Mathematical Analysis 1

Lecture #2

Properties of limits. The Monotone Sequence Theorem and the Squeeze Theorem

Irena Penev

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- This lecture has five parts:
  1. Properties of limits
  2. The Monotone Sequence Theorem
  3. Euler's number
  4. The Squeeze Theorem
  5. Some limits with roots

1. Properties of limits

- Reminder:

Definition

We say that a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers *converges* to a real number L provided that the following holds:

> *For all real numbers $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ s.t. for all $n \in \mathbb{N}$: if $n \geq N$, then $|a_n - L| < \varepsilon$.*

Under such circumstances, we say that L is the *limit* of the sequence $\{a_n\}_{n=1}^{\infty}$, and we write

$$L = \lim_{n\to\infty} a_n \qquad \text{or} \qquad a_n \to L \quad \text{as} \quad n \to \infty.$$

A sequence is *convergent* (or *converges*) if it has a limit. Otherwise, it is *divergent* (or *diverges*).

- **Remark:** Whether a sequence converges or diverges is determined by what happens in the sequence "eventually," i.e. the first few terms do not count. Hence (next slide):

Lemma 2.2.2

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers, and assume that there exits some $N \in \mathbb{N}$ s.t. for all $n \in \mathbb{N}$, if $n \geq N$, then $a_n = b_n$.[a] Then:

(a) $\{a_n\}_{n=1}^{\infty}$ converges iff $\{b_n\}_{n=1}^{\infty}$ converges;

(b) if $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ both converge, then $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n$.

[a] In other words, the two sequences may possibly differ in the first N terms, but after that, they coincide.

- Proof: Exercise.

- In the previous lecture, we defined upper and lower bounds of sets of real numbers, as well as their suprema and infima.
- We can analogously (and straightforwardly) define these concepts for sequences of real numbers, as follows.

Definition

For a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers:

- an *upper bound* of $\{a_n\}_{n=1}^{\infty}$ is any real number x s.t. for all $n \in \mathbb{N}$, we have $a_n \leq x$.
- the *least upper bound* (or *supremum*) of $\{a_n\}_{n=1}^{\infty}$ is an upper bound x of $\{a_n\}_{n=1}^{\infty}$ s.t. for every upper bound y of $\{a_n\}_{n=1}^{\infty}$, we have that $x \leq y$;
- $\{a_n\}_{n=1}^{\infty}$ is *bounded above* if it has an upper bound;
- a *lower bound* of $\{a_n\}_{n=1}^{\infty}$ is any real number x s.t. for all $n \in \mathbb{N}$, we have $x \leq a_n$.
- the *greatest lower bound* (or *infimum*) of $\{a_n\}_{n=1}^{\infty}$ is a lower bound x of $\{a_n\}_{n=1}^{\infty}$ s.t. for every lower bound y of $\{a_n\}_{n=1}^{\infty}$, we have that $y \leq x$;
- $\{a_n\}_{n=1}^{\infty}$ is *bounded below* if it has a lower bound;
- $\{a_n\}_{n=1}^{\infty}$ is *bounded* if it is both bounded above and bounded below.

- **Remarks:**
  - Since the ordered field $\mathbb{R}$ is complete, any sequence of real numbers that is bounded above has a supremum, and any sequence of real numbers that is bounded below has an infimum.
  - Note that if $\{a_n\}_{n=1}^{\infty}$ is a bounded sequence of real numbers, then there exists a real number M s.t. for all $n \in \mathbb{N}$, we have that $|a_n| \leq M$, and in particular, the sequence $\{|a_n|\}_{n=1}^{\infty}$ is also bounded.
    - Indeed, suppose that x is an upper bound of $\{a_n\}_{n=1}^{\infty}$, and that y is a lower bound of $\{a_n\}_{n=1}^{\infty}$. Set $M := \max\{|x|, |y|\}$. Then for all $n \in \mathbb{N}$, we have that $-M \leq y \leq a_n \leq x \leq M$, and consequently, $|a_n| \leq M$.

Lemma 2.2.3

Let $\{a_n\}_{n=1}^{\infty}$ be a **convergent** sequence of real numbers. Then we have the following:

1. $\{a_n\}_{n=1}^{\infty}$ is bounded;
2. for all $m, M \in \mathbb{R}$ s.t. m is a lower bound and M an upper bound of $\{a_n\}_{n=1}^{\infty}$, we have that $m \leq \lim\limits_{n\to\infty} a_n \leq M$.

