MATHEMATICAL ANALYSIS 1

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Contents

Chapter 1

Number systems: rational and real numbers

Notation: In what follows, we will use the following notation:

- $\mathbb{N}$ is the set of all natural numbers (positive integers);
- $\mathbb{N}_0$ is the set of all non-negative integers;
- $\mathbb{Z}$ is the set of all integers;
- $\mathbb{Q}$ is the set of all rational numbers;
- $\mathbb{R}$ is the set of all real numbers;
- $\mathbb{C}$ is the set of all complex numbers.

1.1 A brief introduction to "complete ordered fields"

We begin by reviewing a couple of definitions. You have seen "fields" in Linear Algebra, and you have seen "strict total orders" in Discrete Math. The formal definitions are as follows.

Fields. A *field* is an ordered triple $(\mathbb{F}, +, \cdot)$, where $\mathbb{F}$ is a set, and $+$ and $\cdot$ are binary operations on $\mathbb{F}$ (i.e. functions from $\mathbb{F} \times \mathbb{F}$ to $\mathbb{F}$), called *addition* and *multiplication*, respectively, satisfying the following axioms:

1. [**Associativity of addition and multiplication**] addition and multiplication are associative, that is, for all $a, b, c \in \mathbb{F}$, we have that $a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$;
2. [**Commutativity of addition and multiplication**] addition and multiplication are commutative, that is, for all $a, b \in \mathbb{F}$, we have that $a + b = b + a$ and $a \cdot b = b \cdot a$;

3. [**Additive and multiplicative identity**] there exist distinct elements $0_\mathbb{F}, 1_\mathbb{F} \in \mathbb{F}$ such that for all $a \in \mathbb{F}$, $a + 0_\mathbb{F} = a$ and $a \cdot 1_\mathbb{F} = a$; $0_\mathbb{F}$ is called the *additive identity* of $\mathbb{F}$, and $1_\mathbb{F}$ is called the *multiplicative identity* of $\mathbb{F}$;

4. [**Additive inverses**] for every $a \in \mathbb{F}$, there exists an element in $\mathbb{F}$, denoted by $-a$ and called the *additive inverse* of a, such that $a + (-a) = 0_\mathbb{F}$;

5. [**Multiplicative inverses**] for all $a \in \mathbb{F} \setminus \{0_\mathbb{F}\}$, there exists an element in $\mathbb{F}$, denoted by a^{-1} and called the *multiplicative inverse* of a, such that $a \cdot a^{-1} = 1_\mathbb{F}$;

6. [**Distributivity of multiplication over addition**] multiplication is distributive over addition, that is, for all $a, b, c \in \mathbb{F}$, we have that $a \cdot (b+c) = (a \cdot b) + (a \cdot c)$.

Notation: Usually, we write simply "field $\mathbb{F}$" rather than "field $(\mathbb{F}, +, \cdot)$." When $\mathbb{F}$ is clear from context, we write simply 0 and 1 instead of $0_\mathbb{F}$ and $1_\mathbb{F}$, respectively.

Example 1.1.1. *All the following are fields:*

- $(\mathbb{Q}, +, \cdot)$;
- $(\mathbb{R}, +, \cdot)$;
- $(\mathbb{C}, +, \cdot)$.

*However, the following are **not** fields:*

- $(\mathbb{N}, +, \cdot)$;
- $(\mathbb{N}_0, +, \cdot)$;
- $(\mathbb{Z}, +, \cdot)$.

Strict total orders. A *strict total order* on a non-empty set A is a binary relation $<$ on A that satisfies the following two axioms:

1. [**Transitivity**] For all $a, b \in A$, if $a < b$ and $b < c$, then $a < c$.

2. [**Trichotomy**] For all $a, b \in A$, exactly one of $a < b$, $a = b$, and $b < a$ is true.

Ordered fields. An *ordered field* is a field $\mathbb{F}$ with a strict total order $<$ such that:

- if $a < b$, then $a + c < b + c$;
- if $0 < a$ and $0 < b$, then $0 < ab$.

We define $\leq$ as follows: $a \leq b$ if $a < b$ or $a = b$. Furthermore, we write $a > b$ when $b < a$, and we write $a \geq b$ when $b \leq a$.

Remark: It is not hard to show that, in an ordered field, all the usual algebraic properties of the relations $<$ and $\leq$ hold. We omit the proof.

Fact 1.1.2. *$\mathbb{Q}$ and $\mathbb{R}$ are ordered fields (under the usual $<$ relation). However, $\mathbb{C}$ is **not** an ordered field.*

Upper and lower bounds. If $\mathbb{F}$ is an ordered field, $S \subseteq \mathbb{F}$, $x \in \mathbb{F}$, then

- x is called an *upper bound* for S if for all $s \in S$, we have $s \leq x$;
- x is called the *least upper bound* (or *supremum*) for S if x is an upper bound for S, and every upper bound y for S satisfies $x \leq y$.
- x is called a *lower bound* for S if for all $s \in S$, we have $x \leq s$;
- x is called the *greatest lower bound* (or *infimum*) for S if x is a lower bound for S, and every lower bound y for S satisfies $y \leq x$.

A subset of $\mathbb{F}$ is *bounded above* if it has an upper bound, and it is *bounded below* if it has a lower bound. A subset of $\mathbb{F}$ is *bounded* if it is both bounded above and bounded below.

Remarks:

- An upper or lower bound for a set S may, but need not, belong to S. In particular, the supremum or infimum of a set S (if it exists) may, but need not, belong to S.
- A subset of an ordered field does not necessarily have an upper or a lower bound. For instance, in the ordered field $\mathbb{R}$, the set $\mathbb{R}$ itself has neither an upper nor a lower bound.
- If a set does have a supremum, then that supremum is unique, and the same goes for the infimum (see Proposition 1.1.4 below).

Example 1.1.3. *In* $\mathbb{R}$*:*

- *the set* $[-2, 3)$ *has both a supremum (namely* 3*) and an infimum (namely* -2*);*
- *the set* $(-\infty, 4]$ *has a supremum (namely* 4*), but no infimum;*
- *the set* $(2, \infty)$ *has no supremum, but does have an infimum (namely* 2*);*
- *the set* $(2, 3) \cup (5, 7)$ *has both a supremum (namely* 7*) and an infimum (namely* 2*).*

Proposition 1.1.4. *Let* $\mathbb{F}$ *be an ordered field, and let* $S \subseteq \mathbb{F}$*. Then* S *has at most one supremum and at most one infimum.*[1]

Proof. Suppose that x and y are suprema of S; we must show that $x = y$. Since x is an supremum of S and y is an upper bound of S, we have that $x \leq y$. Similarly, since y is a supremum of S and x is an upper bound of S, we have that $y \leq x$. Since both $x \leq y$ and $y \leq x$ hold, we see that $x = y$. This proves that S does indeed have at most one supremum. Analogously, S has at most one infimum. □

[1] Once again, supremum of S, if it exists, may or may not belong to S, and the same applies to the infimum.

Notation: If $\mathbb{F}$ is an ordered field and the set $S \subseteq \mathbb{F}$ has a supremum, then by Proposition 1.1.4, that supremum is unique, and we denote it by $\sup(S)$. Similarly, if S has an infimum, then by Proposition 1.1.4, that infimum is unique, and we denote it by $\inf(S)$.

Complete ordered fields. An ordered field $\mathbb{F}$ is *complete* if every non-empty subset of $\mathbb{F}$ that is bounded above has the least upper bound (i.e. supremum).

Remark: It is not hard to show that in a complete ordered field, every non-empty set that is bounded below has the greatest lower bound (i.e. infimum).

Fact 1.1.5. *$\mathbb{R}$ is a complete ordered field. In fact, up to "isomorphism" (essentially, a renaming of elements), $\mathbb{R}$ is the **only** complete ordered field.*

Remark: In particular, $\mathbb{Q}$ is **not** a complete ordered field.

We omit the proof of Fact 1.1.5. To give a formal proof, we would need to fully formalize the field of real numbers (e.g. using "Dedekind cuts" or equivalence classes of "Cauchy sequences" of rational numbers), which is beyond the scope of this course. The point is that when we speak of a "complete ordered field," we are in fact referring to $\mathbb{R}$ (equipped with the usual addition, multiplication, and less-than relation).

Remark: The fact that $\mathbb{R}$ is a complete ordered field is essential for formally defining functions such as $\sin, \cos, \exp$ (and many others).

Example 1.1.6. *Note that the set $S = \{x \in \mathbb{Q} \mid x^2 \leq 2\}$ is bounded in $\mathbb{Q}$ (and therefore in $\mathbb{R}$ as well). For example, 2 is an upper bound of S (both in $\mathbb{Q}$ and in $\mathbb{R}$). In $\mathbb{R}$, $\sqrt{2}$ is the least upper bound (i.e. supremum) of S. In $\mathbb{Q}$, S does not have the least upper bound. Here, we are using the well-known fact that $\sqrt{2}$ is an irrational number.*

Actually, let us formally prove that $\sqrt{2}$ is irrational!

Theorem 1.1.7. *$\sqrt{2}$ is irrational.*

Proof. Suppose otherwise, and set $\sqrt{2} = \frac{p}{q}$, where $p \in \mathbb{Z}$, $q \in \mathbb{N}$, and p, q are relatively prime (i.e. they have no common divisor greater than 1). Then $p^2 = 2q^2$, which implies that $2 \mid p^2$. Since 2 is prime, it follows that $2 \mid p$. Thus, there exists some $r \in \mathbb{Z}$ such that $p = 2r$. Now $4r^2 = 2q^2$, and it follows that $2r^2 = q^2$. Now $2 \mid q^2$, and so since 2 is prime, we see that $2 \mid q$. But now 2 is a common divisor of p and q, contrary to the fact that p and q are relatively prime. $\square$

1.2 Cardinality: comparing infinities

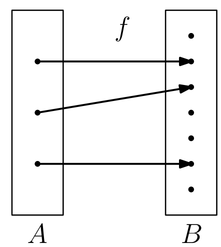
We know that

$$\mathbb{N} \subsetneqq \mathbb{Z} \subsetneqq \mathbb{Q} \subsetneqq \mathbb{R} \subsetneqq \mathbb{C}.$$

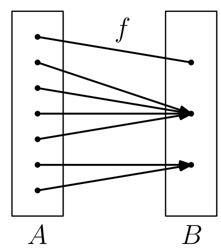
So, in a sense, there are “more” integers than there are natural numbers, “more” rational numbers than integers, “more” real numbers than real numbers, and “more” complex numbers than real numbers. However, there is another way of comparing the sizes of two sets: using bijections. Let us first recall the definition of a bijection.

Injections, surjections, and bijections. A function $f : A \to B$ (where A and B are some sets) is:

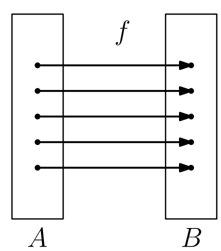
- *one-to-one* (or *injective*, or an *injection*) if for all $a_1, a_2 \in A$ such that $a_1 \neq a_2$, we have that $f(a_1) \neq f(a_2)$;[2]



- *onto* (or *surjective*, or a *surjection*) if for all $b \in B$, there exists some $a \in A$ such that $f(a) = b$;

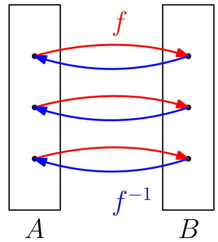


- *bijective* (or a *bijection*) if it is both one-to-one and onto.



Remark: Recall that if $f : A \to B$ is a bijection, then it has an *inverse function* $f^{-1} : B \to A$ such that for all $b \in B$, $f^{-1}(b)$ is the unique element $a \in A$ such that $f(a) = b$. In other words, if $f : A \to B$ is a bijection, then for all $a \in A$ and $b \in B$, we have that $f(a) = b$ if and only if $f^{-1}(b) = a$. Clearly, the inverse of a bijection is also a bijection.

[2] Equivalently: if for all $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$, we have that $a_1 = a_2$.



Cardinality. Sets A and B are said to have *the same cardinality* if there exists a bijection $f : A \to B$, and in this case, we write $|A| = |B|$.

Remark: Thus, we can think of two sets as being of "the same size" (or having "the same number of elements") if we can set up a bijection between them.

Proposition 1.2.1. *All the following hold:*

(a) for all sets A, $|A| = |A|$;

(b) for all sets A and B, if $|A| = |B|$, then $|B| = |A|$;

(c) for all sets A, B, and C, if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

Proof. Fix sets A, B, and C.

For (a), we simply observe that the identity function on A is a bijection.[3]

For (b), we observe that if $f : A \to B$ is a bijection, then so is $f^{-1} : B \to A$.

For (c), we observe that if $f_1 : A \to B$ and $f_2 : B \to C$ are bijections, then $f_2 \circ f_1 : A \to C$ is also a bijection. □

Proposition 1.2.2. $|\mathbb{N}| = |\mathbb{Z}|$.

