

# Mathematical Analysis 1

## Lecture #1

Number systems: rational and real numbers. An introduction to limits of sequences

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- **Notation:**

- $\mathbb{N}$  is the set of all natural numbers (positive integers);
- $\mathbb{N}_0$  is the set of all non-negative integers;
- $\mathbb{Z}$  is the set of all integers;
- $\mathbb{Q}$  is the set of all rational numbers;
- $\mathbb{R}$  is the set of all real numbers;
- $\mathbb{C}$  is the set of all complex numbers.

- This lecture has five parts:
  - ① A brief introduction to “complete ordered fields”
  - ② Cardinality: comparing infinities
  - ③ A (very) brief introduction to metric spaces
  - ④ Bernoulli’s inequality
  - ⑤ An introduction to limits of sequences

- 1 A brief introduction to “complete ordered fields”
  - From Linear Algebra:

### Definition

A *field* is an ordered triple  $(\mathbb{F}, +, \cdot)$ , where  $\mathbb{F}$  is a set, and  $+$  and  $\cdot$  are binary operations on  $\mathbb{F}$  (i.e. functions from  $\mathbb{F} \times \mathbb{F}$  to  $\mathbb{F}$ ), called *addition* and *multiplication*, respectively, satisfying the following axioms:

- 1 **[Associativity of addition and multiplication]** addition and multiplication are associative, that is, for all  $a, b, c \in \mathbb{F}$ , we have that  $a + (b + c) = (a + b) + c$  and  $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ ;
- 2 **[Commutativity of addition and multiplication]** addition and multiplication are commutative, that is, for all  $a, b \in \mathbb{F}$ , we have that  $a + b = b + a$  and  $a \cdot b = b \cdot a$ ;

## Definition (cont.)

- [Additive and multiplicative identity]** there exist distinct elements  $0_{\mathbb{F}}, 1_{\mathbb{F}} \in \mathbb{F}$  such that for all  $a \in \mathbb{F}$ ,  $a + 0_{\mathbb{F}} = a$  and  $a \cdot 1_{\mathbb{F}} = a$ ;  $0_{\mathbb{F}}$  is called the *additive identity* of  $\mathbb{F}$ , and  $1_{\mathbb{F}}$  is called the *multiplicative identity* of  $\mathbb{F}$ ;
- [Additive inverses]** for every  $a \in \mathbb{F}$ , there exists an element in  $\mathbb{F}$ , denoted by  $-a$  and called the *additive inverse* of  $a$ , such that  $a + (-a) = 0_{\mathbb{F}}$ ;
- [Multiplicative inverses]** for all  $a \in \mathbb{F} \setminus \{0_{\mathbb{F}}\}$ , there exists an element in  $\mathbb{F}$ , denoted by  $a^{-1}$  and called the *multiplicative inverse* of  $a$ , such that  $a \cdot a^{-1} = 1_{\mathbb{F}}$ ;
- [Distributivity of multiplication over addition]** multiplication is distributive over addition, that is, for all  $a, b, c \in \mathbb{F}$ , we have that  $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ .

- **Notation:** Usually, we write simply “field  $\mathbb{F}$ ” rather than “field  $(\mathbb{F}, +, \cdot)$ .” When  $\mathbb{F}$  is clear from context, we write simply 0 and 1 instead of  $0_{\mathbb{F}}$  and  $1_{\mathbb{F}}$ , respectively.

### Example 1.1.1

All the following are fields:

- $(\mathbb{Q}, +, \cdot)$ ;                      •  $(\mathbb{R}, +, \cdot)$ ;                      •  $(\mathbb{C}, +, \cdot)$ .

However, the following are **not** fields:

- $(\mathbb{N}, +, \cdot)$ ;                      •  $(\mathbb{N}_0, +, \cdot)$ ;                      •  $(\mathbb{Z}, +, \cdot)$ .

- From Discrete Math:

### Definition

A *strict total order* on a non-empty set  $A$  is a binary relation  $<$  on  $A$  that satisfies the following two axioms:

- 1 **[Transitivity]** For all  $a, b, c \in A$ , if  $a < b$  and  $b < c$ , then  $a < c$ .
- 2 **[Trichotomy]** For all  $a, b \in A$ , exactly one of  $a < b$ ,  $a = b$ , and  $b < a$  is true.

### Definition

An *ordered field* is a field  $\mathbb{F}$  with a strict total order  $<$  such that:

- if  $a < b$ , then  $a + c < b + c$ ;
- if  $0 < a$  and  $0 < b$ , then  $0 < ab$ .

We define  $\leq$  as follows:  $a \leq b$  if  $a < b$  or  $a = b$ . Furthermore, we write  $a > b$  when  $b < a$ , and we write  $a \geq b$  when  $b \leq a$ .

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## Fact 1.1.2

$\mathbb{Q}$  and  $\mathbb{R}$  are ordered fields (under the usual  $<$  relation). However,  $\mathbb{C}$  is **not** an ordered field.

- **Remark:** It is not hard to show that, in an ordered field, all the usual algebraic properties of the relations  $<$  and  $\leq$  hold.
  - Proof: omitted.