Proof. (a) Fix an arbitrary real number $\varepsilon > 0$, and using the fact that $L = \lim\limits_{n\to\infty} a_n$, choose $N \in \mathbb{N}$ s.t. for all $n \in \mathbb{N}$, if $n \geq N$, then $|a_n - L| < \varepsilon$. Now, set

$$m := \min\{a_1, \dots, a_N, L - \varepsilon\} \quad \text{and} \quad M := \max\{a_1, \dots, a_N, L + \varepsilon\}.$$

WTS m and M are a lower and upper bound, respectively, of the sequence $\{a_n\}_{n=1}^{\infty}$. Fix $n \in \mathbb{N}$. WTS $m \leq a_n \leq M$.

Lemma 2.2.3

Let $\{a_n\}_{n=1}^{\infty}$ be a **convergent** sequence of real numbers. Then we have the following:

(a) $\{a_n\}_{n=1}^{\infty}$ is bounded;

(b) for all $m, M \in \mathbb{R}$ s.t. m is a lower bound and M an upper bound of $\{a_n\}_{n=1}^{\infty}$, we have that $m \leq \lim_{n\to\infty} a_n \leq M$.

Proof (continued). Reminder:

- $m = \min\{a_1, \ldots, a_N, L - \varepsilon\}$,
- $M := \max\{a_1, \ldots, a_N, L + \varepsilon\}$.

WTS $m \leq a_n \leq M$.

If $n \leq N$, then by construction, we have that

$$m \leq \min\{a_1, \ldots, a_N\} \leq a_n \leq \max\{a_1, \ldots, a_N\} \leq M,$$

and we are done.

Lemma 2.2.3

Let $\{a_n\}_{n=1}^{\infty}$ be a **convergent** sequence of real numbers. Then we have the following:

1. (a) $\{a_n\}_{n=1}^{\infty}$ is bounded;
2. (b) for all $m, M \in \mathbb{R}$ s.t. m is a lower bound and M an upper bound of $\{a_n\}_{n=1}^{\infty}$, we have that $m \leq \lim_{n\to\infty} a_n \leq M$.

Proof (continued). Reminder:

- $m = \min\{a_1, \ldots, a_N, L - \varepsilon\}$,
- $M := \max\{a_1, \ldots, a_N, L + \varepsilon\}$.

WTS $m \leq a_n \leq M$.

From now on, we assume that $n > N$. Then $|a_n - L| < \varepsilon$, i.e. $a_n \in (L - \varepsilon, L + \varepsilon)$. But now

$$m \quad \leq \quad L - \varepsilon \quad \leq \quad a_n \quad \leq \quad L + \varepsilon \quad \leq \quad M,$$

and again we are done.

Lemma 2.2.3

Let $\{a_n\}_{n=1}^{\infty}$ be a **convergent** sequence of real numbers. Then we have the following:

(a) $\{a_n\}_{n=1}^{\infty}$ is bounded;

(b) for all $m, M \in \mathbb{R}$ s.t. m is a lower bound and M an upper bound of $\{a_n\}_{n=1}^{\infty}$, we have that $m \leq \lim_{n \to \infty} a_n \leq M$.

Proof (continued). (b) Let $m, M \in \mathbb{R}$ be s.t. m is a lower bound and M an upper bound of $\{a_n\}_{n=1}^{\infty}$. Set $L = \lim_{n \to \infty} a_n$. WTS $m \leq L \leq M$.

We prove that $L \leq M$; the proof that $m \leq L$ is similar (see the Lecture Notes).

Lemma 2.2.3

Let $\{a_n\}_{n=1}^{\infty}$ be a **convergent** sequence of real numbers. Then we have the following:

a. $\{a_n\}_{n=1}^{\infty}$ is bounded;
b. for all $m, M \in \mathbb{R}$ s.t. m is a lower bound and M an upper bound of $\{a_n\}_{n=1}^{\infty}$, we have that $m \leq \lim_{n\to\infty} a_n \leq M$.

Proof (continued). Suppose toward a contradiction that $L > M$. Set $\varepsilon_M = L - M$. Fix $N \in \mathbb{N}$ s.t. for all $n \in \mathbb{N}$, if $n \geq N$, then $|a_n - L| < \varepsilon_M$.

Now, fix $n \in \mathbb{N}$ s.t. $n \geq N$. Then

$$\varepsilon_M > |a_n - L| \geq L - a_n = (\underbrace{L - M}_{=\varepsilon_M}) + (\underbrace{M - a_n}_{\geq 0}) \geq \varepsilon_M.$$

We have now proven that $\varepsilon_M > \varepsilon_M$, a contradiction. So, $L \leq M$. $\square$

Lemma 2.2.4

For all $c \in \mathbb{R}$, the constant sequence $c, c, c, \ldots$ converges, and $\lim_{n\to\infty} c = c$.

