Linear Algebra 2: Tutorial 5

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Corollary 6.5.3. *Let V be a finite-dimensional real or complex vector space, equipped with a scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $|| \cdot ||$. Let U be a subspace of V, and let $\mathbf{x} \in V$. Then*

$$\mathbf{x} \quad = \quad \mathbf{x}_U + \mathbf{x}_{U^\perp}.$$

Moreover, this is the unique way of expressing $\mathbf{x}$ as a sum of a vector in U and a vector in $U^\perp$.[1]

Theorem 6.6.1. *Let $A \in \mathbb{R}^{n \times m}$. Then $Row(A)^\perp = Nul(A)$ and $Row(A) = Nul(A)^\perp$.*

Theorem 6.6.3. *Let $A \in \mathbb{R}^{n \times m}$ be a matrix of rank m (i.e. A is a matrix of full column rank). Then the matrix $A(A^T A)^{-1} A^T$ is the standard matrix of orthogonal projection onto $Col(A)$, that is, for all $\mathbf{x} \in \mathbb{R}^n$, the orthogonal projection of $\mathbf{x}$ onto $C := Col(A)$ is given by*

$$\mathbf{x}_C \quad = \quad A(A^T A)^{-1} A^T \mathbf{x}.$$

Theorem 6.7.1. *Let $A \in \mathbb{R}^{n \times m}$ and $\mathbf{b} \in \mathbb{R}^n$. Then the matrix-vector equation*

$$A^T A \mathbf{x} \quad = \quad A^T \mathbf{b}$$

is consistent, and moreover, its solution set is precisely the set of vectors $\mathbf{x}$ in $\mathbb{R}^m$ that minimize the expression

$$||A\mathbf{x} - \mathbf{b}||.$$

The Interpolation Theorem. *For all pairwise distinct $x_0, x_1, \dots, x_n \in \mathbb{R}$ and all (not necessarily distinct) $y_0, y_1, \dots, y_n \in \mathbb{R}$, there exists a unique polynomial $p(x)$ of degree at most n such that $p(x_i) = y_i$ for all $i = 0, 1, \dots, n$.*

> ***Remark:*** *This theorem essentially states that if we are given $n + 1$ data points $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$ (with $x_0, x_1, \dots, x_n$ pairwise distinct), then there is a unique polynomial of degree at most n whose graph passes through those data points.*

Proof. Omitted. □

[1] This means that for all $\mathbf{y} \in U$ and $\mathbf{z} \in U^\perp$, if $\mathbf{x} = \mathbf{y} + \mathbf{z}$, then $\mathbf{y} = \mathbf{x}_U$ and $\mathbf{z} = \mathbf{x}_{U^\perp}$.

Exercise 2 of HW#1. *Consider the following sets of vectors (with entries understood to be in* $\mathbb{Z}_3$*):*

- $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$;
- $\mathcal{C} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Prove that $\mathcal{B}$ *and* $\mathcal{C}$ *are bases of* $\mathbb{Z}_3^3$*, and compute the change of basis matrix* ${}_{\mathcal{C}}\Big[\ Id_{\mathbb{Z}_3^3}\ \Big]_{\mathcal{B}}$.

Problem 1 of HW#1. *Let* U *and* V *be non-trivial, finite-dimensional vector spaces over a field* $\mathbb{F}$*, and let* $f : U \to V$ *be a linear function. Prove that the following are equivalent:*

(1) f *is an isomorphism;*

(2) there exists a positive integer n*, a basis* $\mathcal{B}$ *of* U*, and a basis* $\mathcal{C}$ *of* V *such that* ${}_{\mathcal{C}}\big[\ f\ \big]_{\mathcal{B}} = I_n$.

Exercise 1. *Let*

$$A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1^T \\ \vdots \\ \mathbf{r}_n^T \end{bmatrix}$$

be a matrix in $\mathbb{R}^{n\times m}$*. So, we have that* $Col(A) = Span(\mathbf{a}_1, \dots, \mathbf{a}_m)$ *and* $Row(A) = Col(A^T) = Span(\mathbf{r}_1, \dots, \mathbf{r}_n)$*. Prove that the function* $f : Row(A) \to Col(A)$ *given by* $f(\mathbf{x}) = A\mathbf{x}$ *for all* $\mathbf{x} \in Row(A)$ *is*

(a) well-defined (i.e. that for all $\mathbf{x} \in Row(A)$*, we have that* $f(\mathbf{x}) \in Col(A)$*), and*

(b) an isomorphism.