Proof. We define the function $f : \mathbb{N} \to \mathbb{Z}$ by setting

$$f(n) \quad = \quad \begin{cases} (n-1)/2 & \text{if } n \text{ is odd} \\ -n/2 & \text{if } n \text{ is even} \end{cases}$$

for all $n \in \mathbb{N}$. Thus, we have the following:

- $f(2) = -1$;
- $f(4) = -2$;
- $f(6) = -3$;
- ...

- $f(1) = 0$;
- $f(3) = 1$;
- $f(4) = 2$;
- $f(7) = 3$;
- ...

[3]The *identity function* on A is the function $\mathrm{Id}_A : A \to A$ given by $f(a) = a$ for all $a \in A$.

and it is not difficult to formally check that f is a bijection.[4] This proves that $|\mathbb{N}| = |\mathbb{Z}|$. $\square$

Given sets A and B, we say that the cardinality of A is *no greater than* the cardinality of B, and we write $|A| \leq |B|$ if there exists a one-to-one function $f : A \to B$.

Notation: For sets A and B:

- we write $|A| \geq |B|$ if $|B| \leq |A|$;
- we write $|A| < |B|$ if $|A| \leq |B|$ and $|A| \neq |B|$ (i.e. there exists a one-to-one function from A to B, but there is no bijection between A and B);
- we write $|A| > |B|$ if $|B| < |A|$.

Remark: Note that if sets A, B, and C satisfy $|A| \leq |B|$ and $|B| \leq |C|$, then they also satisfy $|A| \leq |C|$. Indeed, if $f_1 : A \to B$ and $f_2 : B \to C$ are one-to-one functions, then $f_2 \circ f_1 : A \to C$ is also a one-to-one function.

Cantor–Schröder–Bernstein theorem. *If sets A and B satisfy $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.*

Proof. Omitted. $\square$

Remark: The Cantor–Schröder–Bernstein theorem may seem obvious, but it is in fact not! Fully spelled out, it states the following:

> For all sets A and B, if there exist one-to-one functions $f_1 : A \to B$ and $f_2 : B \to A$, then there exists a bijection $f : A \to B$.

The statement is indeed true, but the proof is beyond the scope of this course.

Remark: It is possible to prove that for all sets A and B, either $|A| \leq |B|$ or $|B| \leq |A|$. However, the proof (which we omit) is not easy and uses the so called "Axiom of Choice."

Proposition 1.2.3. *For all sets A and B, if $A \subseteq B$, then $|A| \leq |B|$.*

Proof. Let A and B be sets such that $A \subseteq B$. Then the function $f : A \to B$ given by $f(a) = a$ for all $a \in A$ is one-to-one, and consequently, $|A| \leq |B|$. $\square$

Remark: Since

$$\mathbb{N} \subsetneqq \mathbb{Z} \subsetneqq \mathbb{Q} \subsetneqq \mathbb{R} \subsetneqq \mathbb{C},$$

Proposition 1.2.3 guarantees that

$$|\mathbb{N}| \leq |\mathbb{Z}| \leq |\mathbb{Q}| \leq |\mathbb{R}| \leq |\mathbb{C}|,$$

[4] Check this!

and moreover, by Proposition 1.2.2, we further have that $|\mathbb{N}| = |\mathbb{Z}|$.

Countable and uncountable sets. A set S is *countable* if one of the following holds:

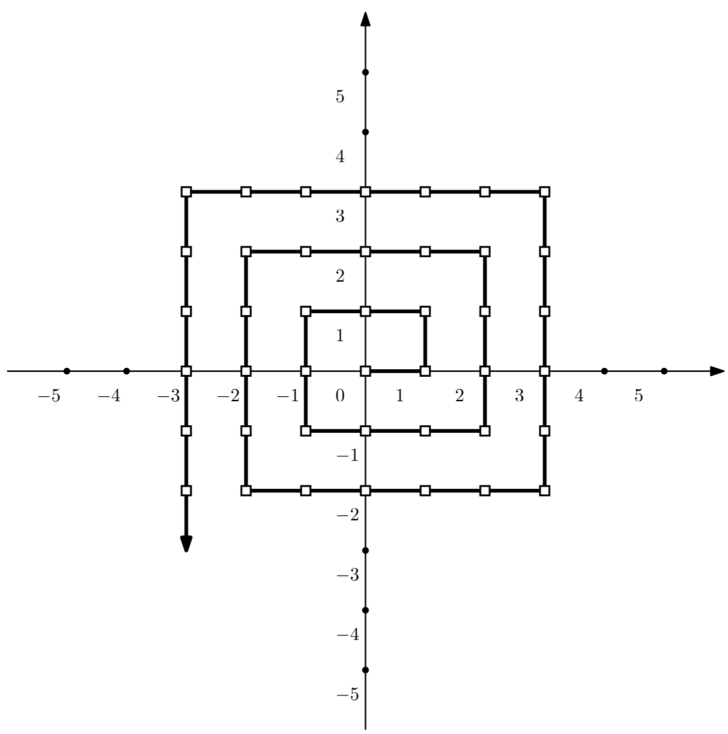
- S is finite;
- there exists a bijection $f : \mathbb{N} \to S$.

A set is *denumerable* (or *countably infinite*) if it is both countable and infinite. So, if S is infinite, then it is countable if and only if its members can be enumerated as $s_1, s_2, s_3, \dots$.[5] A set is *uncountable* if it is not countable.

Remark: By Proposition 1.2.2, $\mathbb{Z}$ is denumerable (i.e. countably infinite).

Theorem 1.2.4. *$\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Q}$ are both denumerable.*

Proof (slightly informal). The picture below shows that $\mathbb{Z} \times \mathbb{Z}$ can be enumerated as $(p_1, q_1), (p_2, q_2), (p_3, q_3), \dots$. So, $\mathbb{Z} \times \mathbb{Z}$ is countable.



Rational numbers correspond to $\mathbb{Z} \times \mathbb{Z}$ in a natural way. Indeed, a fraction $\frac{p}{q}$ corresponds to the ordered pair (p, q). Except that there's a problem: each rational corresponds to more than one ordered pair of integers (because e.g. $\frac{p}{q} = \frac{2p}{2q}$).

[5]Here, the elements, $s_1, s_2, s_3, \dots$ are required to be pairwise distinct, and all the elements of S must appear on the list $s_1, s_2, s_3, \dots$.

Furthermore, the denominator of a fraction can never be zero. We deal with this as follows. First, we take our list of ordered pairs of integers, and we delete from it all pairs (p, q) where $q \leq 0$. Then, we remove all ordered pairs (p, q) where p and q are not relatively prime. Now each rational corresponds to exactly one ordered pair on the remaining list. This proves that $\mathbb{Q}$ is countable. $\square$

Theorem 1.2.5. $\mathbb{R}$ *is uncountable.*

Cantor's diagonal proof. Suppose otherwise. Then in particular, the interval $[0, 1]$ is countable. We now enumerate the members of $[0, 1]$ as follows:[6]

$$x_1 = 0.x_{1,1}x_{1,2}x_{1,3}x_{1,4}\dots$$
$$x_2 = 0.x_{2,1}x_{2,2}x_{2,3}x_{2,4}\dots$$
$$x_3 = 0.x_{3,1}x_{3,2}x_{3,3}x_{3,4}\dots$$
$$x_4 = 0.x_{4,1}x_{4,2}x_{4,3}x_{4,4}\dots$$
$$\vdots$$

We now create a number $a = 0.a_1a_2a_3a_4\dots$ as follows. For each $i \in \mathbb{N}$, we set

$$a_i = \begin{cases} 5 & \text{if} \quad x_{i,i} \neq 5 \\ 6 & \text{if} \quad x_{i,i} = 5 \end{cases}$$

But now $a \in [0, 1]$, and it does not appear on our list (sequence) $x_1, x_2, x_3, \dots$, a contradiction. So, $\mathbb{R}$ is uncountable. $\square$

Remark: It can be shown (but we omit the proof) that $|\mathbb{R}| = |\mathbb{C}|$. This, combined with Propositions 1.2.3 and 1.2.2 and Theorems 1.2.4 and 1.2.5, yield the following:

$$|\mathbb{N}| \quad = \quad |\mathbb{Z}| \quad = \quad |\mathbb{Q}| \quad < \quad |\mathbb{R}| \quad = \quad |\mathbb{C}|.$$

Density of rationals in the reals. Theorems 1.2.4 and 1.2.5 essentially imply that there are "many more" reals than rationals. We do, however, have the following.

Fact 1.2.6. $\mathbb{Q}$ *is* dense *in* $\mathbb{R}$*, i.e. for all* $a, b \in \mathbb{R}$ *such that* $a < b$*, there exists some* $c \in \mathbb{Q}$ *such that* $a < c < b$*.*

Remark: We omit a formal proof of Fact 1.2.6, but we note that this fact is the reason why we are able to approximate real numbers by rationals (with an arbitrarily small error).

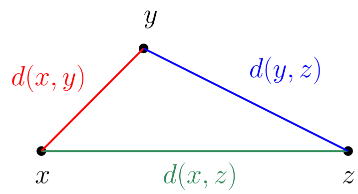
[6] Note that $1 = 0.9999999\dots$.

1.3 A (very) brief introduction to metric spaces

A *metric space* is an ordered pair (M, d), where M is a non-empty set, and $d : M \times M \to \mathbb{R}$ is a *metric* on M, i.e. a function satisfying the following properties:

- for all $x, y \in M$, $d(x, y) \geq 0$,
- for all $x, y \in M$, $d(x, y) = 0$ if and only if $x = y$;
- for all $x, y \in M$, $d(x, y) = d(y, x)$;
- for all $x, y, z \in M$, $d(x, z) \leq d(x, y) + d(y, z)$.

The inequality from the third bullet point is referred to as the *triangle inequality*.



The discrete metric. For any non-empty set M, the function $d : M \times M$ given by

$$d(x, y) \quad = \quad \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

for all $x, y \in M$ is a metric, called the *discrete metric*.

The Euclidean metric. Each of $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ can be turned into a metric space simply by setting $d(x, y) = |x - y|$ for all x and y.

As a matter of fact, $\mathbb{R}^d$ can also be turned into a metric space by setting $d(x, y) = |x - y|$, where for a vector

$$\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_d \end{bmatrix},$$

we have $|\mathbf{v}| = \sqrt{v_1^2 + \cdots + v_d^2}$. This metric is called the *Euclidean metric*.

We will soon start studying sequences. We will develop the theory of sequences for $\mathbb{R}$. However, many (most) of the properties of real sequences are fully generalizable to sequences in general metric spaces.

For future reference, we give the following inequalities.

The Triangle Inequality. *All* $x, y \in \mathbb{R}$ *satisfy* $|x + y| \leq |x| + |y|$.

Corollary 1.3.1. *All* $x, y \in \mathbb{R}$ *satisfy* $|x - y| \geq |x| - |y|$.

Proof. Fix $x, y \in \mathbb{R}$. The triangle inequality applied to $x - y$ and y implies that $\underbrace{|(x - y) + y|}_{=|x|} \leq |x - y| + |y|$, and it follows that $|x - y| \geq |x| - |y|$. □

1.4 Bernoulli's inequality

Bernoulli's inequality. *For all integers $n \geq 0$ and all real numbers $x \geq -1$, we have $(1+x)^n \geq 1+nx$.*

Proof. We fix a real number $x \geq -1$, and we proceed by induction on n.

Base case: For $n = 0$, we have

$$(1+x)^0 = 1 = 1 + 0 \cdot x.$$

Induction step: Fix a non-negative integer n, and assume inductively that

$$(1+x)^n \geq 1 + nx.$$

We must show that

$$(1+x)^{n+1} \geq 1 + (n+1)x.$$

We now compute:

$$\begin{aligned}
(1+x)^{n+1} &= (1+x)^n(1+x) \\
&\geq (1+nx)(1+x) && \text{by the induction hypothesis and the fact that } 1+x \geq 0 \\
&= 1 + (n+1)x + nx^2 \\
&\geq 1 + (n+1)x.
\end{aligned}$$

This completes the induction. □

Chapter 2

Sequences

2.1 Limits of sequences: definition and examples

A *sequence* of real numbers is any function $a : \mathbb{N} \to \mathbb{R}$. By convention, we write a_n instead of $a(n)$. We denote sequences by $\{a_n\}_{n=1}^{\infty}$, by $\{a_n\}_{n\in\mathbb{N}}$, or simply by $a_1, a_2, a_3, \dots$.

Remark: One can also speak of sequences of rational numbers, complex numbers, vectors in $\mathbb{R}^d$, etc. However, we will work almost exclusively with sequences of real numbers.

We now define the "limit" of a sequence.[1] Intuitively, L is the limit of a sequence $\{a_n\}_{n=1}^{\infty}$ if, when n is very large, a_n is very close to L. Let us now formalize this.

