Linear Algebra 2: Tutorial 10

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Problem 2 of HW#5. *Let A be an invertible matrix in $\mathbb{R}^{n\times n}$, and assume that the entries of A are all integers. Prove that the entries of A^{-1} are all integers if and only if $det(A) = \pm 1$.*

> ***Hint:*** *Use the adjugate matrix.*

Definition. *For a field $\mathbb{F}$, a scalar $\lambda_0 \in \mathbb{F}$, and a positive integer t, the* Jordan block *$J_t(\lambda_0)$ is defined to be following $t \times t$ matrix (with entries understood to be in $\mathbb{F}$):*

$$J_t(\lambda_0) \quad = \quad \begin{bmatrix} \lambda_0 & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_0 & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda_0 \end{bmatrix}_{t\times t}.$$

Thus, $J_t(\lambda_0)$ is a matrix in $\mathbb{F}^{t\times t}$, it has all λ_0's on the main diagonal, all 1's on the diagonal right above the main diagonal, and 0's everywhere else.

Exercise 4 of Tutorial 9. *Let $\mathbb{F}$ be a field, let $\lambda_0 \in \mathbb{F}$, and let t be a positive integer. What are the eigenvalues of $J_t(\lambda_0)$? What are their algebraic and geometric multiplicities? Find a basis of each eigenspace.*

Definition. *Let $\mathbb{F}$ be a field. For square matrices $A_1 \in \mathbb{F}^{n_1\times n_1}, A_2 \in \mathbb{F}^{n_2\times n_2}, \dots, A_k \in \mathbb{F}^{n_k\times n_k}$, we define the* direct sum *of $A_1, A_2 \dots, A_k$ to be the $(n_1 + n_2 + \dots + n_k) \times (n_1 + n_2 + \dots + n_k)$ matrix*

$$A_1 \oplus A_2 \oplus \dots \oplus A_k \quad := \quad \left[\begin{array}{c:c:c:c} A_1 & O_{n_1\times n_2} & \dots & O_{n_1\times n_k} \\ \hdashline O_{n_2\times n_1} & A_2 & \dots & O_{n_2\times n_k} \\ \hdashline \vdots & \vdots & \ddots & \vdots \\ \hdashline O_{n_k\times n_1} & O_{n_k\times n_2} & \dots & A_k \end{array}\right].$$

For example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \oplus \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \oplus \begin{bmatrix} 1 \end{bmatrix} \quad = \quad \left[\begin{array}{cc:ccc:c} 1 & 2 & 0 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 4 & 5 & 6 & 0 \\ 0 & 0 & 7 & 8 & 9 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

Exercise 5 of Tutorial 9. *Let $\mathbb{F}$ be a field, let $\lambda_0 \in \mathbb{F}$, and let $t_1, \dots, t_k$ be positive integers. What are the eigenvalues of $J_{t_1}(\lambda_0) \oplus \dots \oplus J_{t_k}(\lambda_0)$? What are the algebraic and geometric multiplicities of those eigenvalues?*

Exercise 6 of Tutorial 9. *Let $\mathbb{F}$ be a field, let $\lambda_1, \lambda_2 \in \mathbb{F}$ be distinct, and let t_1, t_2 be positive integers. What are the eigenvalues of $J_{t_1}(\lambda_1) \oplus J_{t_2}(\lambda_2)$? What are the algebraic and geometric multiplicities of those eigenvalues?*

Exercise 7 of Tutorial 9. *Try to generalize Exercises 5 and 6. What are the eigenvalues of matrix that is a direct sum of arbitrarily many Jordan blocks, when those Jordan blocks can be of arbitrary size and type?*[1] *(Such a matrix is called a "Jordan matrix," or a matrix in "Jordan normal form.") What are the algebraic and geometric multiplicities of those eigenvalues?*

> ***Remark:*** *You don't have to give a fully formal proof, but try to give a reasonable proof outline.*

Exercise 1. *For each of the following matrices A, determine whether the matrix is diagonalizable, and if so, diagonalize it. If A is diagonalizable, then find a formula for A^m for all non-negative integers m.*[2] *Finally, determine whether your formula for A^m also works for negative integers m.*

(a) $A = \begin{bmatrix} -2 & -10 & 0 \\ 0 & 3 & 0 \\ -5 & -10 & 3 \end{bmatrix}$

(b) $A = \begin{bmatrix} -2 & 4 & 1 \\ -2 & 4 & 1 \\ -2 & 2 & 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 6 & 3 & 1 \\ -8 & -4 & -2 \\ 8 & 6 & 4 \end{bmatrix}$

(d) $A = \begin{bmatrix} 4 & -3 & 3 & 0 \\ 0 & 11 & 0 & -6 \\ -2 & 12 & -1 & -6 \\ 0 & 18 & 0 & -10 \end{bmatrix}$

Exercise 2. *For the following matrices D and P (with entries understood to be in $\mathbb{C}$), set $A := PDP^{-1}$ (so, $D = P^{-1}AP$), and then compute the characteristic polynomial and the spectrum of A, specify all the eigenvalues of A along with their algebraic and geometric multiplicities, and find a basis for each eigenspace of A. (In both parts, you may assume that the matrix P is indeed invertible. This can be checked, for example, by computing the determinant of P and seeing that it is non-zero.)*

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[1] So, the matrix is of the form $J_{t_1}(\lambda_1) \oplus \dots \oplus J_{t_k}(\lambda_k)$, where $\lambda_1, \dots, \lambda_k$ are (not necessarily distinct) scalars in the field $\mathbb{F}$ in question, and $t_1, \dots, t_k$ are some positive integers.

[2] If A is not diagonalizable, then you do not need to find a formula for A^m.

(a) $D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 3 \end{bmatrix}$;

(b) $D := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$, $P = \begin{bmatrix} -1 & 0 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 2 & 3 & -2 & -3 \\ 4 & 5 & 6 & 7 \end{bmatrix}$.

Exercise 3. *Determine if the following complex matrices are diagonalizable. Make sure you justify your answer (however, you do not need to actually diagonalize any matrices).*

(a) $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$;

(b) $B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 4 & 5 & 6 \end{bmatrix}$;

(c) $C = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 7 & 3 & 4 & 1 \\ 0 & 0 & 4 & 3 & 1 & 2 & 2 \\ 0 & 0 & 3 & 5 & 6 & 7 & 1 \\ 0 & 0 & 2 & 2 & 2 & 1 & 3 \\ 0 & 0 & 1 & 2 & 1 & 1 & 2 \end{bmatrix}$.