ all belong to $(A - \varepsilon, A + \varepsilon)$. This proves (ii).

### Proposition 2.6.1

- $A$ is an accumulation point of $\{a_n\}_{n=1}^{\infty}$;
- for all real numbers $\varepsilon > 0$, the interval $(A - \varepsilon, A + \varepsilon)$ contains infinitely many terms of the sequence $\{a_n\}_{n=1}^{\infty}$;

*Proof (continued).* Finally, we assume (ii) and prove (i).

Fix $\varepsilon > 0$ and $N \in \mathbb{N}$. Now, using (ii), we know that there exist infinitely many positive integers $n$ s.t. $a_n \in (A - \varepsilon, A + \varepsilon)$, i.e. $|a_n - A| < \varepsilon$. Since there are infinitely many such $n$'s, one of them (in fact, infinitely many of them) must satisfy $n \geq N$. This proves (i). $\square$

### Definition

An *accumulation point* of a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers is a real number $A$ s.t. for all real numbers $\varepsilon > 0$ and all $N \in \mathbb{N}$, there exists some $n \in \mathbb{N}$ s.t. $n \geq N$ and $|a_n - A| < \varepsilon$.

### Proposition 2.6.1

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers, and let $A \in \mathbb{R}$. Then the following are equivalent:

- $A$ is an accumulation point of $\{a_n\}_{n=1}^{\infty}$;
- for all real numbers $\varepsilon > 0$, the interval $(A - \varepsilon, A + \varepsilon)$ contains infinitely many terms of the sequence $\{a_n\}_{n=1}^{\infty}$;
- some subsequence of $\{a_n\}_{n=1}^{\infty}$ converges to $A$.

- **Terminology:** For a real number $\varepsilon > 0$, the open interval $(A - \varepsilon, A + \varepsilon)$ is called the $\varepsilon$-*neighborhood of A*.

### Proposition 2.6.1 (abbreviated)

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers, and let $A \in \mathbb{R}$. Then the following are equivalent:

- ⓘ $A$ is an accumulation point of $\{a_n\}_{n=1}^{\infty}$;
- ⓘ some subsequence of $\{a_n\}_{n=1}^{\infty}$ converges to $A$.

### Theorem 2.6.2

Let $\{a_n\}_{n=1}^{\infty}$ be a **convergent** sequence of real numbers. Then $\{a_n\}_{n=1}^{\infty}$ has exactly one accumulation point, namely $L := \lim\limits_{n \to \infty} a_n$.

*Proof.* First of all, $\{a_n\}_{n=1}^{\infty}$ is a subsequence of itself, and it converges to $L$; so, by Proposition 2.6.1, $L$ is indeed an accumulation point of $\{a_n\}_{n=1}^{\infty}$.

On the other hand, by Proposition 2.5.1, all subsequences of $\{a_n\}_{n=1}^{\infty}$ converge to $L$, and so by Proposition 2.6.2, $\{a_n\}_{n=1}^{\infty}$ has no accumulation points other than $L$. $\square$

### Theorem 2.6.2

Let $\{a_n\}_{n=1}^{\infty}$ be a **convergent** sequence of real numbers. Then $\{a_n\}_{n=1}^{\infty}$ has exactly one accumulation point, namely $L := \lim\limits_{n\to\infty} a_n$.

- **Remark:**
  - By Theorem 2.6.2, if a sequence has more than one accumulation point, then it diverges.
    - For instance, the sequence $\{(-1)^n\}_{n=1}^{\infty}$ has two accumulation points, namely $1$ and $-1$, and so it diverges.
  - However, the converse of Theorem 2.6.2 is false in general, i.e. some sequences that only have one accumulation point nevertheless diverge.
    - One example is the sequence $\{a_n\}_{n=1}^{\infty}$ given by

$$a_n := \begin{cases} \frac{1}{n} & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$$

      for all $n \in \mathbb{N}$. (This is the sequence $1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, 7, \frac{1}{8}, \dots$)
    - Clearly, the only accumulation point of the sequence $\{a_n\}_{n=1}^{\infty}$ is $0$, and yet the sequence diverges.