Proof. This is "obvious," but here's a formal proof. Fix $\varepsilon > 0$, and set $N = 1$. Fix $n \in \mathbb{N}$ s.t. $n \geq N$. Then $|c - c| = 0 < \varepsilon$. So, $\lim_{n\to\infty} c = c$. □

Theorem 2.2.5

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be convergent sequences of real numbers, and let $c \in \mathbb{R}$. Then all the following hold:

(a) $\lim\limits_{n\to\infty}(ca_n) = c \lim\limits_{n\to\infty}(a_n)$;

(b) $\lim\limits_{n\to\infty}(a_n + b_n) = (\lim\limits_{n\to\infty} a_n) + (\lim\limits_{n\to\infty} b_n)$;

(c) $\lim\limits_{n\to\infty}(a_n - b_n) = (\lim\limits_{n\to\infty} a_n) - (\lim\limits_{n\to\infty} b_n)$;

(d) $\lim\limits_{n\to\infty}(a_n b_n) = (\lim\limits_{n\to\infty} a_n)(\lim\limits_{n\to\infty} b_n)$;

(e) if $b_n \neq 0$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} b_n \neq 0$, then

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}.$$

- The full proof is in the Lecture Notes.
- For illustration purposes, let us prove (b).

Theorem 2.2.5

(b) $\lim_{n\to\infty}(a_n + b_n) = (\lim_{n\to\infty} a_n) + (\lim_{n\to\infty} b_n)$;

Proof of (b). Set $a := \lim_{n\to\infty} a_n$ and $b := \lim_{n\to\infty} b_n$. WTS $\lim_{n\to\infty}(a_n + b_n) = a + b$. Fix $\varepsilon > 0$.

Using the fact that $\lim_{n\to\infty} a_n = a$, we choose $N_1 \in \mathbb{N}$ s.t. for all $n \in \mathbb{N}$, if $n \geq N_1$, then $|a_n - a| < \frac{\varepsilon}{2}$.

Using the fact that $\lim_{n\to\infty} b_n = b$, we choose $N_2 \in \mathbb{N}$ s.t. for all $n \in \mathbb{N}$, if $n \geq N_2$, then $|b_n - b| < \frac{\varepsilon}{2}$.

Set $N := \max\{N_1, N_2\}$, and fix $n \in \mathbb{N}$ s.t. $n \geq N$.

Then $n \geq N_1$ and $n \geq N_2$; the former implies that $|a_n - a| < \frac{\varepsilon}{2}$, whereas the later implies that $|b_n - b| < \frac{\varepsilon}{2}$.

Theorem 2.2.5

(b) $\lim_{n\to\infty}(a_n + b_n) = (\lim_{n\to\infty} a_n) + (\lim_{n\to\infty} b_n)$;

Proof of (b). We now compute:

$$\begin{aligned} |(a_n + b_n) - (a + b)| &= |(a_n - a) + (b_n - b)| \\ &\leq |a_n - a| + |b_n - b| \quad \text{by the Triangle Inequality} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

and so $\lim_{n\to\infty}(a_n + b_n) = a + b$. $\square$

Theorem 2.2.5

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be convergent sequences of real numbers, and let $c \in \mathbb{R}$. Then all the following hold:

(a) $\lim_{n\to\infty}(ca_n) = c \lim_{n\to\infty}(a_n)$;

(b) $\lim_{n\to\infty}(a_n + b_n) = (\lim_{n\to\infty} a_n) + (\lim_{n\to\infty} b_n)$;

(c) $\lim_{n\to\infty}(a_n - b_n) = (\lim_{n\to\infty} a_n) - (\lim_{n\to\infty} b_n)$;

(d) $\lim_{n\to\infty}(a_n b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n)$;

(e) if $b_n \neq 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} b_n \neq 0$, then

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}.$$

- Reminder (from last lecture):

Example 2.1.1

Show that $\lim_{n\to\infty} \frac{1}{n} = 0$.

