Linear Algebra 2

Tutorial #11

Tests of positive definiteness

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- This mini-lecture has two parts:
    1. A brief review of bilinear and quadratic forms

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  1. A brief review of bilinear and quadratic forms
  2. The definition and tests of positive definiteness

1. A brief review of bilinear and quadratic forms

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Definition

A *bilinear form* on a vector space V over a field $\mathbb{F}$ is a function $f : V \times V \to \mathbb{F}$ that satisfies the following four axioms:

b.1. $\forall \mathbf{x}_1, \mathbf{x}_2, \mathbf{y} \in V$: $f(\mathbf{x}_1 + \mathbf{x}_2, \mathbf{y}) = f(\mathbf{x}_1, \mathbf{y}) + f(\mathbf{x}_2, \mathbf{y})$;

b.2. $\forall \mathbf{x}, \mathbf{y} \in V$ and $\alpha \in \mathbb{F}$: $f(\alpha\mathbf{x}, \mathbf{y}) = \alpha f(\mathbf{x}, \mathbf{y})$;

b.3. $\forall \mathbf{x}, \mathbf{y}_1, \mathbf{y}_2 \in V$: $f(\mathbf{x}, \mathbf{y}_1 + \mathbf{y}_2) = f(\mathbf{x}, \mathbf{y}_1) + f(\mathbf{x}, \mathbf{y}_2)$;

b.4. $\forall \mathbf{x}, \mathbf{y} \in V, \alpha \in \mathbb{F}$: $f(\mathbf{x}, \alpha\mathbf{y}) = \alpha f(\mathbf{x}, \mathbf{y})$.

The bilinear form f is said to be *symmetric* if it further satisfies the property that $f(\mathbf{x}, \mathbf{y}) = f(\mathbf{y}, \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in V$.

1. A brief review of bilinear and quadratic forms

Definition

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b.2. $\forall \mathbf{x}, \mathbf{y} \in V$ and $\alpha \in \mathbb{F}$: $f(\alpha\mathbf{x}, \mathbf{y}) = \alpha f(\mathbf{x}, \mathbf{y})$;

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The bilinear form f is said to be *symmetric* if it further satisfies the property that $f(\mathbf{x}, \mathbf{y}) = f(\mathbf{y}, \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in V$.

- Remark:
  - Scalar products in **real** vector spaces are bilinear forms.
  - However, scalar products in non-trivial **complex** vector spaces are **not** bilinear forms.

Theorem 9.2.2

Let V be a non-trivial, finite-dimensional vector space over a field $\mathbb{F}$, and let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis of V.

(a) For every matrix $A = \left[\ a_{i,j}\ \right]_{n \times n}$ in $\mathbb{F}^{n \times n}$, the function $f : V \times V \to \mathbb{F}$ given by
$$f(\mathbf{x}, \mathbf{y}) \quad = \quad \left[\ \mathbf{x}\ \right]_{\mathcal{B}}^{T} A \left[\ \mathbf{y}\ \right]_{\mathcal{B}} \qquad \text{for all } \mathbf{x}, \mathbf{y} \in V$$
is a bilinear form on V, and moreover, all the following hold:

(a.1) $f(\mathbf{b}_i, \mathbf{b}_j) = a_{i,j}$ for all $i, j \in \{1, \dots, n\}$,

(a.2) $f\left(\sum_{i=1}^{n} c_i \mathbf{b}_i, \sum_{j=1}^{n} d_j \mathbf{b}_j\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} c_i d_j$ for all $c_1, \dots, c_n, d_1, \dots, d_n \in \mathbb{F}$,

(a.3) f is symmetric iff A is symmetric.