③ Divergence to infinity

### Definition

A sequence $\{a_n\}_{n=1}^{\infty}$ *diverges to infinity*, and we write

$$\lim_{n\to\infty} a_n = \infty \qquad \text{or} \qquad \lim_{n\to\infty} a_n = +\infty,$$

or alternatively,

$$a_n \to \infty \quad \text{as} \quad n \to \infty \qquad \text{or} \qquad a_n \to +\infty \quad \text{as} \quad n \to \infty,$$

if for all $M \in \mathbb{R}$, there exists some $N \in \mathbb{N}$ s.t. for all $n \in \mathbb{N}$, if $n \geq N$, then $a_n > M$.

## Definition

A sequence $\{a_n\}_{n=1}^{\infty}$ *diverges to infinity*, and we write

$$\lim_{n \to \infty} a_n = \infty \qquad \text{or} \qquad \lim_{n \to \infty} a_n = +\infty,$$

or alternatively,

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if for all $M \in \mathbb{R}$, there exists some $N \in \mathbb{N}$ s.t. for all $n \in \mathbb{N}$, if $n \geq N$, then $a_n > M$.

## Example 2.7.1

Using the definition, show that $\lim_{n \to \infty} n^2 = +\infty$.

*Solution.* Fix $M \in \mathbb{R}$. Let $N \in \mathbb{N}$ be s.t. $N > M$. Then $\forall n \in \mathbb{N}$ s.t. $n \geq N$, we have that $n^2 \geq N^2 \geq N > M$. This proves that $\lim_{n \to \infty} n^2 = +\infty$. $\square$

### Definition

A sequence $\{a_n\}_{n=1}^{\infty}$ *diverges to negative infinity*, and we write

$$\lim_{n \to \infty} a_n = -\infty,$$

or alternatively,

$$a_n \to -\infty \quad \text{as} \quad n \to \infty,$$

if for all $M \in \mathbb{R}$, there exists some $N \in \mathbb{N}$ s.t. for all $n \in \mathbb{N}$, if $n \geq N$, then $a_n < M$.

### Example 2.7.2

Using the definition, show that $\lim_{n \to \infty} \left( -\sqrt{n} \right) = -\infty$.

*Solution.* Fix $M \in \mathbb{R}$. Fix $N \in \mathbb{N}$ s.t. $N > M^2$. Then $\sqrt{N} > |M| \geq -M$, and consequently, $-\sqrt{N} < M$. It follows that $\forall n \in \mathbb{N}$ s.t. $n \geq N$, we have that $-\sqrt{n} \leq -\sqrt{N} < M$. This proves that $\lim_{n \to \infty} \left( -\sqrt{n} \right) = -\infty$. $\square$

### Proposition 2.7.3

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then:

- if $\lim_{n\to\infty} a_n = +\infty$, then $\{a_n\}_{n=1}^{\infty}$ is bounded below, but is not bounded above;

- if $\lim_{n\to\infty} a_n = -\infty$, then $\{a_n\}_{n=1}^{\infty}$ is bounded above, but is not bounded below.

- Proof: exercise (easy!).

- Reminder:

### Theorem 2.2.5

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be convergent sequences of real numbers, and let $c \in \mathbb{R}$. Then all the following hold:

- (a) $\lim_{n \to \infty} (c a_n) = c \lim_{n \to \infty} (a_n)$;
- (b) $\lim_{n \to \infty} (a_n + b_n) = (\lim_{n \to \infty} a_n) + (\lim_{n \to \infty} b_n)$;
- (c) $\lim_{n \to \infty} (a_n - b_n) = (\lim_{n \to \infty} a_n) - (\lim_{n \to \infty} b_n)$;
- (d) $\lim_{n \to \infty} (a_n b_n) = (\lim_{n \to \infty} a_n)(\lim_{n \to \infty} b_n)$;
- (e) if $b_n \neq 0$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} b_n \neq 0$, then
  $$\lim_{n \to \infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n \to \infty} a_n}{\lim_{n \to \infty} b_n}.$$

