Linear Algebra 2: Tutorial 6

Irena Penev

Summer 2025

Definition. *The* determinant *of a matrix* $A = \left[\begin{array}{c} a_{i,j} \end{array}\right]_{n \times n}$ *with entries in some field* $\mathbb{F}$*, denoted by* $det(A)$ *or* $|A|$*, is defined by*

$$\begin{aligned} det(A) \quad &:= \quad \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} \\ &= \quad \sum_{\sigma \in S_n} sgn(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \dots a_{n,\sigma(n)}. \end{aligned}$$

Theorem 7.1.3. *Let* $\mathbb{F}$ *be a field. For all* $A \in \mathbb{F}^{n \times n}$*, we have that*

$$det(A^T) \quad = \quad det(A).$$

Proposition 7.3.1. *Let* $\mathbb{F}$ *be a field, and let* $A = \left[\begin{array}{c} a_{i,j} \end{array}\right]_{n \times n}$ *be a triangular matrix in* $\mathbb{F}^{n \times n}$*. Then*

$$det(A) \quad = \quad \prod_{i=1}^{n} a_{i,i} \quad = \quad a_{1,1} a_{2,2} \dots a_{n,n},$$

that is, $det(A)$ *is equal to the product of entries on the main diagonal of* A*.*

Theorem 7.3.2. *Let* $\mathbb{F}$ *be a field, and let* $A = \left[\begin{array}{c} a_{i,j} \end{array}\right]_{n \times n}$ *be a matrix in* $\mathbb{F}^{n \times n}$*. Then all the following hold:*

(a) if a matrix B *is obtained by swapping two rows or swapping two columns of* A*, then*

$$det(B) \quad = \quad -det(A);$$

(b) if a matrix B *is obtained by multiplying some row or some column of* A *by a scalar* $\alpha \in \mathbb{F} \setminus \{0\}$*, then*

$$det(B) \quad = \quad \alpha det(A) \qquad \text{and} \qquad det(A) \quad = \quad \alpha^{-1} det(B);$$

(c) if a matrix B *is obtained from* A *by adding a scalar multiple of one row (resp. column) of* A *to another row (resp. column) of* A*, then*

$$det(B) \quad = \quad det(A).$$

Definition. *For a matrix* $A = \begin{bmatrix} a_{i,j} \end{bmatrix}_{n\times n}$ *(where* $n \geq 2$*) with entries in some field* $\mathbb{F}$*, and for indices* $p, q \in \{1, \dots, n\}$*,* $A_{p,q}$ *is the* $(n-1)\times(n-1)$ *matrix obtained from* A *by deleting the* p*-th row and* q*-th column.*

Laplace expansion. *Let* $\mathbb{F}$ *be a field, and let* $A = \begin{bmatrix} a_{i,j} \end{bmatrix}_{n\times n}$ *(where* $n \geq 2$*) be a matrix in* $\mathbb{F}^{n\times n}$*. Then both the following hold:*

(a) ***[expansion along the*** i***-th row]*** *for all* $i \in \{1, \dots, n\}$*, we have that*

$$det(A) \quad = \quad \sum_{j=1}^{n} (-1)^{i+j}\ a_{i,j}\ det(A_{i,j});$$

(b) ***[expansion along the*** j***-th column]*** *for all* $j \in \{1, \dots, n\}$*, we have that*

$$det(A) \quad = \quad \sum_{i=1}^{n} (-1)^{i+j}\ a_{i,j}\ det(A_{i,j}).$$

Theorem 7.5.2. *Let* $\mathbb{F}$ *be a field, and let* $A, B \in \mathbb{F}^{n\times n}$*. Then*

$$det(AB) \quad = \quad det(A)det(B).$$

Exercise 1. *Using the* ***definition*** *of a determinant (with permutations), compute the determinant of the real matrix below.*

$$A \quad = \quad \begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

Exercise 2. *Compute the determinant of the following* ***permutation*** *matrix (with entries understood to be in* $\mathbb{R}$*):*

$$A \quad = \quad \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Exercise 3. *Consider the matrix* A *below, with entries understood to be in* $\mathbb{R}$*. Determine whether* $det(A)$ *is positive, negative, or zero. Can you tell which it is without actually computing the determinant?*

$$A \quad = \quad \begin{bmatrix} 1 & 1000 & 2 & 3 \\ 1000 & -3 & 5 & 0 \\ 2 & 3 & 5 & 1000 \\ 1 & 2 & 1000 & 4 \end{bmatrix}$$