(b) For every bilinear form f on V, there exists a unique matrix $A = \left[\ a_{i,j}\ \right]_{n \times n}$ in $\mathbb{F}^{n \times n}$, called the *matrix of the bilinear form f with respect to the basis $\mathcal{B}$*, that satisfies the property that
$$f(\mathbf{x}, \mathbf{y}) \quad = \quad \left[\ \mathbf{x}\ \right]_{\mathcal{B}}^{T} A \left[\ \mathbf{y}\ \right]_{\mathcal{B}} \qquad \text{for all } \mathbf{x}, \mathbf{y} \in V.$$
Moreover, the entries of the matrix A are given by $a_{i,j} = f(\mathbf{b}_i, \mathbf{b}_j)$ for all indices $i, j \in \{1, \dots, n\}$.

- As a special case of Theorem 9.2.2 for the special case of $V = \mathbb{F}^n$ (where $\mathbb{F}$ is a field), and $\mathcal{B} = \mathcal{E}_n$ (the standard basis of $\mathbb{F}^n$), we get the following corollary (next slide).

Corollary 9.2.3

Let $\mathbb{F}$ be a field, and let $\mathcal{E}_n = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be the standard basis of $\mathbb{F}^n$.

(a) For every matrix $A = \begin{bmatrix} a_{i,j} \end{bmatrix}_{n \times n}$ in $\mathbb{F}^{n \times n}$, the function $f : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}$ given by

$$f(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T A \mathbf{y} \qquad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{F}^n$$

is a bilinear form on $\mathbb{F}^n$, and moreover, all the following hold:

(a.1) $f(\mathbf{e}_i, \mathbf{e}_j) = a_{i,j}$ for all $i, j \in \{1, \dots, n\}$,

(a.2) $f(\mathbf{x}, \mathbf{y}) = \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} a_{i,j} x_i y_j$ for all vectors $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$ and $\mathbf{y} = \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix}^T$ in $\mathbb{F}^n$,

(a.3) f is symmetric iff A is symmetric.

(b) For every bilinear form f on $\mathbb{F}^n$, there exists a unique matrix $A = \begin{bmatrix} a_{i,j} \end{bmatrix}_{n \times n}$ in $\mathbb{F}^{n \times n}$ that satisfies the property that

$$f(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T A \mathbf{y} \qquad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{F}^n.$$

Moreover, the entries of the matrix A are given by $a_{i,j} = f(\mathbf{e}_i, \mathbf{e}_j)$ for all indices $i, j \in \{1, \dots, n\}$.

- **Remark:** Corollary 9.2.3 implies that, for a field $\mathbb{F}$, the bilinear forms on $\mathbb{F}^n$ are precisely the functions $f : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}$ given by

$$f(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} x_i y_j \qquad \text{for all } \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \text{ in } \mathbb{F}^n,$$

where the $a_{i,j}$'s are some scalars in $\mathbb{F}$.

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  where the $a_{i,j}$'s are some scalars in $\mathbb{F}$.
  - Moreover, such a bilinear form is symmetric iff $a_{i,j} = a_{j,i}$ for all indices $i, j \in \{1, \ldots, n\}$.

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  where the $a_{i,j}$'s are some scalars in $\mathbb{F}$.
  - Moreover, such a bilinear form is symmetric iff $a_{i,j} = a_{j,i}$ for all indices $i, j \in \{1, \dots, n\}$.
  - The matrix of this bilinear form with respect to the standard basis $\mathcal{E}_n$ of $\mathbb{F}^n$ is $\begin{bmatrix} a_{i,j} \end{bmatrix}_{n\times n}$ (so, the i, j-th entry of the matrix is the coefficient in front of $x_i y_j$ from the formula for f above).

- For example, functions $f_1, f_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by the formulas
  - $f_1(\mathbf{x}, \mathbf{y}) = x_1 y_1 - 3x_1 y_2 - 3x_2 y_1 + 7x_2 y_2$,
  - $f_2(\mathbf{x}, \mathbf{y}) = x_1 y_1 - 2x_1 y_2 + 3x_2 y_1 - 3x_2 y_2$,

  for all $\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ and $\mathbf{y} = \begin{bmatrix} y_1 & y_2 \end{bmatrix}^T$ in $\mathbb{R}^2$, are bilinear forms on $\mathbb{R}^2$.