- **Remark:** Properties of limits from Theorem 2.2.5 (from the previous slide) readily generalize to divergence to (positive or negative) infinity.
    - We do not state an analogous theorem formally.
    - Instead, we focus on the "danger zones."
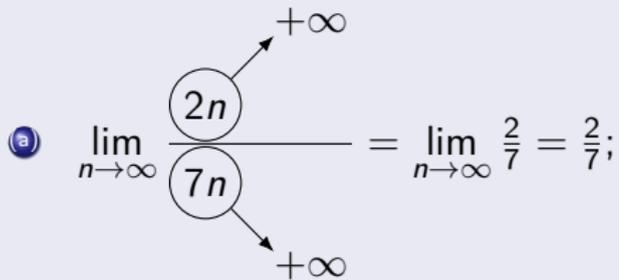    - In particular, the following forms are "indeterminate," i.e. they can in principle be anything:

        $$\frac{0}{0} \qquad \frac{\infty}{\infty} \qquad 0 \cdot \infty \qquad 1^{\infty} \qquad \infty - \infty \qquad 0^{0} \qquad \infty^{0}$$

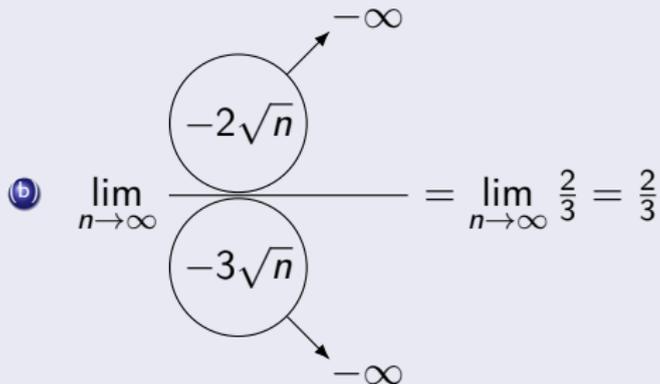- Let us take a look at a few examples!

### Example 2.7.4

Consider the behavior of the following "$\frac{\infty}{\infty}$" forms:

(a) $\lim\limits_{n \to \infty} \dfrac{2n}{7n} = \lim\limits_{n \to \infty} \dfrac{2}{7} = \dfrac{2}{7}$;

where $2n \to +\infty$ and $7n \to +\infty$

(b) $\lim\limits_{n \to \infty} \dfrac{-2\sqrt{n}}{-3\sqrt{n}} = \lim\limits_{n \to \infty} \dfrac{2}{3} = \dfrac{2}{3}$

where $-2\sqrt{n} \to -\infty$ and $-3\sqrt{n} \to -\infty$

## Example 2.7.4 (continued)

Consider the behavior of the following "$\frac{\infty}{\infty}$" forms:

- $\displaystyle \lim_{n \to \infty} \frac{\overset{\displaystyle +\infty}{\overbrace{\big(3n^2\big)}}}{\underset{\displaystyle +\infty}{\underbrace{\big(2n\big)}}} = \lim_{n \to \infty} \frac{3n}{2} = +\infty;$

- $\displaystyle \lim_{n \to \infty} \frac{\overset{\displaystyle -\infty}{\overbrace{\big(-2n^2\big)}}}{\underset{\displaystyle +\infty}{\underbrace{\big(5n^3\big)}}} = \lim_{n \to \infty} \big(-\tfrac{2}{5n}\big) = 0.$

Compute the following limits:

- $\displaystyle\lim_{n\to\infty} \frac{2n^2-3n+7}{-n^2+2}$;

- $\displaystyle\lim_{n\to\infty} \frac{-n^3+5n-1}{2n-1}$;

- $\displaystyle\lim_{n\to\infty} \frac{n^2-1}{n^5+n^4+n^2}$.

(Note that each of the above is of the form "$\frac{\infty}{\infty}$.")