## Definition

If  $\mathbb{F}$  is an ordered field,  $S \subseteq \mathbb{F}$ ,  $x \in \mathbb{F}$ , then

- $x$  is called an *upper bound* for  $S$  if for all  $s \in S$ , we have  $s \leq x$ ;
- $x$  is called the *least upper bound* (or *supremum*) for  $S$  if  $x$  is an upper bound for  $S$ , and every upper bound  $y$  for  $S$  satisfies  $x \leq y$ .
- $x$  is called a *lower bound* for  $S$  if for all  $s \in S$ , we have  $x \leq s$ ;
- $x$  is called the *greatest lower bound* (or *infimum*) for  $S$  if  $x$  is a lower bound for  $S$ , and every lower bound  $y$  for  $S$  satisfies  $y \leq x$ .

A subset of  $\mathbb{F}$  is *bounded above* if it has an upper bound, and it is *bounded below* if it has a lower bound. A subset of  $\mathbb{F}$  is *bounded* if it is both bounded above and bounded below.

### Example 1.1.3

In  $\mathbb{R}$ :

- the set  $[-2, 3)$  has both a supremum (namely 3) and an infimum (namely  $-2$ );
- the set  $(-\infty, 4]$  has a supremum (namely 4), but no infimum;
- the set  $(2, \infty)$  has no supremum, but does have an infimum (namely 2);
- the set  $(2, 3) \cup (5, 7)$  has both a supremum (namely 7) and an infimum (namely 2).

## Definition

If  $\mathbb{F}$  is an ordered field,  $S \subseteq \mathbb{F}$ ,  $x \in \mathbb{F}$ , then

- $x$  is called an *upper bound* for  $S$  if for all  $s \in S$ , we have  $s \leq x$ ;
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A subset of  $\mathbb{F}$  is *bounded above* if it has an upper bound, and it is *bounded below* if it has a lower bound. A subset of  $\mathbb{F}$  is *bounded* if it is both bounded above and bounded below.

● **Remarks:**

- An upper or lower bound for a set  $S$  may, but need not, belong to  $S$ . In particular, the supremum or infimum of a set  $S$  (if it exists) may, but need not, belong to  $S$ .
- A subset of an ordered field does not necessarily have an upper or a lower bound. For instance, in the ordered field  $\mathbb{R}$ , the set  $\mathbb{R}$  itself has neither an upper nor a lower bound.

**Proposition 1.1.4**

Let  $\mathbb{F}$  be an ordered field, and let  $S \subseteq \mathbb{F}$ . Then  $S$  has at most one supremum and at most one infimum.

*Proof.* Suppose that  $x$  and  $y$  are suprema of  $S$ ; WTS  $x = y$ .

Since  $x$  is a supremum of  $S$  and  $y$  is an upper bound of  $S$ , we have that  $x \leq y$ . Similarly, since  $y$  is a supremum of  $S$  and  $x$  is an upper bound of  $S$ , we have that  $y \leq x$ .

Since both  $x \leq y$  and  $y \leq x$  hold, we see that  $x = y$ . This proves that  $S$  does indeed have at most one supremum. Analogously,  $S$  has at most one infimum.  $\square$

### Proposition 1.1.4

Let  $\mathbb{F}$  be an ordered field, and let  $S \subseteq \mathbb{F}$ . Then  $S$  has at most one supremum and at most one infimum.

- **Notation:** For a set  $S$ :
  - $\sup(S)$  = supremum of  $S$  (if it exists);
  - $\inf(S)$  = infimum of  $S$  (if it exists).

## Definition

An ordered field  $\mathbb{F}$  is *complete* if every non-empty subset of  $\mathbb{F}$  that is bounded above has the least upper bound (i.e. supremum).

- **Remark:** It is not hard to show that in a complete ordered field, every non-empty set that is bounded below has the greatest lower bound (i.e. infimum).

## Fact 1.1.5

$\mathbb{R}$  is a complete ordered field. In fact, up to “isomorphism” (essentially, a renaming of elements),  $\mathbb{R}$  is the **only** complete ordered field.

- Proof: Omitted.
- **Remark:** In particular,  $\mathbb{Q}$  is **not** a complete ordered field.
- **Remark:** The fact that  $\mathbb{R}$  is a complete ordered field is essential for formally defining functions such as  $\sin$ ,  $\cos$ ,  $\exp$  (and many others).

### Example 1.1.6

- Note that the set  $S = \{x \in \mathbb{Q} \mid x^2 \leq 2\}$  is bounded in  $\mathbb{Q}$  (and therefore in  $\mathbb{R}$  as well).
  - For example, 2 is an upper bound of  $S$  (both in  $\mathbb{Q}$  and in  $\mathbb{R}$ ).
  - In  $\mathbb{R}$ ,  $\sqrt{2}$  is the least upper bound (i.e. supremum) of  $S$ .
  - In  $\mathbb{Q}$ ,  $S$  does not have the least upper bound.
    - Here, we are using the well-known fact that  $\sqrt{2}$  is an irrational number.
- 
- Actually, let us formally prove that  $\sqrt{2}$  is irrational!

### Theorem 1.1.7

$\sqrt{2}$  is irrational.

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$\sqrt{2}$  is irrational.