- Example 2.1.1, combined with Theorem 2.2.5 yields the following:

Proposition 2.2.6

For all real numbers a and positive integers p, we have that

$$\lim_{n\to\infty} \left(\frac{a}{n^p}\right) = 0.$$

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For all real numbers a and positive integers p, we have that

$$\lim_{n\to\infty} \left(\frac{a}{n^p}\right) = 0.$$

Proof. We compute:

$$\begin{aligned}
\lim_{n\to\infty} \left(\frac{a}{n^p}\right) &= \lim_{n\to\infty} \left(a\left(\frac{1}{n}\right)^p\right) \\
&= a\left(\lim_{n\to\infty} \left(\frac{1}{n}\right)^p\right) && \text{by Theorem 2.2.5(a)} \\
&= a\left(\lim_{n\to\infty} \frac{1}{n}\right)^p && \text{by repeated applicaton of Theorem 2.2.5(d)} \\
&= a0^p && \text{by Example 2.1.1} \\
&= 0.
\end{aligned}$$

□

Example 2.2.7

Compute $\lim\limits_{n\to\infty} \frac{4n^3+2n^2-n+7}{3n^3-3n^2-12}$.

Solution. Here, we have a rational expression, where both the numerator and the denominator have the same degree. The trick is to divide both the numerator and the denominator by n^3 (the highest degree term of both), and then compute, as follows:

$$\lim_{n\to\infty} \frac{4n^3+2n^2-n+7}{3n^3-3n^2-12} \overset{(*)}{=} \lim_{n\to\infty} \frac{4+\overbrace{\frac{2}{n}}^{\to 0}-\overbrace{\frac{1}{n^2}}^{\to 0}+\overbrace{\frac{7}{n^3}}^{\to 0}}{3-\underbrace{\frac{3}{n}}_{\to 0}-\underbrace{\frac{12}{n^3}}_{\to 0}} = \frac{4}{3},$$

where (*) was obtained by dividing both the numerator and the denominator by n^3. (Here, arrows indicate what limit the circled expressions converge to as $n \to \infty$.) □

2. The Monotone Sequence Theorem

Definition

A sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers is

- *non-decreasing* if for all $n \in \mathbb{N}$, we have that $a_n \leq a_{n+1}$;
- *strictly increasing* (or simply *increasing*) if for all $n \in \mathbb{N}$, we have that $a_n < a_{n+1}$;
- *non-increasing* if for all $n \in \mathbb{N}$, we have that $a_n \geq a_{n+1}$;
- *strictly decreasing* (or simply *decreasing*) if for all $n \in \mathbb{N}$, we have that $a_n > a_{n+1}$;
- *monotone* if it is either non-decreasing or non-increasing.

- Our goal is the prove the following theorem:

The Monotone Sequence Theorem

Every monotone and bounded sequence of real numbers is convergent.

- Our goal is to prove the following two lemmas:

Lemma 2.3.1

Let $\{a_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of real numbers bounded above. Then $\{a_n\}_{n=1}^{\infty}$ converges, and its limit is precisely the supremum of the sequence.

Lemma 2.3.2

Let $\{a_n\}_{n=1}^{\infty}$ be a non-increasing sequence of real numbers bounded below. Then $\{a_n\}_{n=1}^{\infty}$ converges, and its limit is precisely the infimum of the sequence.

- Together, they imply the Monotone Sequence Theorem.

The Monotone Sequence Theorem

Every monotone and bounded sequence of real numbers is convergent.

- We prove Lemma 2.3.1. The proof of Lemma 2.3.2 is similar and is left as an exercise.

Lemma 2.3.1

Let $\{a_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of real numbers bounded above. Then $\{a_n\}_{n=1}^{\infty}$ converges, and its limit is precisely the supremum of the sequence.

Proof. Let S be the supremum of the sequence $\{a_n\}_{n=1}^{\infty}$. WTS $\lim\limits_{n\to\infty} a_n = S$.

Fix an arbitrary $\varepsilon > 0$, and fix some $N \in \mathbb{N}$ s.t. $a_N > S - \varepsilon$; such an N exists because otherwise, $S - \varepsilon$ would be an upper bound of $\{a_n\}_{n=1}^{\infty}$, contrary to the fact that S is the supremum of the sequence. Now, fix $n \in \mathbb{N}$ s.t. $n \geq N$. Then

$$S - \varepsilon \quad < \quad a_N \quad \overset{(*)}{\leq} \quad a_n \quad \overset{(**)}{\leq} \quad S,$$

where (*) follows from the fact that our sequence is non-decreasing, wheres (**) follows from the fact that S is the supremum of the sequence. Thus, $-\varepsilon < a_n - S \leq 0$, and consequently, $|a_n - S| < \varepsilon$. This completes the argument. □

Lemma 2.3.1

Let $\{a_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of real numbers bounded above. Then $\{a_n\}_{n=1}^{\infty}$ converges, and its limit is precisely the supremum of the sequence.