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  for all $\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ and $\mathbf{y} = \begin{bmatrix} y_1 & y_2 \end{bmatrix}^T$ in $\mathbb{R}^2$, are bilinear forms on $\mathbb{R}^2$.
    - The bilinear form f_1 is symmetric, whereas the bilinear form f_2 is not.
    - The matrices of the bilinear forms f_1 and f_2 with respect to the standard basis $\mathcal{E}_2$ of $\mathbb{R}^2$ are

$$A_1 = \begin{bmatrix} 1 & -3 \\ -3 & 7 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & -2 \\ 3 & -3 \end{bmatrix},$$

      respectively.

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  for all $\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ and $\mathbf{y} = \begin{bmatrix} y_1 & y_2 \end{bmatrix}^T$ in $\mathbb{R}^2$, are bilinear forms on $\mathbb{R}^2$.
    - The bilinear form f_1 is symmetric, whereas the bilinear form f_2 is not.
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$$A_1 = \begin{bmatrix} 1 & -3 \\ -3 & 7 \end{bmatrix} \qquad \text{and} \qquad A_2 = \begin{bmatrix} 1 & -2 \\ 3 & -3 \end{bmatrix},$$

      respectively.
    - Note that A_1 is symmetric, whereas A_2 is not; this is consistent with the fact that f_1 is symmetric, whereas f_2 is not.

2. The definition and tests of positive definiteness

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Definition

A **symmetric** matrix $A \in \mathbb{R}^{n \times n}$ is said to be *positive definite* if $\mathbf{x}^T A\mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$.

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- Advertisement:

Corollary 10.4.2

For any function $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, the following are equivalent:

(i) $\langle \cdot, \cdot \rangle$ is a scalar product on $\mathbb{R}^n$;

(ii) there exists a positive definite matrix $A \in \mathbb{R}^{n \times n}$ such that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T A\mathbf{y}$.

- Proof: Lecture!

Proposition 10.1.4 (abridged)

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- We present two tests of positive definiteness (without proof, but with examples):
  - the Gaussian elimination test of positive definiteness;
  - Sylvester's criterion of positive definiteness.

Theorem 10.2.6 [The Gaussian elimination test of pos. def.]

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Then the following algorithm correctly determines whether A is positive definite.

- **Step** 0**:** Set $A_1 := A$, and go to Step 1.
- For $j \in \{1, \ldots, n\}$, and assuming the matrix A_j has already been generated, we proceed as follows.

  Step j**:**
  - If the main diagonal of A_j is **not** positive, then the algorithm returns the answer that A is **not** positive definite and terminates.
  - If the main diagonal of A_j is positive and $j = n$, then the algorithm returns the answer that A is positive definite and terminates.
  - If the main diagonal of A_j is positive and $j \leq n - 1$, then for each index $i \in \{j + 1, \ldots, n\}$, we add a suitable scalar multiple of the j-th row of A_j to the i-th row of A_j so that the i, j-th entry of the matrix becomes zero; we call the resulting matrix A_{j+1}, and we go to Step $j + 1$.

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  - It only performs elementary row operations of the form "$R_i \to R_i + \alpha R_j$," where $i > j$ (i.e. row i is below row j), and where α is chosen so that the i, j-th entry of the matrix becomes zero; moreover, these operations (which add scalar multiples of row j to the rows below it) are performed only in Step j.
    - Essentially, we use the j, j-th entry of the matrix A_j to "clean up" the j-th column below the main diagonal, i.e. to turn all entries of the j-th column below the main diagonal into zeros.
    - Note that at the start of Step j, the leftmost $j - 1$ many columns have already been processed, so that they have all zeros below the main diagonal.