The limit of a sequence. We say that a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers *converges* to a real number L provided that the following holds:

> For all real numbers $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$: if $n \geq N$, then $|a_n - L| < \varepsilon$.

Under such circumstances, we say that L is the *limit* of the sequence $\{a_n\}_{n=1}^{\infty}$, and we write

$$L = \lim_{n\to\infty} a_n,$$

or

$$a_n \to L \quad \text{as} \quad n \to \infty.$$

A sequence is *convergent* (or *converges*) if it has a limit. Otherwise, it is *divergent* (or *diverges*).

Remark: Note that

$$\begin{aligned} |a_n - L| < \varepsilon \qquad &\iff \qquad L - \varepsilon < a_n < L + \varepsilon \\ &\iff \qquad a_n \in (L - \varepsilon, L + \varepsilon). \end{aligned}$$

[1]**Warning:** Not all sequences have limits!

Thus, informally, "$L = \lim_{n\to\infty} a_n$" means that no matter how small we choose our real number $\varepsilon > 0$, at some point, the a_n's all start landing in the open interval $(L-\varepsilon, L+\varepsilon)$. Or, more formally, no matter how small we choose our $\varepsilon > 0$, we can find some positive integer N, so that, with the possible exception of $a_1, \dots, a_N$ (the first N terms of our sequence), all the a_n's belong to the interval open interval $(L-\varepsilon, L+\varepsilon)$.

Remark: It turns out that if a sequence converges, then its limit is unique. We will prove this (see Theorem 2.2.1), but first, let us take a look at a couple of examples.

Example 2.1.1. *Show that* $\lim_{n\to\infty} \frac{1}{n} = 0$.

Solution. Fix $\varepsilon > 0$. Let $N \in \mathbb{N}$ be such that $N > \frac{1}{\varepsilon}$. (Thus, $\frac{1}{N} < \varepsilon$.) Fix $n \in \mathbb{N}$ such that $n \geq N$. Then

$$\left|\frac{1}{n} - 0\right| \;=\; \frac{1}{n} \;\leq\; \frac{1}{N} \;<\; \varepsilon.$$

This proves that $\lim_{n\to\infty} \frac{1}{n} = 0$. □

Example 2.1.2. *Show that* $\lim_{n\to\infty} (1 + \frac{1}{n^2}) = 1$.

Solution. Fix $\varepsilon > 0$. Let $N \in \mathbb{N}$ be such that $N > \frac{1}{\sqrt{\varepsilon}}$.[2] Note that this implies that $\frac{1}{N^2} < \varepsilon$. Now, fix $n \in \mathbb{N}$ such that $n \geq N$. We now have the following:

$$\begin{aligned} |(1 + \tfrac{1}{n^2}) - 1| &= \tfrac{1}{n^2} && \text{because } \tfrac{1}{n^2} > 0 \\ &\leq \tfrac{1}{N^2} && \text{because } n \geq N > 0 \\ &< \varepsilon. \end{aligned}$$

This proves that $\lim_{n\to\infty} (1 + \frac{1}{n^2}) = 1$. □

Example 2.1.3. *Show that* $\lim_{n\to\infty} (1 + (-\frac{1}{2})^n) = 1$.

Solution. Fix $\varepsilon > 0$. Let $N \in \mathbb{N}$ be such that $N > \log_2(\frac{1}{\varepsilon})$. (Thus, $\frac{1}{2^N} < \varepsilon$.) Fix $n \in \mathbb{N}$ such that $n \geq N$. Then

$$\left|\Big(1 + (-\tfrac{1}{2})^n\Big) - 1\right| = \frac{1}{2^n} \leq \frac{1}{2^N} < \varepsilon.$$

This proves that $\lim_{n\to\infty} (1 + (-\frac{1}{2})^n) = 1$. □

[2]This footnote is **not** a formal part of the proof (think of it as being on "scratch paper"), but here's how we "guessed" the correct value of N. Our goal is to get $|(1 + \frac{1}{n^2}) - 1| < \varepsilon$. This is equivalent to $\frac{1}{n^2} < \varepsilon$, and that, in turn is equivalent to $n > \frac{1}{\sqrt{\varepsilon}}$. Thus, we should choose $N \in \mathbb{N}$ such that $N > \frac{1}{\sqrt{\varepsilon}}$.

We now take a look at a couple of examples of divergent sequences, i.e. sequences that do **not** have a limit. By definition, a sequence $\{a_n\}_{n=1}^{\infty}$ of real number diverges if the following is satisfied:

> For all $L \in \mathbb{R}$, there exists some real number $\varepsilon > 0$ such that for all $N \in \mathbb{N}$, there exists some $n \in \mathbb{N}$ such that $n \geq N$ and $|a_n - L| \geq \varepsilon$.

In other words (informally), $\{a_n\}_{n=1}^{\infty}$ diverges if no matter which L we choose, we can find some small enough $\varepsilon > 0$ such that infinitely many a_n's land outside of the interval $(L - \varepsilon, L + \varepsilon)$.

Example 2.1.4. *Prove that the sequence* $\{\sqrt{n}\}_{n=1}^{\infty}$ *diverges.*

Remark: Clearly, $\sqrt{n}$ gets very large (arbitrarily large) as n gets very large. So, the sequence $\{\sqrt{n}\}_{n=1}^{\infty}$ "diverges to infinity." However, the goal is to prove formally that a limit does not exist.

Solution. We must prove the following statement:

> For all $L \in \mathbb{R}$, there exists some $\varepsilon > 0$ such that for all $N \in \mathbb{N}$, there exists some $n \in \mathbb{N}$ such that $n \geq N$ and $|\sqrt{n} - L| \geq \varepsilon$.

Fix an arbitrary $L \in \mathbb{R}$. Next, fix some $\varepsilon > 0$.[3] Fix an arbitrary $N \in \mathbb{N}$. Now, choose $n \in \mathbb{N}$ so that $n \geq \max\{N, (L+\varepsilon)^2\}$. Clearly, $n \geq N$. Our goal is to show that $|\sqrt{n} - L| \geq \varepsilon$.

By construction, $n \geq (L+\varepsilon)^2$, and so (since $n \geq 0$) we have that $\sqrt{n} \geq |L+\varepsilon| \geq L + \varepsilon$, and consequently, $\sqrt{n} - L \geq \varepsilon$, which in turn implies that

$$|\sqrt{n} - L| \geq \sqrt{n} - L \geq \varepsilon.$$

This proves that $\{\sqrt{n}\}_{n=1}^{\infty}$ diverges, which is what we needed to show. □

Example 2.1.5. *Prove that the sequence* $\{(-1)^n\}_{n=1}^{\infty}$ *diverges.*

Remark: Clearly, the sequence $\{(-1)^n\}_{n=1}^{\infty}$ "jumps" between -1 and 1, and so it does not have a limit. However, we need to give a formal proof of the the non-existence of a limit.

Solution. We must prove the following statement:

> For all $L \in \mathbb{R}$, there exists some $\varepsilon > 0$ such that for all $N \in \mathbb{N}$, there exists some $n \in \mathbb{N}$ such that $n \geq N$ and $|(-1)^n - L| \geq \varepsilon$.

Fix an arbitrary $L \in \mathbb{R}$. First, note that

[3] In this particular example, any choice of $\varepsilon > 0$ will do. We will later see examples where ε must be chosen more carefully.

$$\begin{aligned} |(-1)-L|+|1-L| &= |L+1|+|1-L| \\ &\geq |(L+1)+(1-L)| \qquad \text{by the Triangle Inequality} \\ &= 2, \end{aligned}$$

and so either $|(-1)-L| \geq 1$ or $|1-L| \geq 1$.

We now set $\varepsilon = 1$. Fix an arbitrary $N \in \mathbb{N}$. Now, we consider two cases: when $|(-1)-L| \geq 1$, and when $|1-L| \geq 1$.

Case 1: $|(-1)-L| \geq 1$. In this case, we fix an odd $n \in \mathbb{N}$ such that $n \geq N$, so that $(-1)^n = -1$. We now have that

$$|(-1)^n - L| \;=\; |(-1)-L| \;\geq\; 1 \;=\; \varepsilon,$$

which is what we needed to show.

Case 2: $|1-L| \geq 1$. In this case, we fix an even $n \in \mathbb{N}$ such that $n \geq N$, so that $(-1)^n = 1$. We now have that

$$|(-1)^n - L| \;=\; |1-L| \;\geq\; 1 \;=\; \varepsilon$$

which is what we needed to show. □

2.2 Properties of limits

The theorems that we prove in this section are stated (and proven) for sequences of real numbers. However, they remain true for sequences in $\mathbb{C}$ or $\mathbb{R}^d$, with very minor (if any) modifications of the proofs.

Theorem 2.2.1. *A sequence of real numbers can have at most one limit. So, every convergent sequence of real numbers has a unique limit.*

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers, and suppose that this sequence has two distinct limits, call them L_1 and L_2. Set $\varepsilon := \frac{1}{2}|L_1 - L_2|$; then $|L_1 - L_2| = 2\varepsilon$.

Using the fact that $\{a_n\}_{n=1}^{\infty}$ converges to L_1, we fix $N_1 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N_1$, then $|a_n - L_1| < \varepsilon$.

Using the fact that $\{a_n\}_{n=1}^{\infty}$ converges to L_2, we fix $N_2 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N_2$, then $|a_n - L_2| < \varepsilon$.

Set $N = \max\{N_1, N_2\}$, and fix $n \in \mathbb{N}$ such that $n \geq N$. Then $|a_n - L_1| < \varepsilon$ and $|a_n - L_2| < \varepsilon$. We now have the following:

$$\begin{aligned} |L_1 - L_2| &= |(L_1 - a_n) + (a_n - L_2)| \\ &\leq |L_1 - a_n| + |a_n - L_2| \qquad \text{by the Triangle Inequality} \end{aligned}$$

$$= \quad |a_n - L_1| + |a_n - L_2|$$
$$< \quad \varepsilon + \varepsilon$$
$$= \quad 2\varepsilon.$$

But this contradicts our choice of ε, i.e. the fact that $|L_1 - L_2| = 2\varepsilon$. □

Remark: Note that whether a sequence of real numbers converges or diverges is determined by what happens in the sequence "eventually," i.e. the first few terms do not count. The same applies to the value of the limit, if it exists. Formally, we have the following lemma.

Lemma 2.2.2. *Let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be sequences of real numbers, and assume that there exits some* $N \in \mathbb{N}$ *such that for all* $n \in \mathbb{N}$*, if* $n \geq N$*, then* $a_n = b_n$*.*[4] *Then:*

(a) $\{a_n\}_{n=1}^{\infty}$ *converges if and only if* $\{b_n\}_{n=1}^{\infty}$ *converges;*

(b) if $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *both converge, then* $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n$*.*

Proof. Exercise. □

In chapter 1, we defined upper and lower bounds of sets of real numbers, as well as their suprema and infima. We can analogously (and straightforwardly) define these concepts for sequences of real numbers, as follows.

Bounded sequences. For a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers:

- an *upper bound* of $\{a_n\}_{n=1}^{\infty}$ is any real number x such that for all $n \in \mathbb{N}$, we have $a_n \leq x$.
- the *least upper bound* (or *supremum*) of $\{a_n\}_{n=1}^{\infty}$ is an upper bound x of $\{a_n\}_{n=1}^{\infty}$ such that for every upper bound y of $\{a_n\}_{n=1}^{\infty}$, we have that $x \leq y$;
- $\{a_n\}_{n=1}^{\infty}$ is *bounded above* if it has an upper bound;
- a *lower bound* of $\{a_n\}_{n=1}^{\infty}$ is any real number x such that for all $n \in \mathbb{N}$, we have $x \leq a_n$.
- the *greatest lower bound* (or *infimum*) of $\{a_n\}_{n=1}^{\infty}$ is a lower bound x of $\{a_n\}_{n=1}^{\infty}$ such that for every lower bound y of $\{a_n\}_{n=1}^{\infty}$, we have that $y \leq x$;
- $\{a_n\}_{n=1}^{\infty}$ is *bounded below* if it has a lower bound;

[4] In other words, the two sequences may possibly differ in the first N terms, but after that, they coincide.

- $\{a_n\}_{n=1}^{\infty}$ is *bounded* if it is both bounded above and bounded below.

Remark: Since the ordered field $\mathbb{R}$ is complete, any sequence of real numbers that is bounded above has a supremum, and any sequence of real numbers that is bounded below has an infimum.

Remark: Note that if $\{a_n\}_{n=1}^{\infty}$ is a bounded sequence of real numbers, then there exists a real number M such that for all $n \in \mathbb{N}$, we have that $|a_n| \leq M$, and in particular, the sequence $\{|a_n|\}_{n=1}^{\infty}$ is also bounded. Indeed, suppose that x is an upper bound of $\{a_n\}_{n=1}^{\infty}$, and that y is a lower bound of $\{a_n\}_{n=1}^{\infty}$. Set $M := \max\{|x|, |y|\}$. Then for all $n \in \mathbb{N}$, we have that $-M \leq y \leq a_n \leq x \leq M$, and consequently, $|a_n| \leq M$.