Lemma 2.3.2

Let $\{a_n\}_{n=1}^{\infty}$ be a non-increasing sequence of real numbers bounded below. Then $\{a_n\}_{n=1}^{\infty}$ converges, and its limit is precisely the infimum of the sequence.

The Monotone Sequence Theorem

Every monotone and bounded sequence of real numbers is convergent.

The Monotone Sequence Theorem

Every monotone and bounded sequence of real numbers is convergent.

- The Monotone Sequence Theorem is often used to prove that a recursively defined sequence converges, and we give an example on the next slide.
- First, we need one more limit law, which we state without proof.

Lemma 2.3.3

Let $\{a_n\}_{n=1}^{\infty}$ be a convergent sequence, and let $p \in \mathbb{R}$. Then

$$\lim_{n\to\infty} (a_n^p) \;=\; \left(\lim_{n\to\infty} a_n\right)^p,$$

as long as both the sequence $\{a_n^p\}_{n=1}^{\infty}$ and the number $\left(\lim_{n\to\infty} a_n\right)^p$ are both defined.

Example 2.3.4

Let $\{a_n\}_{n=1}^{\infty}$ be the sequence defined recursively as follows:

- $a_1 = \sqrt{2}$;
- $a_{n+1} = \sqrt{2a_n}$ for all $n \in \mathbb{N}$.

Show that $\{a_n\}_{n=1}^{\infty}$ converges, and find its limit.

Solution. We first use the Monotone Sequence Theorem to prove that the sequence $\{a_n\}_{n=1}^{\infty}$ converges (i.e. has a limit), and then we compute that limit.

- Reminder: $a_1 = \sqrt{2}$; $a_{n+1} = \sqrt{2a_n}\ \forall n \in \mathbb{N}$.

Solution (continued). **Proving convergence.** We first prove the sequence $\{a_n\}_{n=1}^{\infty}$ is monotone and bounded. More precisely, we will show that for all $n \in \mathbb{N}$, we have that $\sqrt{2} \leq a_n \leq 2$ and $a_n \leq a_{n+1}$. We proceed by induction on n.

Base case: For $n = 1$, we simply observe that $\sqrt{2} = a_1 \leq 2$ and that $a_1 = \sqrt{2} \leq \sqrt{2\sqrt{2}} = \sqrt{2a_1} = a_2$.

Induction step: Fix $n \in \mathbb{N}$, and suppose that $\sqrt{2} \leq a_n \leq 2$ and $a_n \leq a_{n+1}$. WTS $\sqrt{2} \leq a_{n+1} \leq 2$ and $a_{n+1} \leq a_{n+2}$.

First, since $\sqrt{2} \leq a_n \leq 2$, we have that $2\sqrt{2} \leq 2a_n \leq 4$, and so $\sqrt{2\sqrt{2}} \leq \sqrt{2a_n} \leq 2$. Since $\sqrt{2} \leq \sqrt{2\sqrt{2}}$ and $a_{n+1} = \sqrt{2a_n}$, we deduce that $\sqrt{2} \leq a_{n+1} \leq 2$.

On the other hand, since $0 \leq a_{n+1} \leq 2$, we have that $a_{n+1} = \sqrt{a_{n+1}^2} \leq \sqrt{2a_{n+1}} = a_{n+2}$. This completes the induction.

We have now shown that $\{a_n\}_{n=1}^{\infty}$ is monotone and bounded, and so by the Monotone Sequence Theorem, the sequence converges.

- Reminder: $a_1 = \sqrt{2}$; $a_{n+1} = \sqrt{2a_n}\ \forall n \in \mathbb{N}$.

Solution (continued). **Computing the limit.** Set $a = \lim_{n\to\infty} a_n$. (The existence of the limit follows from the convergence of the sequence $\{a_n\}_{n=1}^{\infty}$, proven above.) Then

$$a \;=\; \lim_{n\to\infty} a_{n+1} \;=\; \lim_{n\to\infty} \sqrt{2a_n} \;=\; \sqrt{2\lim_{n\to\infty} a_n} \;=\; \sqrt{2a}.$$

So, $a^2 = 2a$, which implies that $a = 0$ or $a = 2$.

However, since $\{a_n\}_{n=1}^{\infty}$ is bounded below by $\sqrt{2}$, we know by Lemma 2.2.3(b) that $\sqrt{2} \leq a$, and in particular, $a \neq 0$. So, $a = 2$.

Conclusion. We have shown that $\{a_n\}_{n=1}^{\infty}$ converges and satisfies

$$\lim_{n\to\infty} a_n \;=\; 2,$$

and we are done. $\square$

Example 2.3.4

Let $\{a_n\}_{n=1}^{\infty}$ be the sequence defined recursively as follows:

- $a_1 = \sqrt{2}$;
- $a_{n+1} = \sqrt{2a_n}$ for all $n \in \mathbb{N}$.