*Proof.* Suppose otherwise, and set  $\sqrt{2} = \frac{p}{q}$ , where  $p \in \mathbb{Z}$ ,  $q \in \mathbb{N}$ , and  $p, q$  are relatively prime (i.e. they have no common divisor greater than 1).

Then  $p^2 = 2q^2$ , which implies that  $2 \mid p^2$ . Since 2 is prime, it follows that  $2 \mid p$ .

Thus, there exists some  $r \in \mathbb{Z}$  such that  $p = 2r$ . Now  $4r^2 = 2q^2$ , and it follows that  $2r^2 = q^2$ .

Now  $2 \mid q^2$ , and so since 2 is prime, we see that  $2 \mid q$ .

But now 2 is a common divisor of  $p$  and  $q$ , contrary to the fact that  $p$  and  $q$  are relatively prime.  $\square$

## 2 Cardinality: comparing infinities

- We know that

$$\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R} \subsetneq \mathbb{C}.$$

- So, in a sense, there are:
  - “more” integers than there are natural numbers,
  - “more” rational numbers than integers,
  - “more” real numbers than real numbers,
  - “more” complex numbers than real numbers.
- However, there is another way of comparing the sizes of two sets: using bijections.
  - Let us first recall the definition of a bijection.

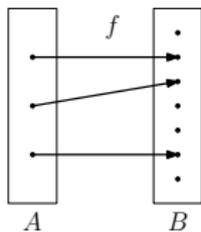
## Definition

A function  $f : A \rightarrow B$  (where  $A$  and  $B$  are some sets) is:

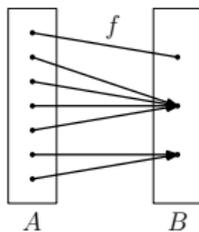
- *one-to-one* (or *injective*, or an *injection*) if for all  $a_1, a_2 \in A$  such that  $a_1 \neq a_ a_2$ , we have that  $f(a_1) \neq f(a_2)$ ;<sup>a</sup>
- *onto* (or *surjective*, or a *surjection*) if for all  $b \in B$ , there exists some  $a \in A$  such that  $f(a) = b$ ;
- *bijective* (or a *bijection*) if it is both one-to-one and onto.

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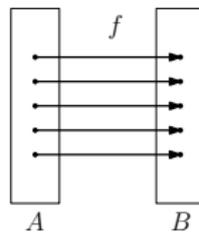
<sup>a</sup>Equivalently: if for all  $a_1, a_2 \in A$  such that  $f(a_1) = f(a_2)$ , we have that  $a_1 = a_2$ .



one-to-one  
(injection)

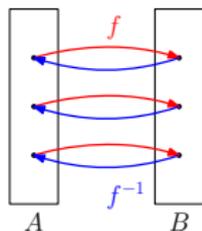


onto  
(surjection)



bijection

- **Remark:** Recall that if  $f : A \rightarrow B$  is a bijection, then it has an *inverse function*  $f^{-1} : B \rightarrow A$  such that for all  $b \in B$ ,  $f^{-1}(b)$  is the unique element  $a \in A$  such that  $f(a) = b$ .
  - In other words, if  $f : A \rightarrow B$  is a bijection, then for all  $a \in A$  and  $b \in B$ , we have that  $f(a) = b$  iff  $f^{-1}(b) = a$ . Clearly, the inverse of a bijection is also a bijection.



### Definition

Sets  $A$  and  $B$  are said to have *the same cardinality* if there exists a bijection  $f : A \rightarrow B$ , and in this case, we write  $|A| = |B|$ .

- **Remark:** Thus, we can think of two sets as being of “the same size” (or having “the same number of elements”) if we can set up a bijection between them.

### Proposition 1.2.1

All the following hold:

- (a) for all sets  $A$ ,  $|A| = |A|$ ;
- (b) for all sets  $A$  and  $B$ , if  $|A| = |B|$ , then  $|B| = |A|$ ;
- (c) for all sets  $A$ ,  $B$ , and  $C$ , if  $|A| = |B|$  and  $|B| = |C|$ , then  $|A| = |C|$ .

*Proof.* Fix sets  $A$ ,  $B$ , and  $C$ .

For (a), we simply observe that the identity function on  $A$  is a bijection.

For (b), we observe that if  $f : A \rightarrow B$  is a bijection, then so is  $f^{-1} : B \rightarrow A$ .

For (c), we observe that if  $f_1 : A \rightarrow B$  and  $f_2 : B \rightarrow C$  are bijections, then  $f_2 \circ f_1 : A \rightarrow C$  is also a bijection.  $\square$

### Proposition 1.2.2

$$|\mathbb{N}| = |\mathbb{Z}|.$$

*Proof.* We define the function  $f : \mathbb{N} \rightarrow \mathbb{Z}$  by setting

$$f(n) = \begin{cases} (n-1)/2 & \text{if } n \text{ is odd} \\ -n/2 & \text{if } n \text{ is even} \end{cases}$$

for all  $n \in \mathbb{N}$ . Thus, we have the following:

- $f(1) = 0;$
- $f(2) = -1;$
- $f(3) = 1;$
- $f(4) = -2;$
- $f(5) = 2;$
- $f(6) = -3;$
- $f(7) = 3;$
- ...
- ...

and it is not difficult to formally check that  $f$  is a bijection. This proves that  $|\mathbb{N}| = |\mathbb{Z}|$ .  $\square$

## Definition

Sets  $A$  and  $B$  are said to have *the same cardinality* if there exists a bijection  $f : A \rightarrow B$ , and in this case, we write  $|A| = |B|$ .