Lemma 2.2.3. *Let $\{a_n\}_{n=1}^{\infty}$ be a convergent sequence of real numbers. Then we have the following:*

(a) $\{a_n\}_{n=1}^{\infty}$ is bounded;

(b) for all $m, M \in \mathbb{R}$ such that m is a lower bound and M an upper bound of $\{a_n\}_{n=1}^{\infty}$, we have that $m \leq \lim\limits_{n\to\infty} a_n \leq M$.

Proof. (a) Fix an arbitrary real number $\varepsilon > 0$,[5] and using the fact that $L = \lim\limits_{n\to\infty} a_n$, choose $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N$, then $|a_n - L| < \varepsilon$. Now, set

$$m := \min\{a_1, \dots, a_N, L - \varepsilon\} \quad \text{and} \quad M := \max\{a_1, \dots, a_N, L + \varepsilon\}.$$

We claim that m and M are a lower and upper bound, respectively, of the sequence $\{a_n\}_{n=1}^{\infty}$. Fix $n \in \mathbb{N}$; we must show that $m \leq a_n \leq M$.

If $n \leq N$, then by construction, we have that

$$m \;\leq\; \min\{a_1, \dots, a_N\} \;\leq\; a_n \;\leq\; \max\{a_1, \dots, a_N\} \;\leq\; M,$$

and we are done.

From now on, we assume that $n > N$. Then $|a_n - L| < \varepsilon$, i.e. $a_n \in (L - \varepsilon, L + \varepsilon)$. But now

$$m \;\leq\; L - \varepsilon \;\leq\; a_n \;\leq\; L + \varepsilon \;\leq\; M,$$

and again we are done.

(b) Let $m, M \in \mathbb{R}$ be such that m is a lower bound and M an upper bound of $\{a_n\}_{n=1}^{\infty}$. Set $L = \lim\limits_{n\to\infty} a_n$. We must show that $m \leq L \leq M$.

We first show that $L \leq M$. Suppose toward a contradiction that $L > M$. Set $\varepsilon_M = L - M$. Fix $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N$, then $|a_n - L| < \varepsilon_M$. Now, fix $n \in \mathbb{N}$ such that $n \geq N$. Then

$$\varepsilon_M \;>\; |a_n - L| \;\geq\; L - a_n \;=\; (\underbrace{L - M}_{=\varepsilon_M}) + (\underbrace{M - a_n}_{\geq 0}) \;\geq\; \varepsilon_M.$$

[5] For the proof of (a), any $\varepsilon > 0$ will do. We could, for example, choose $\varepsilon = 1$.

We have now proven that $\varepsilon_M > \varepsilon_M$, a contradiction. So, $L \leq M$.

The proof that $m \leq L$ is similar. Indeed, suppose that $L < m$, and set $\varepsilon_m = m - L$. Fix $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N$, then $|a_n - L| < \varepsilon_m$. Now, fix $n \in \mathbb{N}$ such that $n \geq N$. Then

$$\varepsilon_m \quad > \quad |a_n - L| \quad \geq \quad a_n - L \quad = \quad \underbrace{(a_n - m)}_{\geq 0} + \underbrace{(m - L)}_{\varepsilon_m} \quad \geq \quad \varepsilon_m.$$

We have now proven that $\varepsilon_m > \varepsilon_m$, a contradiction. So, $m \leq L$. □

Lemma 2.2.4. *For all $c \in \mathbb{R}$, the constant sequence $c, c, c, \dots$ converges, and $\lim\limits_{n\to\infty} c = c$.*

Proof. This is "obvious," but here's a formal proof. Fix $\varepsilon > 0$, and set $N = 1$. Fix $n \in \mathbb{N}$ such that $n \geq N$. Then $|c - c| = 0 < \varepsilon$. So, $\lim\limits_{n\to\infty} c = c$. □

Theorem 2.2.5. *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be convergent sequences of real numbers, and let $c \in \mathbb{R}$. Then all the following hold:*

(a) $\lim\limits_{n\to\infty}(ca_n) = c \lim\limits_{n\to\infty}(a_n)$;

(b) $\lim\limits_{n\to\infty}(a_n + b_n) = \left(\lim\limits_{n\to\infty} a_n\right) + \left(\lim\limits_{n\to\infty} b_n\right)$;

(c) $\lim\limits_{n\to\infty}(a_n - b_n) = \left(\lim\limits_{n\to\infty} a_n\right) - \left(\lim\limits_{n\to\infty} b_n\right)$;

(d) $\lim\limits_{n\to\infty}(a_n b_n) = \left(\lim\limits_{n\to\infty} a_n\right)\left(\lim\limits_{n\to\infty} b_n\right)$;

(e) if $b_n \neq 0$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} b_n \neq 0$, then $\lim\limits_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$.

Proof. For notational convenience, set

$$a := \lim_{n\to\infty} a_n \qquad \text{and} \qquad b := \lim_{n\to\infty} b_n.$$

We must prove:

(a) $\lim\limits_{n\to\infty}(ca_n) = ca$;

(b) $\lim\limits_{n\to\infty}(a_n + b_n) = a + b$;

(c) $\lim\limits_{n\to\infty}(a_n - b_n) = a - b$;

(d) $\lim\limits_{n\to\infty}(a_n b_n) = ab$;

(e) if $b_n \neq 0$ for all $n \in \mathbb{N}$ and $b \neq 0$, then $\lim\limits_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{a}{b}$.

(a) If $c = 0$, then the result follows from Lemma 2.2.4. So assume that $c \neq 0$. Fix $\varepsilon > 0$. Using the fact that $\lim\limits_{n\to\infty} a_n = a$, we fix $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N$, then $|a_n - a| < \frac{\varepsilon}{|c|}$. Now, fix $n \in \mathbb{N}$ such that $n \geq N$. Then

$$|ca_n - ca| \quad = \quad |c||a_n - a| \quad < \quad |c|\tfrac{\varepsilon}{|c|} \quad = \quad \varepsilon.$$

This proves that $\lim\limits_{n\to\infty}(ca_n) = ca$.

(b) Fix $\varepsilon > 0$. Using the fact that $\lim\limits_{n\to\infty} a_n = a$, we choose $N_1 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N_1$, then $|a_n - a| < \frac{\varepsilon}{2}$. Similarly, using the fact that $\lim\limits_{n\to\infty} b_n = b$, we choose $N_2 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N_2$, then $|b_n - b| < \frac{\varepsilon}{2}$. Set $N := \max\{N_1, N_2\}$, and fix $n \in \mathbb{N}$ such that $n \geq N$. Then $n \geq N_1$ and $n \geq N_2$; the former implies that $|a_n - a| < \frac{\varepsilon}{2}$, whereas the later implies that $|b_n - b| < \frac{\varepsilon}{2}$. We now compute:

$$\begin{array}{rcll} |(a_n + b_n) - (a + b)| & = & |(a_n - a) + (b_n - b)| & \\ & \leq & |a_n - a| + |b_n - b| & \text{by the Triangle Inequality} \\ & < & \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad = \quad \varepsilon, & \end{array}$$

and so $\lim\limits_{n\to\infty}(a_n + b_n) = a + b$.

(c) By (a), $\lim\limits_{n\to\infty}(-b_n) = -b$, and so the result follows from (b).[6]

(d) By Lemma 2.2.3(a), since $\{a_n\}_{n=1}^{\infty}$ is convergent, it is bounded. Fix a real number $A > 0$ such that for all $n \in \mathbb{N}$, we have that $|a_n| \leq A$.[7] Now, fix $\varepsilon > 0$. Using the fact that $\lim\limits_{n\to\infty} a_n = a$, we fix $N_1 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ satisfying $n \geq N_1$, we have that $|a_n - a| < \frac{\varepsilon}{2(|b|+1)}$. Further, using the fact that $\lim\limits_{n\to\infty} b_n = b$, we fix $N_2 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ satisfying $n \geq N_2$, we have that $|b_n - b| < \frac{\varepsilon}{2A}$. Set $N := \max\{N_1, N_2\}$, and fix $n \in \mathbb{N}$ such that $n \geq N$. Note that this means that $n \geq N_1$ and $n \geq N_2$; the former implies that $|a_n - a| < \frac{\varepsilon}{2(|b|+1)}$, whereas the latter implies that $|b_n - b| < \frac{\varepsilon}{2A}$. We then have the following:

$$\begin{array}{rcll} |a_nb_n - ab| & = & |(a_nb_n - a_nb) + (a_nb - ab)| & \\ & \leq & |a_nb_n - a_nb| + |a_nb - ab| & \text{by the Triangle Inequality} \end{array}$$

[6]Indeed, we simply apply (b) to the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{-b_n\}_{n=1}^{\infty}$.

[7]To see that such an A exists, fix a lower bound m and an upper bound M of the sequence $\{a_n\}_{n=1}^{\infty}$, and let $A = \max\{|m|, |M|, 1\}$. (Note: we cannot take $A = \max\{|m|, |M|\}$ because it is possible that $m = M = 0$, and we need that $A > 0$.)

$$\begin{aligned} &= |a_n||b_n - b| + |a_n - a||b| \\ &\leq A|b_n - b| + |a_n - a||b| \\ &< A\tfrac{\varepsilon}{2A} + \tfrac{\varepsilon}{2(|b|+1)}|b| \quad \leq \quad \varepsilon. \end{aligned}$$

Thus, $\lim\limits_{n\to\infty}(a_n b_n) = ab$.

(e) Suppose that $b_n \neq 0$ for all $n \in \mathbb{N}$ and $b \neq 0$. Let us first show that $\lim\limits_{n\to\infty}\left(\frac{1}{b_n}\right) = \frac{1}{b}$. We first prove the following.

> ***Claim.*** *There exist some $N_1 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N_1$, then $|b_n| > \frac{|b|}{2}$.*

Proof of Claim. Set $\varepsilon = \frac{|b|}{2}$. Let $N_1 \in \mathbb{N}$ be such that for all $n \in \mathbb{N}$, if $n \geq N_1$, then $|b_n - b| < \varepsilon$ (i.e. $|b_n - b| < \frac{|b|}{2}$). Now, fix $n \in \mathbb{N}$ such that $n \geq N_1$. We claim that $|b_n| > \frac{|b|}{2}$. Suppose otherwise, i.e. suppose that $|b_n| \leq \frac{|b|}{2}$. Then $\frac{|b|}{2} \leq |b| - |b_n|$. But now we have the following:

$$\tfrac{|b|}{2} \quad \leq \quad |b| - |b_n| \quad \overset{(*)}{\leq} \quad |b - b_n| \quad = \quad |b_n - b| \quad < \quad \tfrac{|b|}{2},$$

where (*) follows from Corollary 1.3.1. Thus, $\frac{|b|}{2} < \frac{|b|}{2}$, a contradiction. This proves the Claim. ♦

Let N_1 be as in the Claim. Now, fix $\varepsilon > 0$, and choose $N_2 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N_2$, then $|b_n - b| < \frac{b^2}{2}\varepsilon$. Set $N = \max\{N_1, N_2\}$. Then for all $n \in \mathbb{N}$ such that $n \geq N$, we have the following:

$$\begin{aligned} |\tfrac{1}{b_n} - \tfrac{1}{b}| &= |\tfrac{b - b_n}{b_n b}| \\ &= \tfrac{|b_n - b|}{|b||b_n|} \\ &\leq \tfrac{|b_n - b|}{|b|\cdot\frac{|b|}{2}} && \text{because } n \geq N_1\text{, and so } |b_n| > \tfrac{|b|}{2} \\ &= \tfrac{2}{b^2}|b_n - b| \\ &< \tfrac{2}{b^2}\cdot\tfrac{b^2}{2}\varepsilon && \text{because } n \geq N_2\text{, and so } |b_n - b| < \tfrac{b^2}{2}\varepsilon \\ &= \varepsilon. \end{aligned}$$

Thus, $\lim_{n\to\infty}\left(\frac{1}{b_n}\right)=\frac{1}{b}$, which together with (d) implies that $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)=\frac{a}{b}$.[8] □

Remark: Recall that by Example 2.1.1, we have that $\lim_{n\to\infty}\left(\frac{1}{n}\right)=0$. Combined with Theorem 2.2.5, this readily yields the following proposition.