*Solution.* In each part, we start by factoring out the largest degree term from both the numerator and the denominator, and then we evaluate.

$$\text{(a)} \lim_{n\to\infty} \frac{2n^2-3n+7}{-n^2+2} = \lim_{n\to\infty} \frac{\cancel{n^2}\left(2-\overset{0}{\underbrace{\frac{3}{n}}}+\overset{0}{\underbrace{\frac{7}{n^2}}}\right)}{\cancel{n^2}\left(-1+\underset{0}{\underbrace{\frac{2}{n^2}}}\right)} = -2$$

*Solution (continued).* (b)

$$\lim_{n\to\infty} \frac{-n^3+5n-1}{2n-1} = \lim_{n\to\infty} \frac{n^3\left(-1+\frac{5}{n^2}-\frac{1}{n^3}\right)}{n\left(2-\frac{1}{n}\right)}$$

$$= \lim_{n\to\infty} \left( n^2 \cdot \frac{-1+\frac{5}{n^2}-\frac{1}{n^3}}{2-\frac{1}{n}} \right)$$

$$= -\infty$$

where $n^2 \to +\infty$ and $\frac{-1+\frac{5}{n^2}-\frac{1}{n^3}}{2-\frac{1}{n}} \to -\frac{1}{2}$

*Solution (continued).* (c)

$$\lim_{n \to \infty} \frac{n^2 - 1}{n^5 + n^4 + n^2} = \lim_{n \to \infty} \frac{n^2 \left(1 - \frac{1}{n^2}\right)}{n^5 \left(1 + \frac{1}{n} + \frac{1}{n^3}\right)}$$

$$= \lim_{n \to \infty} \left( \underbrace{\left(\frac{1}{n^3}\right)}_{0} \cdot \underbrace{\left(\frac{1 - \frac{1}{n^2}}{1 + \frac{1}{n} + \frac{1}{n^3}}\right)}_{1} \right)$$

$$= 0$$

Example 2.7.6

Compute the following limits:

(a) $\lim\limits_{n\to\infty} \left(\sqrt{7n^2 + n} - \sqrt{2n^2 - 2n}\right)$;

(b) $\lim\limits_{n\to\infty} \left(\sqrt{n^2 + n} - \sqrt{n^3 - 1}\right)$;

(c) $\lim\limits_{n\to\infty} \left(\sqrt{n^2 + n} - \sqrt{n^2 - 2n}\right)$.

(Note that each of the above is of the form "$\infty - \infty$.")

*Solution.* **Informally**, the idea is as follows:

(a) $\sqrt{7n^2 + n} - \sqrt{2n^2 - 2n} \approx \sqrt{7n^2} - \sqrt{2n^2} = n(\sqrt{7} - \sqrt{2}) \to \infty$;

(b) $\sqrt{n^2 + n} - \sqrt{n^3 - 1} \approx \sqrt{n^2} - \sqrt{n^3} = n - n\sqrt{n} \to -\infty$
   (because $n\sqrt{n}$ increases much faster than $n$);

(c) $\sqrt{n^2 + n} - \sqrt{n^2 - 2n} \approx \sqrt{n^2} - \sqrt{n^2}$ (pure "$\infty - \infty$").

- For $\sqrt{n^2 + n} \approx \sqrt{n^2}$ and $\sqrt{n^2 - 2n} \approx \sqrt{n^2}$, the problem is that the error is only small relative to $\sqrt{n^2}$, and it **cannot** necessarily be made smaller than an arbitrarily small $\varepsilon > 0$.

- Thus, $\lim\limits_{n\to\infty} \left(\sqrt{n^2 + n} - \sqrt{n^2 - 2n}\right) \not\approx 0$, and we need to compute the limit more intelligently.

### Example 2.7.6

Compute the following limits:

a. $\lim\limits_{n\to\infty} \left(\sqrt{7n^2 + n} - \sqrt{2n^2 - 2n}\right)$;

b. $\lim\limits_{n\to\infty} \left(\sqrt{n^2 + n} - \sqrt{n^3 - 1}\right)$;

c. $\lim\limits_{n\to\infty} \left(\sqrt{n^2 + n} - \sqrt{n^2 - 2n}\right)$.