Show that $\{a_n\}_{n=1}^{\infty}$ converges, and find its limit.

- **Remark:** When solving problems similar to Example 2.3.4, **it is imperative that you prove that the limit actually exists**, i.e. that the sequence in question does converge.
  - It is **not** enough to simply compute the limit.
  - Let us explain why (next slide).

- Suppose we are given a sequence $\{a_n\}_{n=1}^{\infty}$, defined recursively as follows:
  - $a_1 = -1$;
  - $a_{n+1} = -a_n$ for all $n \in \mathbb{N}$.
- Suppose we simply set $a := \lim_{n\to\infty} a_n$ and then try to evaluate a.
- We get:

$$a = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} (-a_n) = -\lim_{n\to\infty} a_n = -a.$$

- Thus, $a = -a$, and consequently, $2a = 0$, i.e. $a = 0$.
- But this doesn't work!
- Indeed, our sequence satisfies $a_n = (-1)^n$ for all $n \in \mathbb{N}$, and therefore does not converge (by Example 2.1.5 from the previous lecture).
- What our computation actually showed is that **if** $\lim_{n\to\infty} a_n$ exists, **then** $\lim_{n\to\infty} a_n = 0$.
- It did **not** show that $\lim_{n\to\infty} a_n$ does in fact exist. As a matter of fact, the limit does **not** exist.

3. Euler's number
   - You may recall from high school that *Euler's number* e is an (irrational) real number that satisfies $e \approx 2.71828$.
   - Formally, Euler's number is defined as follows:

$$e \quad := \quad \lim_{n \to \infty} \left(1 + \tfrac{1}{n}\right)^n.$$

   - This immediately raises the following question: how do we even know that this limit exists?
   - To prove that it does, we will use the Monotone Sequence Theorem and Bernoulli's inequality. (Reminder: below.)

The Monotone Sequence Theorem

Every monotone and bounded sequence of real numbers is convergent.

Bernoulli's inequality

For all integers $n \geq 0$ and all real numbers $x \geq -1$, we have $(1 + x)^n \geq 1 + nx$.

- First, we consider the auxiliary sequence $\{a_n\}_{n=1}^{\infty}$ given by

$$a_n \quad := \quad \left(1 + \frac{1}{n}\right)^{n+1} \qquad \forall n \in \mathbb{N}$$

- We will show that $\{a_n\}_{n=1}^{\infty}$ is non-increasing and bounded below, and is consequently convergent (by the Monotone Sequence Theorem).
- It is clear that $a_n > 0$ for all $n \in \mathbb{N}$, and in particular, the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded below.
- Let us show that $\{a_n\}_{n=1}^{\infty}$ is non-increasing (next slide).

- Reminder: $a_n = \left(1+\frac{1}{n}\right)^{n+1}\ \forall n \in \mathbb{N}$
- Next, for all $n \in \mathbb{N}$, we have that

$$\begin{aligned}
\frac{a_n}{a_{n+1}} &= \frac{(1+\frac{1}{n})^{n+1}}{(1+\frac{1}{n+1})^{n+2}} \\
&= \left(\frac{1+\frac{1}{n}}{1+\frac{1}{n+1}}\right)^{n+2} \cdot \frac{1}{1+\frac{1}{n}} \\
&= \left(\frac{n^2+2n+1}{n^2+2n}\right)^{n+2} \cdot \frac{1}{1+\frac{1}{n}} \\
&= \left(1+\frac{1}{n^2+2n}\right)^{n+2} \cdot \frac{1}{1+\frac{1}{n}} \\
&\geq \left(1+(n+2)\cdot\frac{1}{n^2+2n}\right) \cdot \frac{1}{1+\frac{1}{n}} \qquad \text{by Bernoulli's inequality} \\
&= \left(1+\frac{1}{n}\right) \cdot \frac{1}{1+\frac{1}{n}} = 1,
\end{aligned}$$

and consequently (since $a_{n+1} > 0$), that $a_n \geq a_{n+1}$.

- So, the sequence $\{a_n\}_{n=1}^{\infty}$ is non-increasing.