Proposition 2.2.6. *For all real numbers a and positive integers p, we have that*

$$\lim_{n\to\infty}\left(\frac{a}{n^p}\right) \quad = \quad 0.$$

Proof. We compute:

$$\begin{array}{rcll}
\lim_{n\to\infty}\left(\frac{a}{n^p}\right) & = & \lim_{n\to\infty}\left(a(\frac{1}{n})^p\right) & \\
& = & a\left(\lim_{n\to\infty}(\frac{1}{n})^p\right) & \text{by Theorem 2.2.5(a)} \\
& = & a\left(\lim_{n\to\infty}\frac{1}{n}\right)^p & \text{by repeated applicaton of Theorem 2.2.5(d)} \\
& = & a0^p & \text{by Example 2.1.1} \\
& = & 0. &
\end{array}$$

□

Example 2.2.7. *Compute* $\lim_{n\to\infty}\frac{4n^3+2n^2-n+7}{3n^3-3n^2-12}$.

Solution. Here, we have a rational expression, where both the numerator and the denominator have the same degree. The trick is to divide both the numerator and the denominator by n^3 (the highest degree term of both), and then compute. A fully formal proof using Theorem 2.2.5 and Proposition 2.2.6 would look like this:

$$\begin{array}{rcl}
\lim_{n\to\infty}\frac{4n^3+2n^2-n+7}{3n^3-3n^2-12} & \overset{(*)}{=} & \lim_{n\to\infty}\frac{4+\frac{2}{n}-\frac{1}{n^2}+\frac{7}{n^3}}{3-\frac{3}{n}-\frac{12}{n^3}} \\
& = & \frac{\lim_{n\to\infty}\left(4+\frac{2}{n}-\frac{1}{n^2}+\frac{7}{n^3}\right)}{\lim_{n\to\infty}\left(3-\frac{3}{n}-\frac{12}{n^3}\right)} \\
& = & \frac{\lim_{n\to\infty}(4)+\lim_{n\to\infty}(\frac{2}{n})-\lim_{n\to\infty}(\frac{1}{n^2})+\lim_{n\to\infty}(\frac{7}{n^3})}{\lim_{n\to\infty}(3)-\lim_{n\to\infty}(\frac{3}{n})-\lim_{n\to\infty}(\frac{12}{n^3})} \\
& = & \frac{4+0-0+0}{3-0-0} \\
& = & \frac{4}{3},
\end{array}$$

[8] Indeed, we simply apply (d) to the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{\frac{1}{b_n}\}_{n=1}^{\infty}$.

where (*) was obtained by diving both the numerator and the denominator by n^3. However, we do not normally include this level of detail in our computation! Typically, we would evaluate the limit this way:

$$\lim_{n\to\infty} \tfrac{4n^3+2n^2-n+7}{3n^3-3n^2-12} \overset{(*)}{=} \lim_{n\to\infty} \frac{4 + \overset{\nearrow 0}{\boxed{\frac{2}{n}}} - \overset{\nearrow 0}{\boxed{\frac{1}{n^2}}} + \overset{\nearrow 0}{\boxed{\frac{7}{n^3}}}}{3 - \underset{\searrow 0}{\boxed{\frac{3}{n}}} - \underset{\searrow 0}{\boxed{\frac{12}{n^3}}}} = \tfrac{4}{3},$$

where (*) was obtained by dividing both the numerator and the denominator by n^3. (Here, arrows indicate what limit the circled expressions converge to as $n \to \infty$.) □

2.3 The Monotone Sequence Theorem

A sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers is

- *non-decreasing* if for all $n \in \mathbb{N}$, we have that $a_n \leq a_{n+1}$;
- *strictly increasing* (or simply *increasing*) if for all $n \in \mathbb{N}$, we have that $a_n < a_{n+1}$;
- *non-increasing* if for all $n \in \mathbb{N}$, we have that $a_n \geq a_{n+1}$;
- *strictly decreasing* (or simply *decreasing*) if for all $n \in \mathbb{N}$, we have that $a_n > a_{n+1}$;
- *monotone* if it is either non-decreasing or non-increasing.

Lemma 2.3.1. *Let $\{a_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of real numbers bounded above. Then $\{a_n\}_{n=1}^{\infty}$ converges, and its limit is precisely the supremum of the sequence.*[9]

Proof. Let S be the supremum of the sequence $\{a_n\}_{n=1}^{\infty}$. We must show that $\lim\limits_{n\to\infty} a_n = S$. Fix an arbitrary $\varepsilon > 0$, and fix some $N \in \mathbb{N}$ such that $a_N > S - \varepsilon$; such an N exists because otherwise, $S - \varepsilon$ would be an upper bound of $\{a_n\}_{n=1}^{\infty}$, contrary to the fact that S is the supremum of the sequence. Now, fix $n \in \mathbb{N}$ such that $n \geq N$. Then

$$S - \varepsilon \;<\; a_N \;\overset{(*)}{\leq}\; a_n \;\overset{(**)}{\leq}\; S,$$

where (*) follows from the fact that our sequence is non-decreasing, wheres (**) follows from the fact that S is the supremum of the sequence. Thus, $-\varepsilon < a_n - S \leq 0$, and consequently, $|a_n - S| < \varepsilon$. This completes the argument. □

[9]Note that we are using the completeness of the field $\mathbb{R}$, and in particular, the fact that every non-empty subset of $\mathbb{R}$ (and therefore, every sequence in $\mathbb{R}$) bounded above has a (unique) supremum, i.e. the least upper bound. This theorem fails for e.g. $\mathbb{Q}$, since $\mathbb{Q}$ is not complete.

Lemma 2.3.2. *Let* $\{a_n\}_{n=1}^{\infty}$ *be a non-increasing sequence of real numbers bounded below. Then* $\{a_n\}_{n=1}^{\infty}$ *converges, and its limit is precisely the infimum of the sequence.*[10]

Proof. Exercise (similar to the proof of Lemma 2.3.1). □

The Monotone Sequence Theorem. *Every monotone and bounded sequence of real numbers is convergent.*

Proof. This follows immediately from Lemmas 2.3.1 and 2.3.2. □

Remark: The Monotone Sequence Theorem is often used to prove that a recursively defined sequence converges, and we give an example below (see Example 2.3.4). First, we need one more limit law, which we state without proof.

Lemma 2.3.3. *Let* $\{a_n\}_{n=1}^{\infty}$ *be a convergent sequence, and let* $p \in \mathbb{R}$. *Then*

$$\lim_{n\to\infty}(a_n^p) \quad = \quad \big(\lim_{n\to\infty} a_n\big)^p,$$

as long as both the sequence $\{a_n^p\}_{n=1}^{\infty}$ *and the number* $\big(\lim_{n\to\infty} a_n\big)^p$ *are both defined.*

Remark: If the real number p is **not** a non-negative integer, then a_n^p may possibly be undefined for some value(s) of $n \in \mathbb{N}$, in which case, the sequence $\{a_n^p\}_{n=1}^{\infty}$ is undefined. On the other hand, since the sequence $\{a_n\}_{n=1}^{\infty}$ is convergent, we do know that $\lim_{n\to\infty} a_n$ is defined; however, depending on the value of $\lim_{n\to\infty} a_n$ and p, it is possible that $\big(\lim_{n\to\infty} a_n\big)^p$ is undefined. For instance, if $\lim_{n\to\infty} a_n = 0$ and $p < 0$, then $\big(\lim_{n\to\infty} a_n\big)^p$ is undefined.

Proof. Omitted. □

Example 2.3.4. *Let* $\{a_n\}_{n=1}^{\infty}$ *be the sequence defined recursively as follows:*

- $a_1 = \sqrt{2}$;
- $a_{n+1} = \sqrt{2a_n}$ *for all* $n \in \mathbb{N}$.

Show that $\{a_n\}_{n=1}^{\infty}$ *converges, and find its limit.*

Proof. We first use the Monotone Sequence Theorem to prove that the sequence $\{a_n\}_{n=1}^{\infty}$ converges (i.e. has a limit), and then we compute that limit.

Proving convergence. We first prove the sequence $\{a_n\}_{n=1}^{\infty}$ is monotone and bounded. More precisely, we will show that for all $n \in \mathbb{N}$, we have that $\sqrt{2} \leq a_n \leq 2$ and $a_n \leq a_{n+1}$. We proceed by induction on n.

[10] Once again, we are using the completeness of $\mathbb{R}$, and in particular, the fact that every non-empty subset of $\mathbb{R}$ (and therefore, every sequence in $\mathbb{R}$) bounded below has a (unique) infimum, i.e. the greatest lower bound. This theorem fails for e.g. $\mathbb{Q}$, since $\mathbb{Q}$ is not complete.

For $n = 1$, we simply observe that $\sqrt{2} = a_1 \leq 2$ and that $a_1 = \sqrt{2} \leq \sqrt{2\sqrt{2}} = \sqrt{2a_1} = a_2$.

Now, fix $n \in \mathbb{N}$, and suppose inductively that $\sqrt{2} \leq a_n \leq 2$ and $a_n \leq a_{n+1}$. We must show that $\sqrt{2} \leq a_{n+1} \leq 2$ and $a_{n+1} \leq a_{n+2}$. First, since $\sqrt{2} \leq a_n \leq 2$, we have that $2\sqrt{2} \leq 2a_n \leq 4$, and so $\sqrt{2\sqrt{2}} \leq \sqrt{2a_n} \leq 2$. Since $\sqrt{2} \leq \sqrt{2\sqrt{2}}$ and $a_{n+1} = \sqrt{2a_n}$, we deduce that $\sqrt{2} \leq a_{n+1} \leq 2$. On the other hand, since $0 \leq a_{n+1} \leq 2$, we have that $a_{n+1} = \sqrt{a_{n+1}^2} \leq \sqrt{2a_{n+1}} = a_{n+2}$. This completes the induction.

We have now shown that $\{a_n\}_{n=1}^{\infty}$ is monotone and bounded, and so by the Monotone Sequence Theorem, the sequence converges.

Computing the limit. Set $a = \lim\limits_{n\to\infty} a_n$. (The existence of the limit follows from the convergence of the sequence $\{a_n\}_{n=1}^{\infty}$, proven above.) Then

$$a \;=\; \lim_{n\to\infty} a_{n+1} \;=\; \lim_{n\to\infty} \sqrt{2a_n} \;=\; \sqrt{2 \lim_{n\to\infty} a_n} \;=\; \sqrt{2a}.$$

So, $a^2 = 2a$, which implies that $a = 0$ or $a = 2$. However, since $\{a_n\}_{n=1}^{\infty}$ is bounded below by $\sqrt{2}$, we know that $\sqrt{2} \leq a$,[11] and in particular, $a \neq 0$. So, $a = 2$.

Conclusion. We have shown that $\{a_n\}_{n=1}^{\infty}$ converges and satisfies

$$\lim_{n\to\infty} a_n \;=\; 2,$$

and we are done. □

Remark: When solving problems similar to Example 2.3.4, **it is imperative that you prove that the limit actually exists**, i.e. that the sequence in question does converge. It is **not** enough to simply compute the limit. Let us explain why. Suppose we are given a sequence $\{a_n\}_{n=1}^{\infty}$, defined recursively as follows:

- $a_1 = -1$;
- $a_{n+1} = -a_n$ for all $n \in \mathbb{N}$.

Suppose we simply set $a := \lim\limits_{n\to\infty} a_n$ and then try to evaluate a. We get:

$$a \;=\; \lim_{n\to\infty} a_{n+1} \;=\; \lim_{n\to\infty} (-a_n) \;=\; -\lim_{n\to\infty} a_n \;=\; -a.$$

Thus, $a = -a$, and consequently, $2a = 0$, i.e. $a = 0$. But this doesn't work! Indeed, our sequence satisfies $a_n = (-1)^n$ for all $n \in \mathbb{N}$, and therefore does not converge (by Example 2.1.5). What our computation actually showed is that **if** $\lim\limits_{n\to\infty} a_n$ exists, **then** $\lim\limits_{n\to\infty} a_n = 0$. It did **not** show that $\lim\limits_{n\to\infty} a_n$ does in fact exist. As a matter of fact, the limit does **not** exist.

[11] Here, we are using Lemma 2.2.3(b).

2.3.1 Euler's number

You may recall from high school that *Euler's number* e is an (irrational) real number that satisfies $e \approx 2.71828$. Formally, Euler's number is defined as follows:

$$e \;:=\; \lim_{n\to\infty} \left(1+\tfrac{1}{n}\right)^n.$$

This immediately raises the following question: how do we even know that this limit exists? To prove that it does, we will use the Monotone Sequence Theorem and Bernoulli's inequality (see section 1.4), as follows.

