Mathematical Analysis 1: Practice problems #2 (not to be turned in)

Irena Penev
Summer 2026

There will be a quiz on Thursday, March 5. The quiz will contain the following:

- one or two routine limit computations (similar to those from Exercise 1 of Tutorial 1);
- one of Problems 1, 2, 3 (below);
- one of Problems 4, 5, 6 (below).

(Different students will get different problems.) Note that Problems 1, 2, 3 are similar to Example 2.3.4 from the Lecture Notes.

Problems 7 and 8 will **not** be on the quiz. However, similar problems may possibly be on the exam.

You may study any way you like, but on the quiz itself, you will need to solve these problems **without** using any pre-prepared notes or electronic devices. However, I suggest that you prepare by trying to solve the problems by yourself first, and if you aren't able to do so, then get help from someone (such as myself or a classmate) or something (such as AI, though keep in mind that AI occasionally makes mistakes).

Problem 1. *Consider the sequence* $\{a_n\}_{n=1}^{\infty}$ *of real numbers defined recursively as follows:*

- $a_1 = 3$;
- $a_{n+1} = \sqrt{3a_n}$ *for all* $n \in \mathbb{N}$.

Prove that $\{a_n\}_{n=1}^{\infty}$ *converges and compute* $\lim\limits_{n\to\infty} a_n$.

Problem 2. *Consider the sequence* $\{a_n\}_{n=1}^{\infty}$ *of real numbers defined recursively as follows:*

- $a_1 = 3$;
- $a_{n+1} = \sqrt{3 + a_n}$ *for all* $n \in \mathbb{N}$.

Prove that $\{a_n\}_{n=1}^{\infty}$ *converges and compute* $\lim\limits_{n\to\infty} a_n$.

Problem 3. *Consider the sequence* $\{a_n\}_{n=1}^{\infty}$ *of real numbers defined recursively as follows:*

- $a_1 = 5$;
- $a_{n+1} = \frac{2}{\frac{1}{2}+\frac{1}{a_n}}$ *for all* $n \in \mathbb{N}$.

Prove that $\{a_n\}_{n=1}^{\infty}$ *converges and compute* $\lim_{n\to\infty} a_n$.

Problem 4. *Let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be sequences of real numbers, and assume that there exits some* $N \in \mathbb{N}$ *such that for all* $n \in \mathbb{N}$, *if* $n \geq N$, *then* $a_n = b_n$. *Prove that:*

(a) $\{a_n\}_{n=1}^{\infty}$ *converges if and only if* $\{b_n\}_{n=1}^{\infty}$ *converges;*

(b) if $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *both converge, then* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

> ***Remark:*** *You are asked to prove Lemma 2.2.2 from the Lecture Notes. (Its proof was left as an exercise.)*

Problem 5. *Let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence of real numbers, and assume that its subsequences* $\{a_{2n-1}\}_{n=1}^{\infty}$ *and* $\{a_{2n}\}_{n=1}^{\infty}$ *converge to the same limit, say, the real number* L. *Prove that* $\lim_{n\to\infty} a_n = L$.

Problem 6. *Let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence of real numbers, and assume that its subsequences* $\{a_{2n-1}\}_{n=1}^{\infty}$ *and* $\{a_{2n}\}_{n=1}^{\infty}$ *diverge to* $+\infty$. *Prove that* $\lim_{n\to\infty} a_n = +\infty$.

Problem 7. *Let* $\{a_n\}_{n=1}^{\infty}$ *be defined recursively as follows:* $a_1 = 1$, *and* $a_{n+1} = \frac{1}{1+a_n}$ *for all* $n \in \mathbb{N}$. *Prove that* $\lim_{n\to\infty} a_n = \frac{\sqrt{5}-1}{2}$.

> ***Hint:*** *You will probably need to use Problem 5 and the Monotone Sequence Theorem.*

Problem 8. *Let* $\{p_n\}_{n=1}^{\infty}$ *and* $\{q_n\}_{n=1}^{\infty}$ *be sequences of natural numbers defined recursively as follows:* $p_1 = 2$, $q_1 = 1$, $p_{n+1} = 2p_n + 3q_n$, *and* $q_{n+1} = p_n + 2q_n$ *for all* $n \in \mathbb{N}$. *Prove that the sequence* $\{\frac{p_n}{q_n}\}_{n=1}^{\infty}$ *converges, and evaluate* $\lim_{n\to\infty} \frac{p_n}{q_n}$.

> ***Hint:*** *Set* $a_n = \frac{p_n}{q_n}$ *for all* $n \in \mathbb{N}$, *and find a recursive formula for the sequence* $\{a_n\}_{n=1}^{\infty}$. *Then use the Monotone Sequence Theorem.*