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    - It only performs elementary row operations of the form “$R_i \to R_i + \alpha R_j$,” where $i > j$ (i.e. row i is below row j), and where α is chosen so that the i, j-th entry of the matrix becomes zero; moreover, these operations (which add scalar multiples of row j to the rows below it) are performed only in Step j.
        - Essentially, we use the j, j-th entry of the matrix A_j to “clean up” the j-th column below the main diagonal, i.e. to turn all entries of the j-th column below the main diagonal into zeros.
        - Note that at the start of Step j, the leftmost $j - 1$ many columns have already been processed, so that they have all zeros below the main diagonal.
    - We keep modifying our matrix until we either obtain a zero or a negative number on the main diagonal (in this case, our input matrix is not positive definite), or until we transform our matrix into an upper triangular matrix with a positive main diagonal (in this case, our input matrix is positive definite).

Example 10.2.7

Using Theorem 10.2.6, determine whether the matrix

$$A \ := \ \begin{bmatrix} 4 & -2 & 4 \\ -2 & 10 & 1 \\ 4 & 1 & 6 \end{bmatrix}$$

is positive definite.

Solution.

Example 10.2.7

Using Theorem 10.2.6, determine whether the matrix

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Solution. The matrix A is symmetric, and so Theorem 10.2.6 applies.

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is positive definite.

Solution. The matrix A is symmetric, and so Theorem 10.2.6 applies. We perform the modified version of the "forward" part of the row reduction algorithm described in Theorem 10.2.6, as follows (the dotted lines isolate the submatrix in the lower right corner that is still being processed):

Solution (continued).

$$A = \underbrace{\begin{bmatrix} 4 & -2 & 4 \\ -2 & 10 & 1 \\ 4 & 1 & 6 \end{bmatrix}}_{=:A_1} \qquad \text{Step 0}$$

$$\underset{\sim}{\substack{R_2 \to R_2 + \frac{1}{2} R_1 \\ R_3 \to R_3 - R_1}} \quad \underbrace{\begin{bmatrix} 4 & -2 & 4 \\ 0 & 9 & 3 \\ 0 & 3 & 2 \end{bmatrix}}_{=:A_2} \qquad \text{Step 1}$$

$$\underset{\sim}{R_3 \to R_3 - \frac{1}{3} R_2} \quad \underbrace{\begin{bmatrix} 4 & -2 & 4 \\ 0 & 9 & 3 \\ 0 & 0 & 1 \end{bmatrix}}_{=:A_3} \qquad \text{Step 2.}$$

Solution (continued).

$$A = \underbrace{\begin{bmatrix} 4 & -2 & 4 \\ -2 & 10 & 1 \\ 4 & 1 & 6 \end{bmatrix}}_{=:A_1} \qquad \text{Step 0}$$

$$\underset{\sim}{\substack{R_2 \to R_2 + \frac{1}{2} R_1 \\ R_3 \to R_3 - R_1}} \quad \underbrace{\left[\begin{array}{c|cc} 4 & -2 & 4 \\ \hline 0 & 9 & 3 \\ 0 & 3 & 2 \end{array}\right]}_{=:A_2} \qquad \text{Step 1}$$

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We have now obtained an upper triangular matrix with a positive main diagonal. So, by Theorem 10.2.6, A is positive definite. $\square$

Example 10.2.8

Using Theorem 10.2.6, determine whether the matrix

$$A \;:=\; \begin{bmatrix} 2 & -2 & 2 & 0 \\ -2 & 3 & 0 & 1 \\ 2 & 0 & 6 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

is positive definite.

Solution.

Example 10.2.8

Using Theorem 10.2.6, determine whether the matrix

$$A \quad := \quad \begin{bmatrix} 2 & -2 & 2 & 0 \\ -2 & 3 & 0 & 1 \\ 2 & 0 & 6 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

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Solution. The matrix A is symmetric, and so Theorem 10.2.6 applies. We perform the modified version of the "forward" part of the row reduction algorithm described in Theorem 10.2.6, as follows:

Solution (continued).