## Definition

Given sets  $A$  and  $B$ , we say that the cardinality of  $A$  is *no greater than* the cardinality of  $B$ , and we write  $|A| \leq |B|$  if there exists a one-to-one function  $f : A \rightarrow B$ .

- **Notation:** For sets  $A$  and  $B$ :
  - we write  $|A| \geq |B|$  if  $|B| \leq |A|$ ;
  - we write  $|A| < |B|$  if  $|A| \leq |B|$  and  $|A| \neq |B|$  (i.e. there exists a one-to-one function from  $A$  to  $B$ , but there is no bijection between  $A$  and  $B$ );
  - we write  $|A| > |B|$  if  $|B| < |A|$ .
- **Remark:** Note that if sets  $A$ ,  $B$ , and  $C$  satisfy  $|A| \leq |B|$  and  $|B| \leq |C|$ , then they also satisfy  $|A| \leq |C|$ .
  - Indeed, if  $f_1 : A \rightarrow B$  and  $f_2 : B \rightarrow C$  are one-to-one functions, then  $f_2 \circ f_1 : A \rightarrow C$  is also a one-to-one function.

## Cantor–Schröder–Bernstein theorem

If sets  $A$  and  $B$  satisfy  $|A| \leq |B|$  and  $|B| \leq |A|$ , then  $|A| = |B|$ .

- Proof: Omitted.
- **Remark:** The Cantor–Schröder–Bernstein theorem may seem obvious, but it is in fact not! Fully spelled out, it states the following:  
*For all sets  $A$  and  $B$ , if there exist one-to-one functions  $f_1 : A \rightarrow B$  and  $f_2 : B \rightarrow A$ , then there exists a bijection  $f : A \rightarrow B$ .*

The statement is indeed true, but the proof is beyond the scope of this course.

- **Remark:** It is possible to prove that for all sets  $A$  and  $B$ , either  $|A| \leq |B|$  or  $|B| \leq |A|$ . However, the proof (which we omit) is not easy and uses the so called “Axiom of Choice.”

## Definition

Given sets  $A$  and  $B$ , we say that the cardinality of  $A$  is *no greater than* the cardinality of  $B$ , and we write  $|A| \leq |B|$  if there exists a one-to-one function  $f : A \rightarrow B$ .

## Proposition 1.2.3

For all sets  $A$  and  $B$ , if  $A \subseteq B$ , then  $|A| \leq |B|$ .

*Proof.* Let  $A$  and  $B$  be sets such that  $A \subseteq B$ . Then the function  $f : A \rightarrow B$  given by  $f(a) = a$  for all  $a \in A$  is one-to-one, and consequently,  $|A| \leq |B|$ .  $\square$

- **Remark:** Since  $\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R} \subsetneq \mathbb{C}$ , Proposition 1.2.3 guarantees that

$$|\mathbb{N}| \leq |\mathbb{Z}| \leq |\mathbb{Q}| \leq |\mathbb{R}| \leq |\mathbb{C}|,$$

and moreover, by Proposition 1.2.3, we further have that  $|\mathbb{N}| = |\mathbb{Z}|$ .

## Definition

A set  $S$  is *countable* if one of the following holds:

- $S$  is finite;
- there exists a bijection  $f : \mathbb{N} \rightarrow S$ .

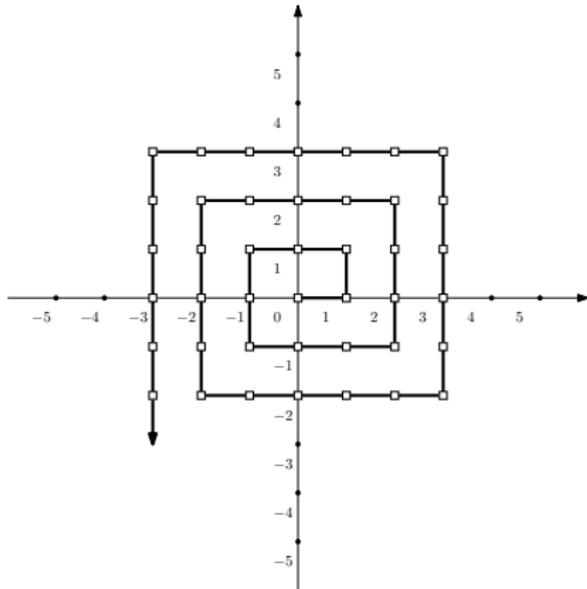
A set is *denumerable* (or *countably infinite*) if it is both countable and infinite. A set is *uncountable* if it is not countable.

- So, if a set  $S$  is infinite, then it is countable iff its members can be enumerated as  $s_1, s_2, s_3, \dots$ 
  - Here, the elements,  $s_1, s_2, s_3, \dots$  are required to be pairwise distinct, and all the elements of  $S$  must appear on the list  $s_1, s_2, s_3, \dots$
- **Remark:** By Proposition 1.2.2,  $\mathbb{Z}$  is denumerable (i.e. countably infinite).

