Mathematical Analysis 1:
Practice problems #4 (not to be turned in)

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There will be a quiz on Thursday, April 9. The quiz will consist of one of the three problems below. (Different students will get different problems.) The solutions of the three problems below are similar to the (partial) Solution#2 of Exercise 3 of Tutorial 6 (see `https://iuuk.mff.cuni.cz/~ipenev/MA1S2026Tutorial06Ex3Ex6c-soln`). You may study any way you like, but on the quiz itself, you will need to solve these problems **without** using any pre-prepared notes or electronic devices. However, I suggest that you prepare by trying to solve the problems by yourself first, and if you aren't able to do so, then get help from someone (such as myself or a classmate) or something (such as AI, though keep in mind that AI occasionally makes mistakes). The Intermediate Value Theorem is stated below, for your convenience. However, on the quiz, you will **not** be given the statement of this theorem, and you will need to have it memorized instead.

The Intermediate Value Theorem. *Let a and b be real numbers such that $a < b$, let $f : [a, b] \to \mathbb{R}$ be a continuous function, and let $M \in \mathbb{R}$ be such that* $\min\{f(a), f(b)\} < M < \max\{f(a), f(b)\}$.[1] *Then there exists some $c \in (a, b)$ such that $f(c) = M$.*

Problem 1. *Let $a, b \in \mathbb{R}$ be such that $a < b$, and let $f : (a, b) \to \mathbb{R}$ be a continuous function. Assume that $\lim\limits_{x \to a^+} f(x) = -\infty$ and $\lim\limits_{x \to b^-} f(x) = +\infty$, and let $M \in \mathbb{R}$. Using the Intermediate Value Theorem, prove that there exists some $c \in (a, b)$ such that $f(c) = M$.*

Problem 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and assume that $L_- := \lim\limits_{x \to -\infty} f(x)$ and $L_+ := \lim\limits_{x \to +\infty} f(x)$ both exist and are real numbers. Assume furthermore that $L_- < L_+$, and let $M \in (L_-, L_+)$. Using the Intermediate Value Theorem, prove that there exists some $c \in \mathbb{R}$ such that $f(c) = M$.*

Problem 3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, assume that $\lim\limits_{x \to -\infty} f(x) = -\infty$ and $\lim\limits_{x \to +\infty} f(x) = +\infty$, and let $M \in \mathbb{R}$. Using the Intermediate Value Theorem, prove that there exists some $c \in \mathbb{R}$ such that $f(c) = M$.*

[1] So, we are assuming that $f(a) \neq f(b)$, and that either $f(a) < M < f(b)$ or $f(b) < M < f(a)$.