Linear Algebra 2: Tutorial 9

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Exercise 1. *Prove or disprove the following each of the following statements.*

(a) For all matrices $A, B \in \mathbb{C}^{2\times 2}$ *and vectors* $\mathbf{v} \in \mathbb{C}^2$*, if* $\mathbf{v}$ *is an eigenvector of both* A *and* B*, then* $\mathbf{v}$ *is an eigenvector of* $A + B$*.*

(b) For all matrices $A, B \in \mathbb{C}^{2\times 2}$ *and scalars* $\lambda \in \mathbb{C}$*, if* λ *is an eigenvalue of both* A *and* B*, then* λ *is an eigenvalue of* $A + B$*.*

For each part, you should first state clearly whether the statement is true or false. If it is true, then prove it. If it is false, then construct a counterexample (and prove that your counterexample really is a counterexample).

Exercise 2. *Construct a matrix* $A \in \mathbb{R}^{2\times 2}$ *such that* A *has no (real) eigenvalues, but* A^2 *does have (real) eigenvalues.*

> ***Hint:*** *Think geometrically.*

Exercise 3. *Let* $\mathbb{F}$ *be a field, and let* $\lambda_0 \in \mathbb{F}$ *be an eigenvalue of a square matrix* $A \in \mathbb{F}^{n\times n}$*.*

(a) Prove that for all non-negative integers m*,* λ_0^m *is an eigenvalue of* A^m*.*

(b) Prove that if A *is invertible, then* λ_0^m *is an eigenvalue of* A^m *for all integers* m*.*

- *This, in particular, means that* $\frac{1}{\lambda_0}$ *is an eigenvalue of* A^{-1} *(again, assuming that* A *is invertible). How do you know that* $\frac{1}{\lambda_0}$ *is even defined, i.e. that you are not dividing by zero?*

(c) Assume that $B := P^{-1}AP$ *for some invertible matrix* $P \in \mathbb{F}^{n\times n}$*. In particular,* A *and* B *are similar, and so by Theorem 8.2.9 of the Lecture Notes,* A *and* B *have the same eigenvalues. If* $\mathbf{v}$ *is an eigenvector of* A *associated with the eigenvalue* λ_0*, can you construct an eigenvector of* B *associated with the eigenvalue* λ_0*?*

Definition. *For a field* $\mathbb{F}$*, a scalar* $\lambda_0 \in \mathbb{F}$*, and a positive integer* t*, the* Jordan block $J_t(\lambda_0)$ *is defined to be following* $t \times t$ *matrix (with entries understood to be in* $\mathbb{F}$*):*

$$J_t(\lambda_0) \quad = \quad \begin{bmatrix} \lambda_0 & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_0 & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda_0 \end{bmatrix}_{t\times t}.$$

Thus, $J_t(\lambda_0)$ *is a matrix in* $\mathbb{F}^{t\times t}$*, it has all* λ_0*'s on the main diagonal, all* 1*'s on the diagonal right above the main diagonal, and* 0*'s everywhere else.*

Exercise 4. *Let $\mathbb{F}$ be a field, let $\lambda_0 \in \mathbb{F}$, and let t be a positive integer. What are the eigenvalues of $J_t(\lambda_0)$? What are their algebraic and geometric multiplicities? Find a basis of each eigenspace.*

Definition. *Let $\mathbb{F}$ be a field. For square matrices $A_1 \in \mathbb{F}^{n_1 \times n_1}, A_2 \in \mathbb{F}^{n_2 \times n_2}, \ldots, A_k \in \mathbb{F}^{n_k \times n_k}$, we define the* direct sum *of $A_1, A_2 \ldots, A_k$ to be the $(n_1 + n_2 + \cdots + n_k) \times (n_1 + n_2 + \cdots + n_k)$ matrix*

$$A_1 \oplus A_2 \oplus \cdots \oplus A_k \quad := \left[\begin{array}{c|c|c|c} A_1 & O_{n_1 \times n_2} & \cdots & O_{n_1 \times n_k} \\ \hline O_{n_2 \times n_1} & A_2 & \cdots & O_{n_2 \times n_k} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline O_{n_k \times n_1} & O_{n_k \times n_2} & \cdots & A_k \end{array}\right].$$

For example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \oplus \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \oplus \begin{bmatrix} 1 \end{bmatrix} \;=\; \left[\begin{array}{cc|ccc|c} 1 & 2 & 0 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 4 & 5 & 6 & 0 \\ 0 & 0 & 7 & 8 & 9 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

Exercise 5. *Let $\mathbb{F}$ be a field, let $\lambda_0 \in \mathbb{F}$, and let $t_1, \ldots, t_k$ be positive integers. What are the eigenvalues of $J_{t_1}(\lambda_0) \oplus \cdots \oplus J_{t_k}(\lambda_0)$? What are the algebraic and geometric multiplicities of those eigenvalues?*

Exercise 6. *Let $\mathbb{F}$ be a field, let $\lambda_1, \lambda_2 \in \mathbb{F}$ be distinct, and let t_1, t_2 be positive integers. What are the eigenvalues of $J_{t_1}(\lambda_1) \oplus J_{t_2}(\lambda_2)$? What are the algebraic and geometric multiplicities of those eigenvalues?*

Exercise 7. *Try to generalize Exercises 5 and 6. What are the eigenvalues of matrix that is a direct sum of arbitrarily many Jordan blocks, when those Jordan blocks can be of arbitrary size and type?*[1] *(Such a matrix is called a "Jordan matrix," or a matrix in "Jordan normal form.") What are the algebraic and geometric multiplicities of those eigenvalues?*

> ***Remark:*** *You don't have to give a fully formal proof, but try to give a reasonable proof outline.*

Exercise 8. *Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{m \times n}$ and $B \in \mathbb{F}^{n \times m}$ be matrices. Define the $(m+n) \times (m+n)$ matrices*

$$C := \left[\begin{array}{c|c} \lambda I_m & A \\ \hline B & I_n \end{array}\right] \qquad \text{and} \qquad D := \left[\begin{array}{c|c} I_m & O_{m \times n} \\ \hline -B & \lambda I_n \end{array}\right].$$

(a) By computing det(CD) and det(DC), find a relationship between $p_{AB}(\lambda)$ and $p_{BA}(\lambda)$.

(b) Using part (a), show that if $m = n$, then $p_{AB}(\lambda) = p_{BA}(\lambda)$.

[1] So, the matrix is of the form $J_{t_1}(\lambda_1) \oplus \cdots \oplus J_{t_k}(\lambda_k)$, where $\lambda_1, \ldots, \lambda_k$ are (not necessarily distinct) scalars in the field $\mathbb{F}$ in question, and $t_1, \ldots, t_k$ are some positive integers.