Linear Algebra 2: Tutorial 11

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Summer 2025

Exercise 1. *Consider the bilinear form f on $\mathbb{R}^3$ given by the formula*

$$f(\mathbf{x},\mathbf{y}) \;=\; x_1y_1 + 2x_1y_2 + x_1y_3 - x_2y_1 + x_2y_2 + x_3y_1 - 5x_3y_3$$

for all $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ and $\mathbf{y} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^T$ in $\mathbb{R}^3$. Compute the matrix of the bilinear form f with respect to the standard basis $\mathcal{E}_3$ of $\mathbb{R}^3$. Is the bilinear form f symmetric?

Exercise 2. *Determine whether the following matrices (with entries in $\mathbb{R}$ are positive definite. Do this in two ways: using the Gaussian elimination test of positive definiteness, and using Sylvester's criterion of positive definiteness.*

(a) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$;

(b) $B = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}$;

(c) $C = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

Exercise 3. *Consider the bilinear form $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^3$ given by*

$$\langle\mathbf{x},\mathbf{y}\rangle \;=\; 3x_1y_1 + 2x_1y_2 + x_1y_3 + 2x_2y_1 + 2x_2y_2 + x_3y_1 + 2x_3y_3$$

for all $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ and $\mathbf{y} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^T$ in $\mathbb{R}^3$.

(a) Compute the matrix of the bilinear form $\langle\cdot,\cdot\rangle$ with respect to the standard basis $\mathcal{E}_3$ of $\mathbb{R}^3$.

(b) Is the bilinear form $\langle\cdot,\cdot\rangle$ symmetric?

(c) Is the bilinear form $\langle\cdot,\cdot\rangle$ a scalar product in $\mathbb{R}^3$?

Exercise 4. *Compute the (symmetric) matrix of the quadratic form q on $\mathbb{R}^3$ given by*

$$q(\mathbf{x}) \;=\; x_1x_2 + 4x_1x_3 + x_2^2 - 6x_2x_3 + 2x_3^2$$

for all $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ in $\mathbb{R}^3$ (with respect to the standard basis $\mathcal{E}_3$ of $\mathbb{R}^3$).

Exercise 5. *Give an example of a quadratic form q on $\mathbb{Z}_2^2$ for which there does **not** exist a symmetric matrix $A \in \mathbb{Z}_2^{2\times 2}$ that satisfies the following:*

$$q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} \qquad \textit{for all } \mathbf{x} \in \mathbb{Z}_2^2.$$

Exercise 6. *Let $\mathbb{F}$ be an algebraically closed field, and let $A \in \mathbb{F}^{m\times m}$ and $B \in \mathbb{F}^{n\times n}$. Using the Jordan normal form, prove that $A \oplus B = \left[\begin{array}{c:c} A & O_{m\times n} \\ \hdashline O_{n\times m} & B \end{array}\right]$ is diagonalizable if and only if A and B are both diagonalizable.*

Exercise 7. *Using Exercise 6, explain why the complex matrix below is **not** diagonalizable.*

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 3 & 1 & 0 \\ 0 & 0 & 5 & 5 & 6 & 6 \\ 0 & 0 & 3 & 1 & 2 & 2 \\ 0 & 0 & 8 & 9 & 9 & 7 \end{bmatrix}$$