Linear Algebra 2: Tutorial 3

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Summer 2025

Exercises 1 of Tutorial 2. *Which (if any) of the following functions* $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ *are scalar products in* $\mathbb{R}^2$*?*

(a) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 - x_2 y_2$ *for all* $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ *and* $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ *in* $\mathbb{R}^2$*;*

(b) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + 2 x_2 y_2$ *for all* $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ *and* $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ *in* $\mathbb{R}^2$*;*

(c) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 x_2 + y_1 y_2$ *for all* $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ *and* $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ *in* $\mathbb{R}^2$*.*

Make sure you justify your answer in each case. You should refer to the definition of the scalar product in a real vector space (above).

Exercises 3 of Tutorial 2. *Find the set of all vectors that are (simultaneously) orthogonal to the following three vectors in* $\mathbb{R}^4$ *(where orthogonality is assumed to be with respect to the standard scalar product* $\cdot$ *in* $\mathbb{R}^4$*):*

$$\mathbf{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}.$$

Exercises 4 of Tutorial 2. *Generalize your answer to Exercise 3 of Tutorial 2. More precisely, suppose that you are given vectors* $\mathbf{a}_1, \dots, \mathbf{a}_k$ *in* $\mathbb{R}^n$*. How would you compute the set of all vectors in* $\mathbb{R}^n$ *that are simultaneously orthogonal (with respect to the standard scalar product* $\cdot$*) to the vectors* $\mathbf{a}_1, \dots, \mathbf{a}_k$*? Must the set that you obtain be a subspace of* $\mathbb{R}^n$*?*

Exercise 1. *Suppose that V is a real of complex vector space, equipped with a scalar product $\langle \cdot, \cdot \rangle$, and suppose that $\mathbf{u} \in V \setminus \{\mathbf{0}\}$.*

(a) If $\mathbf{v} \in V$ is a scalar multiple of $\mathbf{u}$, what is $proj_{\mathbf{u}}(\mathbf{v})$?

(b) If $\mathbf{w} \in V$ is orthogonal to $\mathbf{u}$, what is $proj_{\mathbf{u}}(\mathbf{w})$?

In both parts, justify your answer formally (using the definition of orthogonal projection), and draw a picture to give the intuition behind your answer.

Exercise 2. *Let V be a real vector space, equipped with a scalar product $\langle \cdot, \cdot \rangle$ and the norm $|| \cdot ||$ induced by $\langle \cdot, \cdot \rangle$. Prove that if vectors $\mathbf{x}, \mathbf{y} \in V$ satisfy*

$$||\mathbf{x} + \mathbf{y}||^2 \quad = \quad ||\mathbf{x}||^2 + ||\mathbf{y}||^2,$$

then they are orthogonal.

> ***Remark:*** *So, you are asked to prove the converse of the Pythagorean theorem for the case of* ***real*** *vector spaces.*

Exercise 3. *Consider the standard scalar product $\cdot$ on $\mathbb{C}^2$, and the induced norm $|| \cdot ||$. Find* ***non-orthogonal*** *vectors $\mathbf{x}, \mathbf{y} \in \mathbb{C}^2$ that satisfy*

$$||\mathbf{x} + \mathbf{y}||^2 \quad = \quad ||\mathbf{x}||^2 + ||\mathbf{y}||^2.$$

> ***Remark:*** *This shows that the converse of the Pythagorean theorem need not hold for* ***complex*** *vector spaces.*

Exercise 4. *By carefully examining the proof of the Cauchy-Schwarz inequality, determine when that inequality becomes an equality.*