## Theorem 1.2.4

$\mathbb{Z} \times \mathbb{Z}$  and  $\mathbb{Q}$  are both denumerable.

*Proof (slightly informal).* The picture below shows that  $\mathbb{Z} \times \mathbb{Z}$  can be enumerated as  $(p_1, q_1), (p_2, q_2), (p_3, q_3), \dots$ . So,  $\mathbb{Z} \times \mathbb{Z}$  is countable.



### Theorem 1.2.4

$\mathbb{Z} \times \mathbb{Z}$  and  $\mathbb{Q}$  are both denumerable.

*Proof (slightly informal, continued).* Reminder: We have enumerated  $\mathbb{Z} \times \mathbb{Z}$  as  $(p_1, q_1), (p_2, q_2), (p_3, q_3), \dots$

Rational numbers correspond to  $\mathbb{Z} \times \mathbb{Z}$  in a natural way. Indeed, a fraction  $\frac{p}{q}$  corresponds to the ordered pair  $(p, q)$ .

Except that there's a problem: each rational corresponds to more than one ordered pair of integers (because e.g.  $\frac{p}{q} = \frac{2p}{2q}$ ).

Furthermore, the denominator of a fraction can never be zero.

We deal with this as follows.

- First, we take our list of ordered pairs of integers, and we delete from it all pairs  $(p, q)$  where  $q \leq 0$ .
- Then, we remove all ordered pairs  $(p, q)$  where  $p$  and  $q$  are not relatively prime.

Now each rational corresponds to exactly one ordered pair on the remaining list. This proves that  $\mathbb{Q}$  is countable.  $\square$

## Theorem 1.2.5

$\mathbb{R}$  is uncountable.

*Cantor's diagonal proof.* Suppose otherwise. Then in particular, the interval  $[0, 1]$  is countable. We now enumerate the members of  $[0, 1]$  as follows (note that  $1 = 0.9999999 \dots$ ):

$$x_1 = 0.x_{1,1}x_{1,2}x_{1,3}x_{1,4} \dots$$

$$x_2 = 0.x_{2,1}x_{2,2}x_{2,3}x_{2,4} \dots$$

$$x_3 = 0.x_{3,1}x_{3,2}x_{3,3}x_{3,4} \dots$$

$$x_4 = 0.x_{4,1}x_{4,2}x_{4,3}x_{4,4} \dots$$

$\vdots$

We now create a number  $a = 0.a_1a_2a_3a_4 \dots$  as follows. For each  $i \in \mathbb{N}$ , we set

$$a_i = \begin{cases} 5 & \text{if } x_{i,i} \neq 5 \\ 6 & \text{if } x_{i,i} = 5 \end{cases}$$

But now  $a \in [0, 1]$ , and it does not appear on our list (sequence)  $x_1, x_2, x_3, \dots$ , a contradiction. So,  $\mathbb{R}$  is uncountable.  $\square$

### Proposition 1.2.2

$$|\mathbb{N}| = |\mathbb{Z}|.$$

### Theorem 1.2.4

$\mathbb{Z} \times \mathbb{Z}$  and  $\mathbb{Q}$  are both denumerable.

### Theorem 1.2.5

$\mathbb{R}$  is uncountable.

- **Remark:** It can be shown that  $|\mathbb{R}| = |\mathbb{C}|$ .
  - Proof: omitted.
- Putting our results together, we obtain:

$$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| < |\mathbb{R}| = |\mathbb{C}|.$$

- **Reminder:**

$$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| < |\mathbb{R}| = |\mathbb{C}|.$$

- So, there are “many more” reals than rationals.
- However:

### Fact 1.2.6

$\mathbb{Q}$  is *dense* in  $\mathbb{R}$ , i.e. for all  $a, b \in \mathbb{R}$  such that  $a < b$ , there exists some  $c \in \mathbb{Q}$  such that  $a < c < b$ .

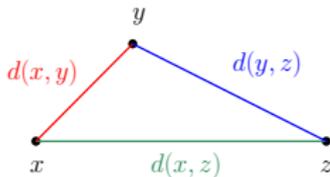
- **Remark:** We omit a formal proof of Fact 1.2.6, but we note that this fact is the reason why we are able to approximate real numbers by rationals (with an arbitrarily small error).

### 3 A (very) brief introduction to metric spaces

#### Definition

A *metric space* is an ordered pair  $(M, d)$ , where  $M$  is a non-empty set, and  $d : M \times M \rightarrow \mathbb{R}$  is a *metric* on  $M$ , i.e. a function satisfying the following properties:

- for all  $x, y \in M$ ,  $d(x, y) \geq 0$ ,
  - for all  $x, y \in M$ ,  $d(x, y) = 0$  iff  $x = y$ ;
  - for all  $x, y \in M$ ,  $d(x, y) = d(y, x)$ ;
  - for all  $x, y, z \in M$ ,  $d(x, z) \leq d(x, y) + d(y, z)$ .
- The inequality from the third bullet point is referred to as the *triangle inequality*.