- Reminder: $a_n = \left(1 + \frac{1}{n}\right)^{n+1}\ \forall n \in \mathbb{N}$
- We have now shown that $\{a_n\}_{n=1}^{\infty}$ is monotone and bounded, and consequently, it is convergent (by the Monotone Sequence Theorem).
  - In other words, $\lim_{n\to\infty} a_n$ exists.
- Finally, we compute:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\frac{a_n}{1+\frac{1}{n}} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty}(1+\frac{1}{n})} = \frac{\lim_{n\to\infty} a_n}{1} = \lim_{n\to\infty} a_n.$$

- This proves that the sequence $\left\{(1+\frac{1}{n})^n\right\}_{n=1}^{\infty}$ is in fact convergent, and that Euler's number

$$e := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

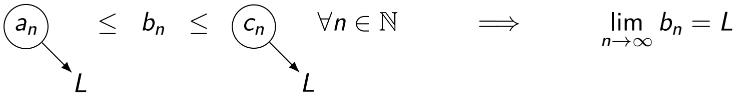
is indeed well defined.

4. The Squeeze Theorem

The Squeeze Theorem

Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, and $\{c_n\}_{n=1}^{\infty}$ be sequences of real numbers s.t. $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$. Assume that $\{a_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ converge to the same limit L. Then $\{b_n\}_{n=1}^{\infty}$ also converges to L.

- **Remark:** Schematically (and informally), we can represent the Squeeze Theorem like this:



- We will prove the Squeeze Theorem, but let us first take a look at an example.

Example 2.4.1

Compute $\lim_{n\to\infty}\left((-1)^n\frac{1}{n}\right)$, or prove that the limit does not exist.

Solution.

$$\underbrace{-\frac{1}{n}}_{\to 0} \quad \leq \quad (-1)^n\frac{1}{n} \quad \leq \quad \underbrace{\frac{1}{n}}_{\to 0} \qquad \forall n\in\mathbb{N} \qquad \overset{(*)}{\Longrightarrow} \qquad \lim_{n\to\infty}\left((-1)^n\frac{1}{n}\right)=0.$$

where (*) follows from the Squeeze Theorem. □

The Squeeze Theorem

Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, and $\{c_n\}_{n=1}^{\infty}$ be sequences of real numbers s.t. $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$. Assume that $\{a_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ converge to the same limit L. Then $\{b_n\}_{n=1}^{\infty}$ also converges to L.

Proof. Fix $\varepsilon > 0$.

Using the fact that $\lim_{n\to\infty} a_n = L$, we fix some $N_1 \in \mathbb{N}$ s.t. for all $n \in \mathbb{N}$, if $n \geq N_1$, then $|a_n - L| < \varepsilon$.

Similarly, using the fact that $\lim_{n\to\infty} c_n = L$, we fix some $N_2 \in \mathbb{N}$ s.t. for all $n \in \mathbb{N}$, if $n \geq N_2$, then $|c_n - L| < \varepsilon$.

Let $N = \max\{N_1, N_2\}$. Fix $n \in \mathbb{N}$ s.t. $n \geq N$. Then $n \geq N_1$, and it follows that $|a_n - L| < \varepsilon$, and so $-\varepsilon < a_n - L < \varepsilon$. Similarly, $n \geq N_2$, and it follows that $|c_n - L| < \varepsilon$, and so $-\varepsilon < c_n - L < \varepsilon$. Since $a_n \leq b_n \leq c_n$, we have that

$$-\varepsilon \quad < \quad a_n - L \quad \leq \quad b_n - L \quad \leq \quad c_n - L \quad < \quad \varepsilon,$$

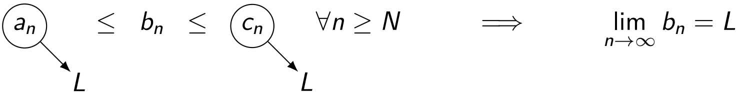
and we deduce that $|b_n - L| < \varepsilon$. This proves that $\lim_{n\to\infty} b_n = L$. $\square$

- **Remark:** Recall that the first few terms of a sequence have no effect on the existence or value of the limit.
- This allows us to state a slightly stronger version of the Squeeze Theorem, as follows.
  - Try to give a formal proof by yourself, using the original Squeeze Theorem, plus Lemma 2.2.2!

The Squeeze Theorem for sequences (stronger version)

Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, and $\{c_n\}_{n=1}^{\infty}$ be sequences of real numbers. Assume that there exists an integer $N \in \mathbb{N}$ s.t. for all $n \in \mathbb{N}$ satisfying $n \geq N$, we have that $a_n \leq b_n \leq c_n$. Assume that $\{a_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ converge to the same limit L. Then $\{b_n\}_{n=1}^{\infty}$ also converges to L.