First, we consider the auxiliary sequence $\{a_n\}_{n=1}^{\infty}$ given by

$$a_n \;:=\; \left(1+\tfrac{1}{n}\right)^{n+1}$$

for all $n \in \mathbb{N}$. We will show that $\{a_n\}_{n=1}^{\infty}$ is non-increasing and bounded below, and is consequently convergent (by the Monotone Sequence Theorem).[12]

It is clear that $a_n > 0$ for all $n \in \mathbb{N}$, and in particular, the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded below. Next, for all $n \in \mathbb{N}$, we have that

$$\begin{aligned}
\frac{a_n}{a_{n+1}} &= \frac{(1+\frac{1}{n})^{n+1}}{(1+\frac{1}{n+1})^{n+2}} \\
&= \left(\frac{1+\frac{1}{n}}{1+\frac{1}{n+1}}\right)^{n+2} \cdot \frac{1}{1+\frac{1}{n}} \\
&= \left(\frac{n^2+2n+1}{n^2+2n}\right)^{n+2} \cdot \frac{1}{1+\frac{1}{n}} \\
&= \left(1+\frac{1}{n^2+2n}\right)^{n+2} \cdot \frac{1}{1+\frac{1}{n}} \\
&\geq \left(1+(n+2)\cdot\frac{1}{n^2+2n}\right) \cdot \frac{1}{1+\frac{1}{n}} \qquad \text{by Bernoulli's inequality} \\
&= \left(1+\tfrac{1}{n}\right) \cdot \frac{1}{1+\frac{1}{n}} \\
&= 1,
\end{aligned}$$

and consequently (since $a_{n+1} > 0$), that $a_n \geq a_{n+1}$. So, the sequence $\{a_n\}_{n=1}^{\infty}$ is non-increasing. We have now shown that $\{a_n\}_{n=1}^{\infty}$ is monotone and bounded, and consequently, it is convergent (by the Monotone Sequence Theorem). In other words, $\lim\limits_{n\to\infty} a_n$ exists.

[12]Technically, we are using Lemma 2.3.2. However, note also that any non-increasing sequence is automatically bounded above (for example, by the first term of the sequence). So, any sequence that is non-increasing and bounded below, is in particular monotone and bounded, and consequently convergent (by the Monotone Sequence Theorem).

Finally, we compute:

$$\lim_{n\to\infty}\left(1+\tfrac{1}{n}\right)^n \;=\; \lim_{n\to\infty}\frac{a_n}{1+\frac{1}{n}} \;=\; \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty}(1+\frac{1}{n})} \;=\; \frac{\lim\limits_{n\to\infty} a_n}{1} \;=\; \lim_{n\to\infty} a_n.$$

This proves that the sequence $\left\{(1+\frac{1}{n})^n\right\}_{n=1}^{\infty}$ is in fact convergent, and that Euler's number

$$e \;:=\; \lim_{n\to\infty}\left(1+\tfrac{1}{n}\right)^n$$

is indeed well defined.

Remark: Note that for all $n \in \mathbb{N}$, we have that

$$(1+\tfrac{1}{n})^n \;\overset{(*)}{\geq}\; 1+n\tfrac{1}{n} \;=\; 2,$$

where (*) follows from Bernoulli's inequality. So, 2 is a lower bound of the sequence $\left\{(1+\frac{1}{n})^n\right\}_{n=1}^{\infty}$. So, by Lemma 2.2.3(b), we have that $e = \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n \geq 2$. To obtain an upper bound for e, we recall that $e = \lim\limits_{n\to\infty} a_n$, and that the sequence $\{a_n\}_{n=1}^{\infty}$ is non-increasing and is therefore bounded above by a_1. Therefore,

$$e \;=\; \lim_{n\to\infty} a_n \;\overset{(*)}{\leq}\; a_1 \;=\; 4,$$

where (*) follows from Lemma 2.2.3(b). Thus, we have obtained the following estimate for Euler's number e:

$$2 \;\leq\; e \;\leq\; 4.$$

By using numerical methods, it is possible to obtain an ever more precise estimate of e. For example, it is known that $e \approx 2.71828$.

2.4 The squeeze theorem for sequences

The Squeeze Theorem for sequences. *Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, and $\{c_n\}_{n=1}^{\infty}$ be sequences of real numbers such that $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$. Assume that $\{a_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ converge to the same limit L. Then $\{b_n\}_{n=1}^{\infty}$ also converges to L.*

Remark: Schematically (and informally), we can represent the Squeeze Theorem like this:

$$\underset{\searrow L}{\boxed{a_n}} \;\leq\; b_n \;\leq\; \underset{\searrow L}{\boxed{c_n}} \quad \forall n \in \mathbb{N} \qquad \Longrightarrow \qquad \lim_{n\to\infty} b_n \;=\; L$$

Proof. Fix $\varepsilon > 0$. Using the fact that $\lim\limits_{n\to\infty} a_n = L$, we fix some $N_1 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N_1$, then $|a_n - L| < \varepsilon$. Similarly, using the fact that $\lim\limits_{n\to\infty} c_n = L$, we fix some $N_2 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N_2$, then $|c_n - L| < \varepsilon$. Let $N = \max\{N_1, N_2\}$. Fix $n \in \mathbb{N}$ such that $n \geq N$. Then $n \geq N_1$, and it follows that $|a_n - L| < \varepsilon$, and so $-\varepsilon < a_n - L < \varepsilon$. Similarly, $n \geq N_2$, and it follows that $|c_n - L| < \varepsilon$, and so $-\varepsilon < c_n - L < \varepsilon$. Since $a_n \leq b_n \leq c_n$, we have that

$$-\varepsilon \;<\; a_n - L \;\leq\; b_n - L \;\leq\; c_n - L \;<\; \varepsilon,$$

and we deduce that $|b_n - L| < \varepsilon$. This proves that $\lim\limits_{n\to\infty} b_n = L$, as we had claimed. □

Example 2.4.1. *Compute* $\lim\limits_{n\to\infty} \big((-1)^n \frac{1}{n}\big)$*, or prove that the limit does not exist.*

Solution. Clearly, we have that

$$-\tfrac{1}{n} \;\leq\; (-1)^n \tfrac{1}{n} \;\leq\; \tfrac{1}{n}$$

for all $n \in \mathbb{N}$. Since

- $\lim\limits_{n\to\infty} \big(\frac{1}{n}\big) = 0$,
- $\lim\limits_{n\to\infty} \big(-\frac{1}{n}\big) = -\lim\limits_{n\to\infty} \big(\frac{1}{n}\big) = -0 = 0$,

the Squeeze theorem implies that

$$\lim_{n\to} \big((-1)^n \tfrac{1}{n}\big) \;=\; 0.$$

Remark: The argument above was fully formal. In practice, though, it is enough to write something like this:

$$\underset{\searrow 0}{\boxed{-\tfrac{1}{n}}} \;\leq\; (-1)^n \tfrac{1}{n} \;\leq\; \underset{\searrow 0}{\boxed{\tfrac{1}{n}}} \quad \forall n \in \mathbb{N} \quad \overset{(*)}{\Longrightarrow} \quad \lim_{n\to\infty} \big((-1)^n \tfrac{1}{n}\big) \;=\; 0,$$

where (*) follows from the Squeeze Theorem. □

Remark: Recall that the first few terms of a sequence have no effect on the existence or value of the limit (formally, we have Lemma 2.2.2). This allows us to state a slightly stronger version of the Squeeze Theorem, as follows. (Try to give a formal proof by yourself, using the original Squeeze Theorem, plus Lemma 2.2.2.)

The Squeeze Theorem for sequences (stronger version). *Let* $\{a_n\}_{n=1}^{\infty}$*,* $\{b_n\}_{n=1}^{\infty}$*, and* $\{c_n\}_{n=1}^{\infty}$ *be sequences of real numbers. Assume that there exists an integer* $N \in \mathbb{N}$ *such that for all* $n \in \mathbb{N}$ *satisfying* $n \geq N$*, we have that* $a_n \leq b_n \leq c_n$*. Assume that* $\{a_n\}_{n=1}^{\infty}$ *and* $\{c_n\}_{n=1}^{\infty}$ *converge to the same limit* L*. Then* $\{b_n\}_{n=1}^{\infty}$ *also converges to* L*.*

Remark: A schematic (and informal) representation of the theorem above looks like this:

$$\underset{\searrow L}{\boxed{a_n}} \quad \leq \quad b_n \quad \leq \quad \underset{\searrow L}{\boxed{c_n}} \quad \forall n \geq N \qquad \Longrightarrow \qquad \lim_{n\to\infty} b_n \quad = \quad L$$

2.4.1 Some limits with roots

Proposition 2.4.2. $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$.

Proof. Clearly, for all $n \in \mathbb{N}$, we have that $\sqrt[n]{n} > 1$, and consequently,

$$r_n \quad := \quad \sqrt[n]{n} - 1 \quad > \quad 0.$$

Our goal is to show that $\lim\limits_{n\to\infty} r_n = 0$. This is enough, for it will imply that

$$\lim_{n\to\infty} \sqrt[n]{n} \quad = \quad \lim_{n\to\infty} (1 + r_n) \quad = \quad 1 + \lim_{n\to\infty} r_n \quad = \quad 1 + 0 \quad = \quad 1,$$

which is what we need.

> **Claim.** $0 < r_n < \sqrt{\frac{2}{n-1}}$ for all integers $n \geq 2$.

Proof of the Claim. Fix an integer $n \geq 2$. We already saw that $r_n > 0$. For the other inequality, we observe that

$$n \quad \overset{(*)}{=} \quad (r_n + 1)^n \quad \overset{(**)}{=} \quad \sum_{k=0}^{n} \binom{n}{k} r_n^k \quad \overset{(***)}{>} \quad \binom{n}{2} r_n^2 \quad = \quad \frac{n(n-1)}{2} r_n^2,$$

and consequently $r_n < \sqrt{\frac{2}{n-1}}$, where

- (*) follows from the definition of r_n;
- (**) follows from the Binomial Theorem;
- (***) follows from the fact that each summand in the sum $\sum\limits_{k=0}^{n} \binom{n}{k} r_n^k$ is strictly positive (because $r_n > 0$), and so each summand is strictly smaller than the whole sum (because there is more than one summand).[13]

This proves the Claim. ♦

[13] Note that there are $n + 1 \geq 3$ summands, and $\binom{n}{2} r_n^2$ is the third summand (the one we get for $k = 2$.

Clearly, $\lim\limits_{n\to\infty} \sqrt{\frac{2}{n-1}} = 0$. So, by the Claim and the Squeeze Theorem, we have that $\lim\limits_{n\to\infty} r_n = 0$, and we are done.[14] $\square$

Proposition 2.4.3. *For all real numbers $a > 0$, we have that $\lim\limits_{n\to\infty} \sqrt[n]{a} = 1$.*

Remark: We could prove Proposition 2.4.3 similarly to the way we proved Proposition 2.4.2.[15] However, since we have already proven Proposition 2.4.2, we can simply use it, together with the Squeeze Theorem, to prove Proposition 2.4.3.

Proof. Fix a real number $a > 0$. We consider two cases: when $a \geq 1$, and when $0 < a < 1$.

Case 1: $a \geq 1$. Fix $N \in \mathbb{N}$ such that $a \leq N$. Then for all $n \in \mathbb{N}$ such that $n \geq N$, we have that

$$1 \quad \leq \quad \sqrt[n]{a} \quad \leq \quad \sqrt[n]{n}.$$

Since $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$ (by Proposition 2.4.2), the Squeeze Theorem guarantees that $\lim\limits_{n\to\infty} \sqrt[n]{a} = 1$.[16]

Case 2: $0 < a < 1$. Then $\frac{1}{a} > 1$, and so by Case 1, we have that $\lim\limits_{n\to\infty} \sqrt[n]{\frac{1}{a}} = 1$. We now compute:

$$\lim_{n\to\infty} \sqrt[n]{a} \quad = \quad \lim_{n\to\infty} \frac{1}{\sqrt[n]{\frac{1}{a}}} \quad = \quad \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{\frac{1}{a}}} \quad = \quad \frac{1}{1} \quad = \quad 1,$$

and we are done. $\square$

2.5 Subsequences and the Bolzano-Weierstrass Theorem

A *subsequence* of a sequence $\{a_n\}_{n=1}^{\infty}$ is a sequence of the form $\{a_{n_j}\}_{j=1}^{\infty}$, where $\{n_j\}_{j=1}^{\infty}$ is a strictly increasing sequence of positive integers.

Informally, a subsequence of a sequence $\{a_n\}_{n=1}^{\infty}$ is any sequence that can be obtained from $\{a_n\}_{n=1}^{\infty}$ by possibly deleting some terms, but so that infinitely many terms still remain. (In particular, every sequence is a subsequence of itself.)

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[14] Indeed, the Claim guarantees that for all integers $n \geq 2$, we have that

$$0 \quad < \quad r_n \quad < \quad \underbrace{\sqrt{\frac{2}{n-1}}}_{\to\, 0} \quad ,$$

and so by the Squeeze Theorem, $\lim\limits_{n\to\infty} r_n = 0$. (Here, we are using our "stronger version" of the Squeeze Theorem, for $N = 2$.)

[15] Try it!

[16] Note that we are using our "stronger version" of the Squeeze Theorem.

