Linear Algebra 2: Tutorial 2

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Exercises 3 of Tutorial 1.

(a) Prove that there exists a linear function $f : \mathbb{Z}_2^{2\times 2} \to \mathbb{Z}_2^3$ *that satisfies the following properties:*

- $f(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T$;
- $f(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$;
- $f(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$.

(b) Is the linear function f *from part (a) unique?*

(c) Can a linear function f *satisfying the properties from part (a) be one-to-one? Can it be onto?*

(d) Find a formula for some linear function f *satisfying the properties from part (a). Can you find more than one correct answer? Can you come up with examples of different rank?*

Definition. *A* scalar product *(also called* inner product*) in a real vector space* V *is a function* $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ *that satisfies the following four axioms:*

r.1. for all $\mathbf{x} \in V$, $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, *and equality holds if and only if* $\mathbf{x} = \mathbf{0}$;

r.2. for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$;

r.3. for all $\mathbf{x}, \mathbf{y} \in V$ *and* $\alpha \in \mathbb{R}$, $\langle \alpha\mathbf{x}, \mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle$;

r.4. for all $\mathbf{x}, \mathbf{y} \in V$, $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$.

Exercise 1. *Which (if any) of the following functions* $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ *are scalar products in* $\mathbb{R}^2$*?*

(a) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 - x_2 y_2$ *for all* $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ *and* $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ *in* $\mathbb{R}^2$*;*

(b) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + 2 x_2 y_2$ *for all* $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ *and* $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ *in* $\mathbb{R}^2$*;*

(c) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 x_2 + y_1 y_2$ *for all* $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ *and* $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ *in* $\mathbb{R}^2$.

Make sure you justify your answer in each case. You should refer to the definition of the scalar product in a real vector space (above).

Definition. *The* trace *of a square matrix* $A = \left[\ a_{i,j}\ \right]_{n \times n}$ *is defined to be*

$$\mathit{trace}(A) \quad := \quad \sum_{i=1}^{n} a_{i,i} \quad = \quad a_{1,1} + a_{2,2} + \cdots + a_{n,n}.$$

In other words, the trace of a square matrix is the sum of its entries on the main diagonal.[1]

Exercise 2. *Consider the function* $\langle \cdot, \cdot \rangle : \mathbb{R}^{n \times m} \times \mathbb{R}^{n \times m} \to \mathbb{R}$ *given by*

$$\langle A, B \rangle \quad = \quad \mathit{trace}\big(A^T B\big)$$

for all $A, B \in \mathbb{R}^{n \times m}$*. Prove that* $\langle \cdot, \cdot \rangle$ *is a scalar product on* $\mathbb{R}^{n \times m}$*.*

Exercise 3. *Find the set of all vectors that are (simultaneously) orthogonal to the following three vectors in* $\mathbb{R}^4$ *(where orthogonality is assumed to be with respect to the standard scalar product* $\cdot$ *in* $\mathbb{R}^4$*):*

$$\mathbf{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \qquad \mathbf{u}_3 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}.$$

Exercise 4. *Generalize your answer to Exercise 3. More precisely, suppose that you are given vectors* $\mathbf{a}_1, \ldots, \mathbf{a}_k$ *in* $\mathbb{R}^n$*. How would you compute the set of all vectors in* $\mathbb{R}^n$ *that are simultaneously orthogonal (with respect to the standard scalar product* $\cdot$*) to the vectors* $\mathbf{a}_1, \ldots, \mathbf{a}_k$*? Must the set that you obtain be a subspace of* $\mathbb{R}^n$*?*

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[1] For example, $\mathrm{trace}\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\right) = 1 + 5 + 9 = 15.$