(Note that each of the above is of the form "$\infty - \infty$.")

*Solution (continued).* The above only gives the intuition, and it does **not** count as a proper proof! Let us try to formalize this, i.e. give a proper solution.

*Solution (continued).* (a)

$$\lim_{n\to\infty} \left( \sqrt{7n^2 + n} - \sqrt{2n^2 - 2n} \right)$$

$$= \lim_{n\to\infty} \left( \overset{+\infty}{\textcircled{n}} \left( \underbrace{\sqrt{7 + \frac{1}{n}} - \sqrt{2 - \frac{2}{n}}}_{\longrightarrow \sqrt{7} - \sqrt{2}} \right) \right) = \infty$$

*Solution (continued).* (b)

$$\lim_{n\to\infty} \left( \sqrt{n^2 + n} - \sqrt{n^3 - 1} \right)$$

$$= \lim_{n\to\infty} \left( \underbrace{\left(n\sqrt{n}\right)}_{} \quad \left( \underbrace{\sqrt{\frac{1}{n} + \frac{1}{n^2}}}_{\longrightarrow 0} - \underbrace{\sqrt{1 - \frac{1}{n^3}}}_{\longrightarrow 1} \right) \right) = -\infty$$

$+\infty$

*Solution.* (c) Here, the trick is to multiply and divide by $\sqrt{n^2 + n} + \sqrt{n^2 - 2n}$, and then make use of the familiar formula $(x - y)(x + y) = x^2 - y^2$. This way, we will eliminate square roots in the numerator, while obtaining the sum (rather than difference) of square roots in the denominator. Formally, we have the following.

$$
\begin{aligned}
\lim_{n \to \infty} \left( \sqrt{n^2 + n} - \sqrt{n^2 - 2n} \right) &= \lim_{n \to \infty} \frac{\left(\sqrt{n^2+n} - \sqrt{n^2-2n}\right)\left(\sqrt{n^2+n} + \sqrt{n^2-2n}\right)}{\sqrt{n^2+n} + \sqrt{n^2-2n}} \\[2mm]
&= \lim_{n \to \infty} \frac{(n^2+n) - (n^2-2n)}{\sqrt{n^2+n} + \sqrt{n^2-2n}} \\[2mm]
&= \lim_{n \to \infty} \frac{3n}{n\left(\sqrt{1+\frac{1}{n}} + \sqrt{1-\frac{2}{n}}\right)} \\[2mm]
&= \lim_{n \to \infty} \frac{3}{\sqrt{1+\frac{1}{n}} + \sqrt{1-\frac{2}{n}}} \\[2mm]
&= \frac{3}{2}
\end{aligned}
$$

### Theorem 2.7.7

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers s.t.
$a_n \leq b_n$ for all $n \in \mathbb{N}$. Then

- if $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ both converge, then
  $\lim\limits_{n \to \infty} a_n \leq \lim\limits_{n \to \infty} b_n$;
- if $\lim\limits_{n \to \infty} a_n = +\infty$, then $\lim\limits_{n \to \infty} b_n = +\infty$;
- if $\lim\limits_{n \to \infty} b_n = -\infty$, then $\lim\limits_{n \to \infty} a_n = -\infty$.

- First an example, then a proof.

### Theorem 2.7.7

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers s.t. $a_n \le b_n$ for all $n \in \mathbb{N}$. Then

- (a) if $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ both converge, then $\lim\limits_{n \to \infty} a_n \le \lim\limits_{n \to \infty} b_n$;
- (b) if $\lim\limits_{n \to \infty} a_n = +\infty$, then $\lim\limits_{n \to \infty} b_n = +\infty$;
- (c) if $\lim\limits_{n \to \infty} b_n = -\infty$, then $\lim\limits_{n \to \infty} a_n = -\infty$.

