Linear Algebra 2: Tutorial 4

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Remark: Unless explicitly specified otherwise,[1] in the following exercises, $\mathbb{R}^n$ is understood to be equipped with the standard scalar product $\cdot$ and the induced norm $||\cdot||$, and in particular, orthogonality and orthonormality in $\mathbb{R}^n$ are understood to be with respect to the standard scalar product and the induced norm.

Exercise 1.

(a) Let V be a real vector space, equipped with the scalar product $\langle\cdot,\cdot\rangle$ and the induced norm $||\cdot||$, and let $\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be an orthonormal basis of V. Let $\cdot$ be the standard scalar product in $\mathbb{R}^n$. Prove that for all $\mathbf{x}, \mathbf{y} \in V$, we have that

$$\langle \mathbf{x}, \mathbf{y} \rangle \;=\; \left[\ \mathbf{x}\ \right]_{\mathcal{B}} \cdot \left[\ \mathbf{y}\ \right]_{\mathcal{B}}.$$

***Hint:** The first thing you should do is find a formula for $\left[\ \mathbf{x}\ \right]_{\mathcal{B}}$ and $\left[\ \mathbf{y}\ \right]_{\mathcal{B}}$ using the orthonormal basis $\mathcal{B}$ and the scalar product $\langle\cdot,\cdot\rangle$ in V.*

(b) Same question as in part (a), only for a complex vector space V.

Exercise 2. *Consider the following linearly independent vectors in $\mathbb{R}^4$:*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix}.$$

Compute an orthonormal basis of $U := Span(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ in two different ways: using Gram-Schmidt orthogonalization (version 1) and using Gram-Schmidt orthogonalization (version 2).

Exercise 3. *Suppose that V is a real or complex vector space, equipped with a scalar product $\langle\cdot,\cdot\rangle$ and the induced norm $||\cdot||$, and suppose that $\mathbf{v}_1, \dots, \mathbf{v}_k$ are linearly independent vectors in V. Now suppose that you perform the two versions of Gram-Schmidt orthogonalization on the vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ in order to obtain orthonormal bases of $U := Span(\mathbf{v}_1, \dots, \mathbf{v}_k)$. Under what circumstances will your two orthonormal bases be different? (Will they ever be different?) Justify your answer.*

[1] In some exercises, it is indeed specified otherwise.

Exercise 4. *Consider the vectors* $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ *from Exercise 2. How would you obtain an orthonormal basis for* $U := \mathit{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ *that is* ***different*** *from the one(s) that you obtained in Exercise 2?*

> ***Remark:*** *Your basis should differ from the one(s) from Exercise 2 not merely in the order in which the vectors appear in the basis, but also in terms of the actual vectors that the basis contains. Don't perform the whole calculation: just explain how you would compute.*

Exercise 5. *Suppose that* V *is a real or complex vector space, equipped with a scalar product* $\langle \cdot, \cdot \rangle$ *and the induced norm* $|| \cdot ||$*. Suppose that* $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ *is an orthogonal set of non-zero vectors in* V*. What do you obtain if you perform the Gram-Schmidt orthogonalization process (version 1) on* $\mathbf{v}_1, \dots, \mathbf{v}_k$ ***without*** *normalizing at the end? And what happens if you do normalize at the end? What happens if you perform the Gram-Schmidt orthogonalization process (version 2) on* $\mathbf{v}_1, \dots, \mathbf{v}_k$*?*

Exercise 6. *Consider the following vectors in* $\mathbb{R}^5$*:*

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 9 \\ 27 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} -2 \\ 4 \\ -6 \\ 8 \\ 0 \end{bmatrix}.$$

Compute an orthonormal basis of $U := \mathit{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$*. Can you tell what the answer will be without actually performing Gram-Schmidt orthogonalization (either version)?*

Exercise 7. *Consider the scalar product* $\langle \cdot, \cdot \rangle$ *on* $\mathbb{R}^2$ *defined by*

$$\langle \mathbf{x}, \mathbf{y} \rangle \quad = \quad x_1 y_1 + 2 x_2 y_2$$

for all $\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ *and* $\mathbf{y} = \begin{bmatrix} y_1 & y_2 \end{bmatrix}^T$ *in* $\mathbb{R}^2$*. (You may assume that* $\langle \cdot, \cdot \rangle$ *really is a scalar product in* $\mathbb{R}^2$*.) Consider the following vectors in* $\mathbb{R}^2$*:*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Explain why $\{\mathbf{v}_1, \mathbf{v}_2\}$ *is a basis of* $\mathbb{R}^2$*, and then perform Gram-Schmidt orthogonalization with input* $\mathbf{v}_1, \mathbf{v}_2$ *to obtain an orthonormal basis of* $\mathbb{R}^2$*, where orthonormality is understood to be with respect to our scalar product* $\langle \cdot, \cdot \rangle$ *and norm* $|| \cdot ||$ *induced by it.*

> ***Remark:*** *Note that* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *are orthogonal with respect to the standard scalar product* $\cdot$*, but they are* ***not*** *orthogonal with respect to the scalar product* $\langle \cdot, \cdot \rangle$ *defined above.*