## Definition

A *metric space* is an ordered pair  $(M, d)$ , where  $M$  is a non-empty set, and  $d : M \times M \rightarrow \mathbb{R}$  is a *metric* on  $M$ , i.e. a function satisfying the following properties:

- for all  $x, y \in M$ ,  $d(x, y) \geq 0$ ,
  - for all  $x, y \in M$ ,  $d(x, y) = 0$  iff  $x = y$ ;
  - for all  $x, y \in M$ ,  $d(x, y) = d(y, x)$ ;
  - for all  $x, y, z \in M$ ,  $d(x, z) \leq d(x, y) + d(y, z)$ .
- For any non-empty set  $M$ , the function  $d : M \times M$  given by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

for all  $x, y \in M$  is a metric, called the *discrete metric*.

## Definition

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  - for all  $x, y \in M$ ,  $d(x, y) = d(y, x)$ ;
  - for all  $x, y, z \in M$ ,  $d(x, z) \leq d(x, y) + d(y, z)$ .
- 
- Each of  $\mathbb{Q}$ ,  $\mathbb{R}$ ,  $\mathbb{C}$  can be turned into a metric space simply by setting  $d(x, y) = |x - y|$  for all  $x$  and  $y$ .

- $\mathbb{R}^d$  can also be turned into a metric space by setting  $d(x, y) = |x - y|$ , where for a vector

$$\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_d \end{bmatrix},$$

we have  $|\mathbf{v}| = \sqrt{v_1^2 + \cdots + v_d^2}$ . This metric is called the *Euclidean metric*.

## Definition

A *metric space* is an ordered pair  $(M, d)$ , where  $M$  is a non-empty set, and  $d : M \times M \rightarrow \mathbb{R}$  is a *metric* on  $M$ , i.e. a function satisfying the following properties:

- for all  $x, y \in M$ ,  $d(x, y) \geq 0$ ,
  - for all  $x, y \in M$ ,  $d(x, y) = 0$  iff  $x = y$ ;
  - for all  $x, y \in M$ ,  $d(x, y) = d(y, x)$ ;
  - for all  $x, y, z \in M$ ,  $d(x, z) \leq d(x, y) + d(y, z)$ .
- 
- We will soon start studying sequences. We will develop the theory of sequences for  $\mathbb{R}$ .
  - However, many (most) of the properties of real sequences are fully generalizable to sequences in general metric spaces.

- For future reference:

### Triangle inequality

All  $x, y \in \mathbb{R}$  satisfy  $|x + y| \leq |x| + |y|$ .

### Corollary 1.3.1

All  $x, y \in \mathbb{R}$  satisfy  $|x - y| \geq |x| - |y|$ .

*Proof.* Fix  $x, y \in \mathbb{R}$ . The triangle inequality applied to  $x - y$  and  $y$  implies that

$$\underbrace{|(x - y) + y|}_{=|x|} \leq |x - y| + |y|,$$

and it follows that  $|x - y| \geq |x| - |y|$ .  $\square$

#### 4 Bernoulli's inequality

### Bernoulli's inequality

For all integers  $n \geq 0$  and all real numbers  $x \geq -1$ , we have  $(1 + x)^n \geq 1 + nx$ .

*Proof.* We fix a real number  $x \geq -1$ , and we proceed by induction on  $n$ .

**Base case:** For  $n = 0$ , we have

$$(1 + x)^0 = 1 = 1 + 0 \cdot x.$$

## Bernoulli's inequality

For all integers  $n \geq 0$  and all real numbers  $x \geq -1$ , we have  $(1 + x)^n \geq 1 + nx$ .

*Proof, continued.* **Induction step:** Fix a non-negative integer  $n$ , and assume inductively that

$$(1 + x)^n \geq 1 + nx.$$

We must show that

$$(1 + x)^{n+1} \geq 1 + (n + 1)x.$$

We now compute:

$$\begin{aligned}(1 + x)^{n+1} &= (1 + x)^n(1 + x) \\ &\geq (1 + nx)(1 + x) \\ &= 1 + (n + 1)x + nx^2 \\ &\geq 1 + (n + 1)x.\end{aligned}$$

by the induction hypothesis  
and the fact that  $1 + x \geq 0$

This completes the induction.  $\square$

5 An introduction to limits of sequences

- A *sequence* of real numbers is any function  $a : \mathbb{N} \rightarrow \mathbb{R}$ .
  - By convention, we write  $a_n$  instead of  $a(n)$ .
  - We denote sequences by  $\{a_n\}_{n=1}^{\infty}$ , by  $\{a_n\}_{n \in \mathbb{N}}$ , or simply by  $a_1, a_2, a_3, \dots$ .
- **Remark:** One can also speak of sequences of rational numbers, complex numbers, vectors in  $\mathbb{R}^d$ , etc. However, we will work almost exclusively with sequences of real numbers.

- We now define the “limit” of a sequence.
  - **Warning:** Not all sequences have limits!
- Intuitively,  $L$  is the limit of a sequence  $\{a_n\}_{n=1}^{\infty}$  if, when  $n$  is very large,  $a_n$  is very close to  $L$ .
  - Let us now formalize this.