- **Remark:** A schematic (and informal) representation of the theorem above looks like this:



5. Some limits with roots

Proposition 2.4.2

$\lim_{n\to\infty} \sqrt[n]{n} = 1.$

Proof. Clearly, for all $n \in \mathbb{N}$, we have that $\sqrt[n]{n} > 1$, and consequently,

$$r_n \quad := \quad \sqrt[n]{n} - 1 \quad > \quad 0.$$

Our goal is to show that $\lim_{n\to\infty} r_n = 0$. This is enough, for it will imply that

$$\lim_{n\to\infty} \sqrt[n]{n} \quad = \quad \lim_{n\to\infty}(1 + r_n) \quad = \quad 1 + \lim_{n\to\infty} r_n \quad = \quad 1 + 0 \quad = \quad 1,$$

which is what we need.

Proof (continued). Reminder: $r_n = \sqrt[n]{n} - 1 > 0$; WTS $\lim_{n\to\infty} r_n = 0$.

Claim. *$0 < r_n < \sqrt{\frac{2}{n-1}}$ for all integers $n \geq 2$.*

Proof of the Claim. Fix an integer $n \geq 2$. We already saw that $r_n > 0$. For the other inequality, we observe that

$$n \overset{(*)}{=} (r_n + 1)^n \overset{(**)}{=} \sum_{k=0}^{n} \binom{n}{k} r_n^k \overset{(***)}{>} \binom{n}{2} r_n^2 = \frac{n(n-1)}{2} r_n^2,$$

and consequently $r_n < \sqrt{\frac{2}{n-1}}$, where

- (*) follows from the definition of r_n;
- (**) follows from the Binomial Theorem;
- (***) follows from the fact that each summand in the sum $\sum_{k=0}^{n} \binom{n}{k} r_n^k$ is strictly positive (because $r_n > 0$), and so each summand is strictly smaller than the whole sum (because there is more than one summand).

This proves the Claim. ♦

Proposition 2.4.2

$\lim_{n\to\infty} \sqrt[n]{n} = 1.$

Proof (continued). Reminder: $r_n = \sqrt[n]{n} - 1 > 0$; WTS $\lim_{n\to\infty} r_n = 0$.

> ***Claim.*** *$0 < r_n < \sqrt{\frac{2}{n-1}}$ for all integers $n \geq 2$.*

Clearly, $\lim_{n\to\infty} \sqrt{\frac{2}{n-1}} = 0$. So, by the Claim and the Squeeze Theorem, we have that $\lim_{n\to\infty} r_n = 0$, and we are done.

- Indeed, the Claim guarantees that for all integers $n \geq 2$, we have that

$$0 \quad < \quad r_n \quad < \quad \underbrace{\sqrt{\frac{2}{n-1}}}_{\to 0} \quad ,$$

 and so by the Squeeze Theorem, $\lim_{n\to\infty} r_n = 0$. (Here, we are using our "stronger version" of the Squeeze Theorem, for $N = 2$.) □

Proposition 2.4.2

$\lim_{n\to\infty} \sqrt[n]{n} = 1.$

Proposition 2.4.3

For all real numbers $a > 0$, we have that $\lim_{n\to\infty} \sqrt[n]{a} = 1.$

- **Remark:** We could prove Proposition 2.4.3 similarly to the way we proved Proposition 2.4.2 (try it!).
  - However, since we have already proven Proposition 2.4.2, we can simply use it, together with the Squeeze Theorem to prove Proposition 2.4.3.

Proposition 2.4.2

$\lim_{n\to\infty} \sqrt[n]{n} = 1.$

Proposition 2.4.3

For all real numbers $a > 0$, we have that $\lim_{n\to\infty} \sqrt[n]{a} = 1.$

Proof. Fix a real number $a > 0$. We consider two cases: when $a \geq 1$, and when $0 < a < 1$.

Case 1: $a \geq 1$. Fix $N \in \mathbb{N}$ s.t. $a \leq N$. Then $\forall n \in \mathbb{N}$ s.t. $n \geq N$, we have that $1 \leq \sqrt[n]{a} \leq \sqrt[n]{n}$. Since $\lim_{n\to\infty} \sqrt[n]{n} = 1$ (by Proposition 2.4.2), the Squeeze Theorem guarantees that $\lim_{n\to\infty} \sqrt[n]{a} = 1$.

Case 2: $0 < a < 1$. Then $\frac{1}{a} > 1$, and so by Case 1, $\lim_{n\to\infty} \sqrt[n]{\frac{1}{a}} = 1$. We now compute:

$$\lim_{n\to\infty} \sqrt[n]{a} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{\frac{1}{a}}} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{\frac{1}{a}}} = \frac{1}{1} = 1.$$

□