### Example 2.7.8

Compute $\lim\limits_{n \to \infty} (n + (-1)^n \sqrt{n})$.

*Solution.* Clearly, $n - \sqrt{n} \le n + (-1)^n \sqrt{n} \quad \forall n \in \mathbb{N}$. Since

$$\lim_{n \to \infty} (n - \sqrt{n}) = \lim_{n \to \infty} \left( \overset{+\infty}{\boxed{n}} \ \ \left(1 - \overset{0}{\left(\boxed{\tfrac{1}{\sqrt{n}}}\right)}\right)\right) = +\infty,$$

Theorem 2.7.7(b) implies that $\lim\limits_{n \to \infty} (n + (-1)^n \sqrt{n}) = +\infty$. $\square$

### Theorem 2.7.7

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers s.t. $a_n \leq b_n$ for all $n \in \mathbb{N}$. Then

- (a) if $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ both converge, then
  $$\lim_{n \to \infty} a_n \leq \lim_{n \to \infty} b_n;$$
- (b) if $\lim\limits_{n \to \infty} a_n = +\infty$, then $\lim\limits_{n \to \infty} b_n = +\infty$;
- (c) if $\lim\limits_{n \to \infty} b_n = -\infty$, then $\lim\limits_{n \to \infty} a_n = -\infty$.

*Proof.* We prove (a) and (b). The proof of (c) is similar to the proof of (b).

### Theorem 2.7.7

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers s.t. $a_n \leq b_n$ for all $n \in \mathbb{N}$. Then

- if $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ both converge, then
  $\lim\limits_{n\to\infty} a_n \leq \lim\limits_{n\to\infty} b_n$;

*Proof of (a).* Assume that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ both converge, and set $a := \lim\limits_{n\to\infty} a_n$ and $b := \lim\limits_{n\to\infty} b_n$. WTS $a \leq b$. Suppose otherwise, so that $b < a$. Set $\varepsilon := \frac{a-b}{2}$, so that $a - \varepsilon = b + \varepsilon$.

Using the fact that $a = \lim\limits_{n\to\infty} a_n$, we fix some $N_1 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$, if $n \geq N_1$, then $|a_n - a| < \varepsilon$, i.e. $a - \varepsilon < a_n < a + \varepsilon$.

Similarly, using the fact that $b = \lim\limits_{n\to\infty} b_n$, we fix some $N_2 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$, if $n \geq N_2$, then $|b_n - b| < \varepsilon$, i.e. $b - \varepsilon < b_n < b + \varepsilon$.

Set $N := \max\{N_1, N_2\}$. Then

$$b_N \ < \ b + \varepsilon \ = \ a - \varepsilon \ < \ a_N,$$

contrary to the fact that $a_N \leq b_N$. $\square$

## Theorem 2.7.7

Let $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ be sequences of real numbers s.t. $a_n \le b_n$ for all $n \in \mathbb{N}$. Then

**(b)** if $\lim\limits_{n\to\infty} a_n = +\infty$, then $\lim\limits_{n\to\infty} b_n = +\infty$;

*Proof of (b).* Assume that $\lim\limits_{n\to\infty} a_n = +\infty$. WTS $\lim\limits_{n\to\infty} b_n = +\infty$.

Fix $M \in \mathbb{R}$.

Using the fact that $\lim\limits_{n\to\infty} a_n = +\infty$, we fix some $N \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$, if $n \ge N$, then $a_N > M$.

But now $\forall n \in \mathbb{N}$ s.t. $n \ge N$, we have that $b_n \ge a_n > M$. This proves that $\lim\limits_{n\to\infty} b_n = +\infty$. $\square$

### Theorem 2.7.7

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers s.t.
$a_n \leq b_n$ for all $n \in \mathbb{N}$. Then

a. if $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ both converge, then
$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n;$$

b. if $\lim_{n\to\infty} a_n = +\infty$, then $\lim_{n\to\infty} b_n = +\infty$;

c. if $\lim_{n\to\infty} b_n = -\infty$, then $\lim_{n\to\infty} a_n = -\infty$.