Proposition 2.5.1. *For all sequences $\{a_n\}_{n=1}^{\infty}$ of real numbers, the following hold:*

(a) $\{a_n\}_{n=1}^{\infty}$ converges if and only if all subsequences of $\{a_n\}_{n=1}^{\infty}$ converge;

(b) if $\{a_n\}_{n=1}^{\infty}$ converges, then all subsequences of $\{a_n\}_{n=1}^{\infty}$ converge to $\lim\limits_{n\to\infty} a_n$.

Proof. By definition, $\{a_n\}_{n=1}^{\infty}$ is a subsequence of itself. Consequently, if all subsequences of $\{a_n\}_{n=1}^{\infty}$ converge, then in particular, $\{a_n\}_{n=1}^{\infty}$ also converges.[17]

Suppose now that $\{a_n\}_{n=1}^{\infty}$ converges, and set $L := \lim\limits_{n\to\infty} a_n$. We will show that all subsequences of $\{a_n\}_{n=1}^{\infty}$ converge to L.[18] Fix any subsequence $\{a_{n_j}\}_{j=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$, where $\{n_j\}_{j=1}^{\infty}$ is a strictly increasing sequence of positive integers. We must show that $\lim\limits_{j\to\infty} a_{n_j} = L$. Fix any $\varepsilon > 0$, and using the fact that $L := \lim\limits_{n\to\infty} a_n$, fix $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N$, then $|a_n - L| < \varepsilon$. Since $\{n_j\}_{j=1}^{\infty}$ is a strictly increasing sequence of positive integers, we see that for all $j \in \mathbb{N}$ such that $j \geq N$, we have that $n_j \geq n_N \geq N$, and consequently, $|a_{n_j} - L| < \varepsilon$. This proves that $L = \lim\limits_{j\to\infty} a_{n_j}$. □

Lemma 2.5.2. *Every sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers has a monotone subsequence.*

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Let us say that a positive integer m is a *peak* of the sequence $\{a_n\}_{n=1}^{\infty}$ if for all $n \in \mathbb{N}$ such that $n > m$, we have that $a_m > a_n$ (i.e. a_m is strictly greater than every subsequent term of the sequence).

Suppose first that $\{a_n\}_{n=1}^{\infty}$ has infinitely many peaks, and let $\{n_j\}_{j=1}^{\infty}$ be the sequence of all the peaks, arranged in increasing order (i.e. $n_1 < n_2 < n_3 < \dots$). But now $a_{n_1} > a_{n_2} > a_{n_3} > \dots$, i.e. $\{a_{n_j}\}_{j=1}^{\infty}$ is a strictly decreasing (and therefore monotone) subsequence of $\{a_n\}_{n=1}^{\infty}$.

From now on, we assume that $\{a_n\}_{n=1}^{\infty}$ has only finitely many (if any) peaks. Fix some $N \in \mathbb{N}$ such that all the peaks of $\{a_n\}_{n=1}^{\infty}$ are strictly smaller than N.[19] Note that this means that for all $m \in \mathbb{N}$ such that $m \geq N$, the integer m is **not** a peak, i.e. there exists some $n \in \mathbb{N}$ such that $m < n$ and $a_m \leq a_n$.

Our goal is now to (recursively) form an increasing sequence $\{n_j\}_{j=1}^{\infty}$ of positive integers such that for all $j \in \mathbb{N}$, we have that $a_{n_j} \leq a_{n_{j+1}}$. Note that once we have formed such a sequence $\{n_j\}_{j=1}^{\infty}$, we will have that $\{a_{n_j}\}_{j=1}^{\infty}$ is a non-decreasing (and therefore monotone) subsequence of $\{a_n\}_{n=1}^{\infty}$, which is what we need. First, fix $n_1 = N$. Next, fix $j \in \mathbb{N}$, and suppose we have constructed positive integers $n_1, \dots, n_j$ so that $n_1 < \dots < n_j$ and $a_{n_1} \leq \dots \leq a_{n_j}$. Since n_j is not a peak (because

[17] Note that this proves the "$\Longleftarrow$" part of (a).

[18] This will prove the "$\Longrightarrow$" part of (a), as well as all of part (b).

[19] Let us check that such an N exists. If $\{a_n\}_{n=1}^{\infty}$ has no peaks, then we can choose N to be any positive integer (for example, $N = 1$). If $\{a_n\}_{n=1}^{\infty}$ has at least one peak, then (using the fact that $\{a_n\}_{n=1}^{\infty}$ has only finitely many peaks) we let $m_1, \dots, m_k$ be all the peaks of $\{a_n\}_{n=1}^{\infty}$, and we let N be any integer greater than $\max\{m_1, \dots, m_k\}$ (for example, $N = \max\{m_1, \dots, m_k\} + 1$).

$n_j \geq N$), there exists some $n \in \mathbb{N}$ such that $n_j < n$ and $a_{n_j} \leq a_n$; choose n_{j+1} to be the smallest such n.[20]

By construction, $\{n_j\}_{j=1}^{\infty}$ is a strictly increasing sequence of positive integers, and $\{a_{n_j}\}_{j=1}^{\infty}$ is a non-decreasing (and therefore monotone) subsequence of $\{a_n\}_{n=1}^{\infty}$. □

The Bolzano-Weierstrass Theorem. *Every bounded sequence of real numbers has a convergent subsequence.*

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers. By Lemma 2.5.2, $\{a_n\}_{n=1}^{\infty}$ has a monotone subsequence, say $\{a_{n_j}\}_{j=1}^{\infty}$, where $\{n_j\}_{j=1}^{\infty}$ is an increasing sequence of positive integers. Since $\{a_n\}_{n=1}^{\infty}$ is bounded, so is its subsequence $\{a_{n_j}\}_{j=1}^{\infty}$.[21] Now $\{a_{n_j}\}_{j=1}^{\infty}$ is a monotone and bounded sequence of real numbers, and so by the Monotone Sequence Theorem, it converges. □

2.6 Accumulation points

An *accumulation point* of a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers is a real number A such that for all real numbers $\varepsilon > 0$ and all $N \in \mathbb{N}$, there exists some $n \in \mathbb{N}$ such that $n \geq N$ and $|a_n - A| < \varepsilon$.

Proposition 2.6.1. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers, and let $A \in \mathbb{R}$. Then the following are equivalent:*

(i) A is an accumulation point of $\{a_n\}_{n=1}^{\infty}$;

(ii) for all real numbers $\varepsilon > 0$, the interval $(A - \varepsilon, A + \varepsilon)$ contains infinitely many terms of the sequence $\{a_n\}_{n=1}^{\infty}$;

(iii) some subsequence of $\{a_n\}_{n=1}^{\infty}$ converges to A.

Terminology: For a real number $\varepsilon > 0$, the open interval $(A - \varepsilon, A + \varepsilon)$ is called the *ε-neighborhood of A*.

Proof. It suffices to prove the following sequence of implications: "(i) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i)."

We first assume (i) and prove (iii). We define a strictly increasing sequence $\{n_j\}_{j=1}^{\infty}$ recursively as follows. First, using the fact that A is an accumulation point of $\{a_n\}_{n=1}^{\infty}$, we fix $n_1 \in \mathbb{N}$ such that $|a_{n_1} - A| < 1$.[22] Next, assume that for some

[20] Actually, we could have chosen n_{j+1} to be **any** such n, and the argument would still go through. It would, however, rely on the (once controversial) Axiom of Choice, which we will not discuss in any detail in this course, but which (very roughly) states that we are allowed to make infinitely many arbitrary choices. By choosing n_{j+1} to be the **minimal** n having the desired property, our choice of n_{j+1} stops being arbitrary, and so the Axiom of Choice is not needed.

[21] Indeed, any upper bound of $\{a_n\}_{n=1}^{\infty}$ is automatically an upper bound of the subsequence $\{a_{n_j}\}_{j=1}^{\infty}$, and similarly, any lower bound of $\{a_n\}_{n=1}^{\infty}$ is a lower bound of $\{a_{n_j}\}_{j=1}^{\infty}$.

[22] Here, we are using the definition of an accumulation point for $\varepsilon = 1$ and $N = 1$.

$j \in \mathbb{N}$, we have defined the positive integer n_j; we then define $n_{j+1} \in \mathbb{N}$ as follows. Since A is an accumulation point of $\{a_n\}_{n=1}^{\infty}$, we let n_{j+1} be the smallest positive integer satisfying $n_{j+1} \geq n_j + 1$ and $|A - a_{n_{j+1}}| < \frac{1}{j+1}$.[23] We have now constructed a strictly increasing sequence $\{n_j\}_{j=1}^{\infty}$ of positive integers such that for all $j \in \mathbb{N}$, we have that $|a_{n_j} - A| < \frac{1}{j}$.

It is now easy to verify that $\lim\limits_{j\to\infty} a_{n_j} = A$. Indeed, fix $\varepsilon > 0$. Let $J \in \mathbb{N}$ be such that $J > \frac{1}{\varepsilon}$. Fix $j \in \mathbb{N} \in \mathbb{N}$ such that $j \geq J$. Then

$$\begin{array}{rcll} |a_{n_j} - A| & < & \frac{1}{j} & \text{by the construction of } n_j \\ & \leq & \frac{1}{J} & \text{because } j \geq J \\ & < & \varepsilon & \text{because } J > \frac{1}{\varepsilon}. \end{array}$$

This proves that $\lim\limits_{j\to\infty} a_{n_j} = A$, i.e. (iii) holds.

Next, we assume (iii) and prove (ii). Using (iii), we fix a strictly increasing sequence $\{n_j\}_{n=1}^{\infty}$ of positive integers such that $\lim\limits_{j\to\infty} a_{n_j} = A$. Now fix $\varepsilon > 0$. Using the definition of a limit, we now fix $J \in \mathbb{N}$ such that for all $j \in \mathbb{N}$, if $j \geq J$, then $|a_{n_j} - A| < \varepsilon$, i.e. $a_{n_j} \in (A - \varepsilon, A + \varepsilon)$. But now $a_{n_J}, a_{n_{J+1}}, a_{n_{J+2}}, \dots$ all belong to $(A - \varepsilon, A + \varepsilon)$. This proves (ii).

Finally, we assume (ii) and prove (i). Fix $\varepsilon > 0$ and $N \in \mathbb{N}$. Now, using (ii), we know that there exist infinitely many positive integers n such that $a_n \in (A-\varepsilon, A+\varepsilon)$, i.e. $|a_n - A| < \varepsilon$. Since there are infinitely many such n's, one of them (in fact, infinitely many of them) must satisfy $n \geq N$. This proves (i). □

Theorem 2.6.2. *Let $\{a_n\}_{n=1}^{\infty}$ be a convergent sequence of real numbers. Then $\{a_n\}_{n=1}^{\infty}$ has exactly one accumulation point, namely $L := \lim\limits_{n\to\infty} a_n$.*

Proof. First of all, $\{a_n\}_{n=1}^{\infty}$ is a subsequence of itself, and it converges to L; so, by Proposition 2.6.1, L is indeed an accumulation point of $\{a_n\}_{n=1}^{\infty}$. On the other hand, by Proposition 2.5.1, all subsequences of $\{a_n\}_{n=1}^{\infty}$ converge to L, and so by Proposition 2.6.1, $\{a_n\}_{n=1}^{\infty}$ has no accumulation points other than L. □

Remark: By Theorem 2.6.2, if a sequence has more than one accumulation point, then it diverges. For instance, the sequence $\{(-1)^n\}_{n=1}^{\infty}$ (which we saw in Example 2.1.5) has two accumulation points, namely 1 and -1, and so it diverges. However, the converse of Theorem 2.6.2 is false in general, i.e. some sequences that only have one accumulation point nevertheless diverge. One example is the sequence $\{a_n\}_{n=1}^{\infty}$

[23] Here, we are using the definition of an accumulation point for $\varepsilon = \frac{1}{j+1}$ and $N = n_j + 1$.

given by

$$a_n \;:=\; \begin{cases} \frac{1}{n} & \text{if} \quad n \text{ is even} \\ n & \text{if} \quad n \text{ is odd} \end{cases}$$

for all $n \in \mathbb{N}$.[24] Clearly, the only accumulation point of the sequence $\{a_n\}_{n=1}^{\infty}$ is 0, and yet the sequence diverges.

2.7 Divergence to infinity

Divergence to (positive) infinity. A sequence $\{a_n\}_{n=1}^{\infty}$ *diverges to infinity*, and we write

$$\lim_{n\to\infty} a_n = \infty \qquad \text{or} \qquad \lim_{n\to\infty} a_n = +\infty,$$

or alternatively,

$$a_n \to \infty \quad \text{as} \quad n \to \infty \qquad \text{or} \qquad a_n \to +\infty \quad \text{as} \quad n \to \infty,$$

if for all $M \in \mathbb{R}$, there exists some $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N$, then $a_n > M$.