## Definition

We say that a sequence  $\{a_n\}_{n=1}^{\infty}$  of real numbers *converges* to a real number  $L$  provided that the following holds:

*For all real numbers  $\varepsilon > 0$ , there exists some  $N \in \mathbb{N}$  such that for all  $n \in \mathbb{N}$ : if  $n \geq N$ , then  $|a_n - L| < \varepsilon$ .*

Under such circumstances, we say that  $L$  is the *limit* of the sequence  $\{a_n\}_{n=1}^{\infty}$ , and we write

$$L = \lim_{n \rightarrow \infty} a_n,$$

or

$$a_n \rightarrow L \quad \text{as} \quad n \rightarrow \infty.$$

A sequence is *convergent* (or *converges*) if it has a limit. Otherwise, it is *divergent* (or *diverges*).

- **Remark:** Note that

$$|a_n - L| < \varepsilon \iff L - \varepsilon < a_n < L + \varepsilon$$

$$\iff a_n \in (L - \varepsilon, L + \varepsilon).$$

- Thus, informally, “ $L = \lim_{n \rightarrow \infty} a_n$ ” means that no matter how small we choose our real number  $\varepsilon > 0$ , at some point, the  $a_n$ 's all start landing in the open interval  $(L - \varepsilon, L + \varepsilon)$ .
- Or, more formally, no matter how small we choose our  $\varepsilon > 0$ , we can find some positive integer  $N$ , so that, with the possible exception of  $a_1, \dots, a_N$  (the first  $N$  terms of our sequence), all the  $a_n$ 's belong to the interval open interval  $(L - \varepsilon, L + \varepsilon)$ .
- **Remark:** It turns out that if a sequence converges, then its limit is unique.
  - We will prove this (see Theorem 2.2.1), but first, let's take a look at a few examples.

### Example 2.1.1

Show that  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ .

*Solution.* Fix  $\varepsilon > 0$ . Let  $N \in \mathbb{N}$  be such that  $N > \frac{1}{\varepsilon}$ . (Thus,  $\frac{1}{N} < \varepsilon$ .) Fix  $n \in \mathbb{N}$  such that  $n \geq N$ . Then

$$\left| \frac{1}{n} - 0 \right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

This proves that  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ .  $\square$

### Example 2.1.2

Show that  $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n^2}\right) = 1$ .

*Solution.* Fix  $\varepsilon > 0$ . Let  $N \in \mathbb{N}$  be such that  $N > \frac{1}{\sqrt{\varepsilon}}$ . Note that this implies that  $\frac{1}{N^2} < \varepsilon$ . Now, fix  $n \in \mathbb{N}$  such that  $n \geq N$ . We now have the following:

$$\begin{aligned} \left| \left(1 + \frac{1}{n^2}\right) - 1 \right| &= \frac{1}{n^2} && \text{because } \frac{1}{n^2} > 0 \\ &\leq \frac{1}{N^2} && \text{because } n \geq N > 0 \\ &< \varepsilon. \end{aligned}$$

This proves that  $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n^2}\right) = 1$ .  $\square$

### Example 2.1.3

Show that  $\lim_{n \rightarrow \infty} (1 + (-\frac{1}{2})^n) = 1$ .

*Solution.* Fix  $\varepsilon > 0$ . Let  $N \in \mathbb{N}$  be such that  $N > \log_2(\frac{1}{\varepsilon})$ . (Thus,  $\frac{1}{2^N} < \varepsilon$ .) Fix  $n \in \mathbb{N}$  such that  $n \geq N$ . Then

$$\left| \left(1 + \left(-\frac{1}{2}\right)^n\right) - 1 \right| = \frac{1}{2^n} \leq \frac{1}{2^N} < \varepsilon.$$

This proves that  $\lim_{n \rightarrow \infty} (1 + (-\frac{1}{2})^n) = 1$ .  $\square$

- We now take a look at a couple of examples of divergent sequences, i.e. sequences that do **not** have a limit.
- By definition, a sequence  $\{a_n\}_{n=1}^{\infty}$  of real number diverges if the following is satisfied:

*For all  $L \in \mathbb{R}$ , there exists some real number  $\varepsilon > 0$  such that for all  $N \in \mathbb{N}$ , there exists some  $n \in \mathbb{N}$  such that  $n \geq N$  and  $|a_n - L| \geq \varepsilon$ .*
- In other words (informally),  $\{a_n\}_{n=1}^{\infty}$  diverges if no matter which  $L$  we choose, we can find some small enough  $\varepsilon > 0$  such that infinitely many  $a_n$ 's land outside of the interval  $(L - \varepsilon, L + \varepsilon)$ .

### Example 2.1.4

Prove that the sequence  $\{\sqrt{n}\}_{n=1}^{\infty}$  diverges.

- **Remark:** Clearly,  $\sqrt{n}$  gets very large (arbitrarily large) as  $n$  gets very large. So, the sequence  $\{\sqrt{n}\}_{n=1}^{\infty}$  “diverges to infinity.” However, the goal is to prove formally that a limit does not exist.

### Example 2.1.4

Prove that the sequence  $\{\sqrt{n}\}_{n=1}^{\infty}$  diverges.

*Solution.* We must prove the following statement:

*For all  $L \in \mathbb{R}$ , there exists some  $\varepsilon > 0$  such that for all  $N \in \mathbb{N}$ , there exists some  $n \in \mathbb{N}$  such that  $n \geq N$  and  $|\sqrt{n} - L| \geq \varepsilon$ .*

Fix an arbitrary  $L \in \mathbb{R}$ . Next, fix some  $\varepsilon > 0$ .