Exercise 4. *Compute the determinants of the following 3×3 real matrices.*

1. $A_1 = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 8 & 16 \\ 0 & 0 & 32 \end{bmatrix}$;

2. $A_2 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$;

3. $A_3 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 3 & 6 & 3 \end{bmatrix}$;

4. $A_4 = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 2 & 4 & 5 \end{bmatrix}$;

5. $A_5 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 2 & 5 \end{bmatrix}$;

6. $A_6 = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 2 & -2 \\ 3 & 4 & 1 \end{bmatrix}$;

Exercise 5. *Let $a, b, c, d, e, f, g, h, i$ be real numbers such that*

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7.$$

Compute the following determinants.

1. $\begin{vmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{vmatrix}$

2. $\begin{vmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{vmatrix}$

3. $\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix}$

4. $\begin{vmatrix} g & h & i \\ a & b & c \\ d & e & f \end{vmatrix}$

5. $\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix}$

6. $\begin{vmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{vmatrix}$

Exercise 6. *Compute the determinant of the matrix below, with entries understood to be in $\mathbb{Z}_3$.*

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

Exercise 7. *Compute the determinant of the following real matrix:*

$$A = \begin{bmatrix} 0 & 1 & 2 & 0 & 3 \\ -2 & 2 & 1 & 0 & -1 \\ 2 & -2 & 1 & 3 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 3 & 1 & -3 & 0 & 2 \end{bmatrix}.$$

Start via Laplace expansion along the most convenient row or column, and then continue using any method you like.

Exercise 8. *Using determinants, determine for which (if any) values of k the matrix A (below) is invertible. (Entries of A are understood to be in $\mathbb{R}$).*

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & k & 1 \\ 3 & 1 & 0 \end{bmatrix}$$

Exercise 9. *Compute the determinants of the real matrices below.*

- $A_1 = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 5 \end{bmatrix}$
- $A_2 = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 4 & 6 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix}$
- $A_3 = \begin{bmatrix} 1 & 0 & -1 & 3 \\ 2 & 0 & -2 & 4 \\ 3 & 3 & -3 & 3 \\ 4 & 4 & -4 & 4 \end{bmatrix}$
- $A_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$
- $A_5 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{bmatrix}$
- $A_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 & 5 \end{bmatrix}$

Exercise 10. *Let A be an $n \times n$ matrix of the following form:*

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & a_{1,n} \\ 0 & 0 & \cdots & a_{2,n-1} & a_{2,n} \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & a_{n-1,2} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n-1} & a_{n,n} \end{bmatrix}$$

Find a formula for det(A).

Exercise 11. *Let $A \in \mathbb{R}^{n\times n}$. If det$(A) = 3$, then what is det$(A^T A)$?*

Exercise 12. *Let $A \in \mathbb{R}^{n\times n}$. What can you say about the sign of det$(A^T A)$?*

Definition. *Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times n}$.*

- *A is* symmetric *if $A^T = A$.*
- *A is* skew-symmetric *if $A^T = -A$.*

Exercise 13. *Let $\mathbb{F}$ be a field of characteristic other than 2, i.e. assume that in the field $\mathbb{F}$, $1 + 1 \neq 0$.*

(a) Show that if n is odd positive integer, then any skew-symmetric matrix $A \in \mathbb{F}^{n\times n}$ is non-invertible. (Use determinants.)

(b) For each even positive integer n, construct an invertible skew-symmetric matrix in $\mathbb{F}^{n\times n}$.