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A = \begin{bmatrix} 2 & -2 & 2 & 0 \\ -2 & 3 & 0 & 1 \\ 2 & 0 & 6 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}
$$

$$
\underset{\substack{R_2 \to R_2 + R_1 \\ R_3 \to R_3 - R_1}}{\sim} \left[\begin{array}{c|ccc} 2 & -2 & 2 & 0 \\ \hline 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 0 & 2 \end{array}\right]
$$

$$
\underset{\substack{R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 - R_2}}{\sim} \left[\begin{array}{cc|cc} 2 & -2 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ \hline 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & 1 \end{array}\right].
$$

Solution (continued).

$$A = \begin{bmatrix} 2 & -2 & 2 & 0 \\ -2 & 3 & 0 & 1 \\ 2 & 0 & 6 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

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$$\underset{\sim}{\substack{R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 - R_2}} \left[\begin{array}{cc|cc} 2 & -2 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ \hline 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & 1 \end{array}\right].$$

We have now obtained a zero on the main diagonal of our matrix, and so by Theorem 10.2.6, the matrix A is **not** positive definite. □

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- Given any $n \times n$ matrix A, and any index $k \in \{1, \ldots, n\}$, we let $A^{(k)}$ be the $k \times k$ matrix in the upper left corner of A.
- For example, if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix},$$

then we have that

$$A^{(1)} = \begin{bmatrix} 1 \end{bmatrix}, \qquad A^{(2)} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}, \qquad A^{(3)} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

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- Clearly, for any $n \times n$ matrix A, we have that $A^{(n)} = A$.

Theorem 10.2.9 [Sylvester's criterion of positive definiteness]

For all symmetric matrices $A \in \mathbb{R}^{n \times n}$, the following are equivalent:

1. (i) A is positive definite;
2. (ii) $\det(A^{(1)}), \ldots, \det(A^{(n)}) > 0$.

Example 10.2.10

Using Sylvester's criterion of positive definiteness, determine whether the matrix

$$A \quad := \quad \begin{bmatrix} 4 & -2 & 4 \\ -2 & 10 & 1 \\ 4 & 1 & 6 \end{bmatrix}$$

is positive definite.

Solution.

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Solution. The matrix A is symmetric, and so we can use Sylvester's criterion of positive definiteness.

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is positive definite.

Solution. The matrix A is symmetric, and so we can use Sylvester's criterion of positive definiteness. We compute:

- $\det(A^{(1)}) = \begin{vmatrix} 4 \end{vmatrix} = 4 > 0;$
- $\det(A^{(2)}) = \begin{vmatrix} 4 & -2 \\ -2 & 10 \end{vmatrix} = 36 > 0;$
- $\det(A^{(3)}) = \begin{vmatrix} 4 & -2 & 4 \\ -2 & 10 & 1 \\ 4 & 1 & 6 \end{vmatrix} = 36 > 0.$

All three determinants are positive, and so by Sylvester's criterion of positive definiteness, the matrix A is positive definite. $\square$

Example 10.2.11

Using Sylvester's criterion of positive definiteness, determine whether the matrix

$$A := \begin{bmatrix} 2 & -2 & 2 & 0 \\ -2 & 3 & 0 & 1 \\ 2 & 0 & 6 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

is positive definite.

Solution.

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is positive definite.

Solution. The matrix A is symmetric, and so we can use Sylvester's criterion of positive definiteness.

Solution (continued). We compute:

- $\det(A^{(1)}) = \begin{vmatrix} 2 \end{vmatrix} = 2 > 0$;
- $\det(A^{(2)}) = \begin{vmatrix} 2 & -2 \\ -2 & 3 \end{vmatrix} = 2 > 0$;
- $\det(A^{(3)}) = \begin{vmatrix} 2 & -2 & 2 \\ -2 & 3 & 0 \\ 2 & 0 & 6 \end{vmatrix} = 0$.

Since $\det(A^{(3)})$ is not positive, Sylvester's criterion of positive definiteness guarantees that A is **not** positive definite.

Solution (continued). We compute:

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Since $\det(A^{(3)})$ is not positive, Sylvester's criterion of positive definiteness guarantees that A is **not** positive definite.

Remark: Note that we did not need to compute $\det(A^{(4)})$. In general, if we obtain $\det(A^{(k)}) \leq 0$ for some k, then we do not need to compute further, and we simply deduce that A is not positive definite. $\square$