Divergence to negative infinity. A sequence $\{a_n\}_{n=1}^{\infty}$ *diverges to negative infinity*, and we write

$$\lim_{n\to\infty} a_n = -\infty,$$

or alternatively,

$$a_n \to -\infty \quad \text{as} \quad n \to \infty,$$

if for all $M \in \mathbb{R}$, there exists some $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N$, then $a_n < M$.

Example 2.7.1. *Using the definition, show that* $\lim\limits_{n\to\infty} n^2 = +\infty$.

Solution. Fix $M \in \mathbb{R}$. Let $N \in \mathbb{N}$ be such that $N > M$. Then for all $n \in \mathbb{N}$ such that $n \geq N$, we have that $n^2 \geq N^2 \geq N > M$. This proves that $\lim\limits_{n\to\infty} n^2 = +\infty$. □

Example 2.7.2. *Using the definition, show that* $\lim\limits_{n\to\infty} \big(-\sqrt{n}\big) = -\infty$.

Proof. Fix $M \in \mathbb{R}$. Fix $N \in \mathbb{N}$ such that $N > M^2$. Then $\sqrt{N} > |M| \geq -M$, and consequently, $-\sqrt{N} < M$. It follows that for all $n \in \mathbb{N}$ such that $n \geq N$, we have that $-\sqrt{n} \leq -\sqrt{N} < M$. This proves that $\lim\limits_{n\to\infty} \big(-\sqrt{n}\big) = -\infty$. □

Proposition 2.7.3. *Let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence of real numbers. Then:*

[24]This is the sequence $1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, 7, \frac{1}{8}, \dots$.

(a) if $\lim\limits_{n\to\infty} a_n = +\infty$*, then* $\{a_n\}_{n=1}^{\infty}$ *is bounded below, but is not bounded above;*

(b) if $\lim\limits_{n\to\infty} a_n = -\infty$*, then* $\{a_n\}_{n=1}^{\infty}$ *is bounded above, but is not bounded below.*

Proof. Exercise. □

Remark: Properties of limits from Theorem 2.2.5 readily generalize to divergence to (positive or negative) infinity. We do not state an analogous theorem formally. Instead, we focus on the "danger zones." In particular, the following forms are "indeterminate," i.e. they can in principle be anything:

$$\tfrac{0}{0} \qquad \tfrac{\infty}{\infty} \qquad 0\cdot\infty \qquad 1^{\infty} \qquad \infty-\infty \qquad 0^0 \qquad \infty^0$$

We now consider a few examples involving these indeterminate forms.

Example 2.7.4. *Consider the behavior of the following "*$\frac{\infty}{\infty}$*" forms:*

(a) $$\lim_{n\to\infty}\frac{\overset{\nearrow +\infty}{\boxed{2n}}}{\underset{\searrow +\infty}{\boxed{7n}}} = \lim_{n\to\infty}\tfrac{2}{7} = \tfrac{2}{7};$$

(b) $$\lim_{n\to\infty}\frac{\overset{\nearrow -\infty}{\boxed{-2\sqrt{n}}}}{\underset{\searrow -\infty}{\boxed{-3\sqrt{n}}}} = \lim_{n\to\infty}\tfrac{2}{3} = \tfrac{2}{3}$$

(c) $$\lim_{n\to\infty}\frac{\overset{\nearrow +\infty}{\boxed{3n^2}}}{\underset{\searrow +\infty}{\boxed{2n}}} = \lim_{n\to\infty}\tfrac{3n}{2} = +\infty;$$

(d) $$\lim_{n\to\infty}\frac{\overset{\nearrow -\infty}{\boxed{-2n^2}}}{\underset{\searrow +\infty}{\boxed{5n^3}}} = \lim_{n\to\infty}\big(-\tfrac{2}{5n}\big) = 0.$$

Example 2.7.5. *Compute the following limits:*

(a) $\lim\limits_{n\to\infty} \frac{2n^2-3n+7}{-n^2+2}$;

(b) $\lim\limits_{n\to\infty} \frac{-n^3+5n-1}{2n-1}$;

(c) $\lim\limits_{n\to\infty} \frac{n^2-1}{n^5+n^4+n^2}$.

(Note that each of the above is of the form "$\frac{\infty}{\infty}$.")

Solution. In each part, we start by factoring out the largest degree term from both the numerator and the denominator, and then we evaluate.

(a) $$\lim_{n\to\infty} \frac{2n^2-3n+7}{-n^2+2} = \lim_{n\to\infty} \frac{\cancel{n^2}\left(2-\underbrace{\frac{3}{n}}_{\to 0}+\underbrace{\frac{7}{n^2}}_{\to 0}\right)}{\cancel{n^2}\left(-1+\underbrace{\frac{2}{n^2}}_{\to 0}\right)} = -2$$

(b)

$$\begin{aligned}
\lim_{n\to\infty} \frac{-n^3+5n-1}{2n-1} &= \lim_{n\to\infty} \frac{n^3\left(-1+\frac{5}{n^2}-\frac{1}{n^3}\right)}{n\left(2-\frac{1}{n}\right)} \\
&= \lim_{n\to\infty} \left(\underbrace{n^2}_{\to +\infty} \cdot \underbrace{\frac{-1+\frac{5}{n^2}-\frac{1}{n^3}}{2-\frac{1}{n}}}_{\to -\frac{1}{2}}\right) \\
&= -\infty
\end{aligned}$$

(c)

$$\begin{aligned}
\lim_{n\to\infty} \frac{n^2-1}{n^5+n^4+n^2} &= \lim_{n\to\infty} \frac{n^2\left(1-\frac{1}{n^2}\right)}{n^5\left(1+\frac{1}{n}+\frac{1}{n^3}\right)} \\
&= \lim_{n\to\infty} \left(\underbrace{\frac{1}{n^3}}_{\to 0} \cdot \underbrace{\frac{1-\frac{1}{n^2}}{1+\frac{1}{n}+\frac{1}{n^3}}}_{\to 1}\right)
\end{aligned}$$

$$= \quad 0$$

□

Example 2.7.6. *Compute the following limits:*

(a) $\lim_{n\to\infty}\left(\sqrt{7n^2+n}-\sqrt{2n^2-2n}\right)$*;*

(b) $\lim_{n\to\infty}\left(\sqrt{n^2+n}-\sqrt{n^3-1}\right)$*;*

(c) $\lim_{n\to\infty}\left(\sqrt{n^2+n}-\sqrt{n^2-2n}\right)$*.*

(Note that each of the above is of the form "$\infty-\infty$.")

Solution. Informally, the idea is as follows:

(a) $\sqrt{7n^2+n}-\sqrt{2n^2-2n}\approx\sqrt{7n^2}-\sqrt{2n^2}=n(\sqrt{7}-\sqrt{2})\to\infty$;

(b) $\sqrt{n^2+n}-\sqrt{n^3-1}\approx\sqrt{n^2}-\sqrt{n^3}=n-n\sqrt{n}\to-\infty$ (because $n\sqrt{n}$ increases much faster than n);

(c) $\sqrt{n^2+n}-\sqrt{n^2-2n}\approx\sqrt{n^2}-\sqrt{n^2}$ (pure "$\infty-\infty$").

- Here, we do indeed have $\sqrt{n^2+n}\approx\sqrt{n^2}$ and $\sqrt{n^2-2n}\approx\sqrt{n^2}$, but the problem is that the error is only small relative to $\sqrt{n^2}$, and **cannot** necessarily be made smaller than an arbitrarily small $\varepsilon>0$. Thus, we **cannot** deduce that $\lim_{n\to\infty}\left(\sqrt{n^2+n}-\sqrt{n^2-2n}\right)=0$, and we need to compute more intelligently in order to figure out what the limit is.

However, the above only gives the intuition, and it does **not** count as a proper proof! Let us try to formalize this, i.e. give a proper solution.

(a)

$$\lim_{n\to\infty}\left(\sqrt{7n^2+n}-\sqrt{2n^2-2n}\right) = \lim_{n\to\infty}\Big(\underset{\nearrow +\infty}{\boxed{n}}\underbrace{\Big(\sqrt{7+\frac{1}{n}}-\sqrt{2-\frac{2}{n}}\Big)}_{\longrightarrow\sqrt{7}-\sqrt{2}}\Big)$$

$$= \quad \infty$$

(b)

$$\lim_{n\to\infty}\left(\sqrt{n^2+n}-\sqrt{n^3-1}\right) = \lim_{n\to\infty}\Big(\underbrace{n\sqrt{n}}_{\to+\infty}\Big(\underbrace{\sqrt{\frac{1}{n}+\frac{1}{n^2}}}_{\longrightarrow 0}-\underbrace{\sqrt{1-\frac{1}{n^3}}}_{\longrightarrow 1}\Big)\Big)$$

$$= -\infty$$

(c) Here, the trick is to multiply and divide by $\sqrt{n^2+n}+\sqrt{n^2-2n}$, and then make use of the familiar formula $(x-y)(x+y)=x^2-y^2$. This way, we will eliminate square roots in the numerator, while obtaining the sum (rather than difference) of square roots in the denominator. Formally, we have the following.

$$\begin{aligned}
\lim_{n\to\infty}\left(\sqrt{n^2+n}-\sqrt{n^2-2n}\right) &= \lim_{n\to\infty}\frac{\left(\sqrt{n^2+n}-\sqrt{n^2-2n}\right)\left(\sqrt{n^2+n}+\sqrt{n^2-2n}\right)}{\sqrt{n^2+n}+\sqrt{n^2-2n}}\\
&= \lim_{n\to\infty}\frac{(n^2+n)-(n^2-2n)}{\sqrt{n^2+n}+\sqrt{n^2-2n}}\\
&= \lim_{n\to\infty}\frac{3n}{n\left(\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{2}{n}}\right)}\\
&= \lim_{n\to\infty}\frac{3}{\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{2}{n}}}\\
&= \frac{3}{2}
\end{aligned}$$

□

Theorem 2.7.7. *Let $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ be sequences of real numbers such that $a_n \le b_n$ for all $n \in \mathbb{N}$. Then*

(a) if $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ both converge, then $\lim\limits_{n\to\infty} a_n \le \lim\limits_{n\to\infty} b_n$;

(b) if $\lim\limits_{n\to\infty} a_n = +\infty$, then $\lim\limits_{n\to\infty} b_n = +\infty$;

(c) if $\lim\limits_{n\to\infty} b_n = -\infty$, then $\lim\limits_{n\to\infty} a_n = -\infty$.

Proof. We prove (a) and (b). The proof of (c) is similar to the proof of (b).

(a) Assume that $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ both converge, and set $a := \lim\limits_{n\to\infty} a_n$ and $b := \lim\limits_{n\to\infty} b_n$. We must show that $a \le b$. Suppose otherwise, so that $b < a$. Set $\varepsilon := \frac{a-b}{2}$, and note that this implies that $a-\varepsilon = b+\varepsilon$. Using the fact that $a = \lim\limits_{n\to\infty} a_n$, we fix some $N_1 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \ge N_1$, then $|a_n - a| < \varepsilon$, i.e. $a-\varepsilon < a_N < a+\varepsilon$. Similarly, using the fact that $b = \lim\limits_{n\to\infty} b_n$, we fix some

$N_2 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N_2$, then $|b_n - b| < \varepsilon$, i.e. $b - \varepsilon < b_N < b + \varepsilon$. Set $N := \max\{N_1, N_2\}$. Then $N \geq N_1$ and $N \geq N_2$, and consequently,

$$b_N \quad < \quad b + \varepsilon \quad = \quad a - \varepsilon \quad < \quad a_N,$$

contrary to the fact that $a_N \leq b_N$.

(b) Assume that $\lim\limits_{n\to\infty} a_n = +\infty$. We must show that $\lim\limits_{n\to\infty} b_n = +\infty$. Fix $M \in \mathbb{R}$. Using the fact that $\lim\limits_{n\to\infty} a_n = +\infty$, we fix some $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N$, then $a_N > M$. But now for all $n \in \mathbb{N}$ such that $n \geq N$, we have that $b_n \geq a_n > M$. This proves that $\lim\limits_{n\to\infty} b_n = +\infty$. □

Example 2.7.8. *Compute* $\lim\limits_{n\to\infty} \big(n + (-1)^n\sqrt{n}\big)$.

Solution. Clearly, we have that $n - \sqrt{n} \leq n + (-1)^n\sqrt{n}$ for all $n \in \mathbb{N}$. Since

$$\lim_{n\to\infty} \big(n - \sqrt{n}\big) \quad = \quad \lim_{n\to\infty} \Big(\overset{+\infty}{\nearrow}\!\!\!\!\!\! \boxed{n} \;\; \big(1 - \overset{0}{\nearrow}\!\!\!\!\!\!\boxed{\tfrac{1}{\sqrt{n}}}\;\big)\Big) \quad = \quad +\infty,$$

Theorem 2.7.7(b) implies that $\lim\limits_{n\to\infty} \big(n + (-1)^n\sqrt{n}\big) = +\infty$, and we are done. □