> ***Remark:*** *Prove this* ***without*** *relying on rank. Instead, you should use Corollary 6.5.3 and 6.6.1 (and possibly some other results, none of which should rely on rank). This should be an alternative proof of the fact that* $Row(A) \cong Col(A)$ *when* A *is a* ***real*** *matrix.*[2]
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> ***Hint:*** *For one-to-oneness, examine* $Ker(f)$*. For ontoness, fix* $\mathbf{b} \in Col(A)$*, and explain why there exists some* $\mathbf{x} \in \mathbb{R}^m$ *such that* $A\mathbf{x} = \mathbf{b}$*; then use Corollary 6.5.3 and Theorem 6.6.1 to show that there exists some* $\mathbf{x}_R \in Row(A)$ *such that* $A\mathbf{x}_R = \mathbf{b}$.

Exercise 2. *Exercise 1(b) crucially relies on the fact that* A *is a* ***real*** *matrix, and it becomes false if we replace* $\mathbb{R}$ *by an arbitrary field* $\mathbb{F}$*. Come up with a counterexample for* $\mathbb{F} = \mathbb{Z}_2$.

> ***Remark/Hint:*** *There are some* ***very*** *simple examples.*

[2] Here's our original proof. By Corollary 3.3.11(a), we have that $\dim(\text{Col}(A)) = \dim(\text{Row}(A)) = \text{rank}(A)$. Since finite-dimensional vector spaces (over the same field) are isomorphic if and only if they have the same dimension, this implies that $\text{Row}(A) \cong \text{Col}(A)$.

Exercise 3. *In what follows, we assume that* $\mathbb{R}^3$ *is equipped with the standard scalar product* $\cdot$ *and the induced norm* $||\cdot||$. *Consider the following vectors in* $\mathbb{R}^3$*:*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Set $C := Span(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

(a) Compute the standard matrix P *of the orthogonal projection* $proj_C : \mathbb{R}^3 \to \mathbb{R}^3$ *onto* C.

> ***Warning:*** *The matrix* $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$ *does* ***not*** *have rank* 3. *So, you cannot use Theorem 6.6.3 directly.*

(b) Using the matrix P *from part (a), compute the vector* $\mathbf{x}_C$ *(the projection of* $\mathbf{x}$ *onto* C*). Does* $\mathbf{x}$ *belong to* C*?*

Exercise 4. *Find a polynomial of degree at most* 3 *that passes through the points* $(1,3)$, $(-1,13)$, $(2,1)$, $(-2,33)$.

Exercise 5. *Fit a linear function of the the form* $f(x) = ax + b$ *to the data points* $(0,0)$, $(0,1)$, $(1,1)$ *using least squares.*

Exercise 6. *The following table gives world population in 10-year intervals, starting with 1950 and ending with 2000.*

year	*population (in* 10^9*)*
1950	*2.519*
1960	*2.982*
1970	*3.692*
1980	*4.435*
1990	*5.263*
2000	*6.070*

We would like to use the least-squares method to find the linear function that best fits these data. Set up the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ *to which we need to apply the least-squares method.*

Exercise 7. *Using the least-squares method, fit a quadratic function to the four data points* $(a_1, b_1) = (-1, 8)$, $(a_2, b_2) = (0, 8)$, $(a_3, b_3) = (1, 4)$, *and* $(a_4, b_4) = (2, 16)$.

Exercise 8. *Explain how one would a trigonometric function of the form* $f(t) = c_0 + c_1 \sin t + c_2 \cos t$ *that best fits the data points* $(0,0)$, $(1,1)$, $(2,2)$, $(3,3)$.

> ***Remark:*** *Here, you should set up the matrix-vector equation* $A\mathbf{c} = \mathbf{b}$ *(where* A *and* $\mathbf{b}$ *are known and* $\mathbf{c}$ *is the unknown) to which the least-squares method should be applied. Don't compute the actual least-squares solution.*