Linear Algebra 2: Tutorial 8

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Exercise 1. *Prove or disprove the following statement:*

> *For all matrices* $A = \begin{bmatrix} \mathbf{a}_1 & \dots \mathbf{a}_n \end{bmatrix}$ *in* $\mathbb{R}^{n \times n}$, *we have that* $|det(A)| \leq \prod_{i=1}^{n} ||\mathbf{a}_i||$.
>
> ***Hint:*** *Volume.*

Exercise 2. *What is the maximum possible value of* $det(A)$ *if* A *is a matrix in* $\mathbb{R}^{4 \times 4}$, *all of whose entries are* 1, 0, *or* -1? *Exhibit a matrix* A *for which this maximum is reached.*

Exercise 3. *Prove of disprove the following statement:*

> *For all invertible matrices* $A \in \mathbb{R}^{2 \times 2}$ *and* $\mathbf{v} \in \mathbb{R}^2$, *we have that* $||A\mathbf{v}|| \leq |det(A)| \, ||\mathbf{v}||$.

Exercise 4. *Show that the area of the triangle with vertices* (x_1, y_1), (x_2, y_2), *and* (x_3, y_3) *in* $\mathbb{R}^2$ *is equal to*

$$\frac{1}{2} \left| \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right|.$$

Definition. *Let* $\mathbb{F}$ *be a field, and let* $A \in \mathbb{F}^{n \times n}$ *be a square matrix. An* eigenvector *of* A *is a vector* $\mathbf{v} \in \mathbb{F}^n \setminus \{\mathbf{0}\}$ *for which there exists a scalar* $\lambda \in \mathbb{F}$, *called the* eigenvalue *of* A *associated with the eigenvector* $\mathbf{v}$, *such that*

$$A\mathbf{v} \quad = \quad \lambda\mathbf{v}.$$

Under these circumstances, we also say that $\mathbf{v}$ *is an eigenvector of* A *associated with the eigenvalue* λ.

Definition. *For a square matrix $A \in \mathbb{F}^{n\times n}$ (where $\mathbb{F}$ is some field), and for a scalar $\lambda \in \mathbb{F}$, we define*

$$E_\lambda(A) \quad := \quad \{\mathbf{v} \in \mathbb{F}^n \mid A\mathbf{v} = \lambda\mathbf{v}\}.$$

If λ is an eigenvalue of A, then $E_\lambda(A)$ is called the eigenspace *of A associated with the eigenvalue λ.*

Theorem 8.2.2. *Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n\times n}$, and let $\lambda_0 \in \mathbb{F}$. Then*

$$E_{\lambda_0}(A) \quad = \quad Nul\big(\lambda_0 I_n - A\big) \quad = \quad Nul\big(A - \lambda_0 I_n\big).$$

Moreover, the following are equivalent:

(1) λ_0 is an eigenvalue of A;

(2) λ_0 is a root of the characteristic polynomial of A, i.e. $p_A(\lambda_0) = 0$;

(3) λ_0 is a solution of the characteristic equation of A, i.e. $det\big(\lambda_0 I_n - A\big) = 0$.

Exercise 5. *For each of the following matrices A in $\mathbb{C}^{3\times 3}$, compute the characteristic polynomial $p_A(\lambda)$ and the spectrum of A. For each eigenvalue λ of A, determine both the algebraic and the geometric multiplicity of λ, and compute a basis of the eigenspace E_λ.*

(a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix};$

(b) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix};$

(c) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix};$

(d) $A = \begin{bmatrix} 3 & 0 & -2 \\ -7 & 0 & 4 \\ 4 & 0 & -3 \end{bmatrix};$

(e) $A = \begin{bmatrix} -1 & 0 & 1 \\ -3 & 0 & 1 \\ -4 & 0 & 3 \end{bmatrix};$