- **Remark:** In this particular example, **any** choice of  $\varepsilon > 0$  will do. In many other examples,  $\varepsilon$  must be chosen more carefully.

Fix an arbitrary  $N \in \mathbb{N}$ . Now, choose  $n \in \mathbb{N}$  so that  $n \geq \max\{N, (L + \varepsilon)^2\}$ . Clearly,  $n \geq N$ . Our goal is to show that  $|\sqrt{n} - L| \geq \varepsilon$ .

By construction,  $n \geq (L + \varepsilon)^2$ , and so (since  $n \geq 0$ ) we have that  $\sqrt{n} \geq |L + \varepsilon| \geq L + \varepsilon$ , and consequently,  $\sqrt{n} - L \geq \varepsilon$ , which in turn implies that  $|\sqrt{n} - L| \geq \sqrt{n} - L \geq \varepsilon$ . This proves that  $\{\sqrt{n}\}_{n=1}^{\infty}$  diverges, which is what we needed to show.  $\square$

### Example 2.1.5

Prove that the sequence  $\{(-1)^n\}_{n=1}^{\infty}$  diverges.

- **Remark:** Clearly, the sequence  $\{(-1)^n\}_{n=1}^{\infty}$  “jumps” between  $-1$  and  $1$ , and so it does not have a limit. However, we need to give a formal proof of the non-existence of a limit.

### Example 2.1.5

Prove that the sequence  $\{(-1)^n\}_{n=1}^{\infty}$  diverges.

*Solution.* We must prove the following statement:

*For all  $L \in \mathbb{R}$ , there exists some  $\varepsilon > 0$  such that for all  $N \in \mathbb{N}$ , there exists some  $n \in \mathbb{N}$  such that  $n \geq N$  and  $|(-1)^n - L| \geq \varepsilon$ .*

Fix an arbitrary  $L \in \mathbb{R}$ . First, note that

$$\begin{aligned} |(-1) - L| + |1 - L| &= |L + 1| + |1 - L| \\ &\geq |(L + 1) + (1 - L)| && \text{by the Triangle} \\ &= 2, && \text{Inequality} \end{aligned}$$

and so either  $|(-1) - L| \geq 1$  or  $|1 - L| \geq 1$ .

We now set  $\varepsilon = 1$ . Fix an arbitrary  $N \in \mathbb{N}$ . Now, we consider two cases: when  $|(-1) - L| \geq 1$ , and when  $|1 - L| \geq 1$ .

### Example 2.1.5

Prove that the sequence  $\{(-1)^n\}_{n=1}^{\infty}$  diverges.

*Solution (continued).* **Case 1:**  $|(-1) - L| \geq 1$ . In this case, we fix an odd  $n \in \mathbb{N}$  such that  $n \geq N$ , so that  $(-1)^n = -1$ . We now have that

$$|(-1)^n - L| = |(-1) - L| \geq 1 = \varepsilon,$$

which is what we needed to show.

**Case 2:**  $|1 - L| \geq 1$ . In this case, we fix an even  $n \in \mathbb{N}$  such that  $n \geq N$ , so that  $(-1)^n = 1$ . We now have that

$$|(-1)^n - L| = |1 - L| \geq 1 = \varepsilon$$

which is what we needed to show.  $\square$

### Theorem 2.2.1

A sequence of real numbers can have at most one limit. So, every convergent sequence of real numbers has a unique limit.

*Proof.* Let  $\{a_n\}_{n=1}^{\infty}$  be a sequence of real numbers, and suppose that this sequence has two distinct limits, call them  $L_1$  and  $L_2$ . Let  $\varepsilon := \frac{1}{2}|L_1 - L_2|$ ; then  $|L_1 - L_2| = 2\varepsilon$ .

Using the fact that  $\{a_n\}_{n=1}^{\infty}$  converges to  $L_1$ , we fix  $N_1 \in \mathbb{N}$  such that for all  $n \in \mathbb{N}$ , if  $n \geq N_1$ , then  $|a_n - L_1| < \varepsilon$ .

Using the fact that  $\{a_n\}_{n=1}^{\infty}$  converges to  $L_2$ , we fix  $N_2 \in \mathbb{N}$  such that for all  $n \in \mathbb{N}$ , if  $n \geq N_2$ , then  $|a_n - L_2| < \varepsilon$ .

Set  $N = \max\{N_1, N_2\}$ , and fix  $n \in \mathbb{N}$  such that  $n \geq N$ . Then  $|a_n - L_1| < \varepsilon$  and  $|a_n - L_2| < \varepsilon$ . We now have the following:

$$\begin{aligned} |L_1 - L_2| &= |(L_1 - a_n) + (a_n - L_2)| \\ &\leq |L_1 - a_n| + |a_n - L_2| && \text{by the Triangle Inequality} \\ &= |a_n - L_1| + |a_n - L_2| \\ &< \varepsilon + \varepsilon = 2\varepsilon. \end{aligned}$$

But this contradicts our choice of  $\varepsilon$ .  $\square$