

# Linear Algebra 2

## Lecture #23

### Diagonalization

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- This lecture has two parts:

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  - 1 Eigenvectors and linear independence. Eigenbases

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  - ① Eigenvectors and linear independence. Eigenbases
  - ② Diagonalization

## ① Eigenvectors and linear independence. Eigenbases

## 1 Eigenvectors and linear independence. Eigenbases

### Definition

For a finite-dimensional vector space  $V$  over a field  $\mathbb{F}$  and a linear function  $f : V \rightarrow V$ , an *eigenbasis* of  $V$  associated with  $f$  is a basis  $\mathcal{B}$  of  $V$  s.t. all vectors in  $\mathcal{B}$  are eigenvectors of  $f$ .

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For an field  $\mathbb{F}$  and a matrix  $A \in \mathbb{F}^{n \times n}$ , an *eigenbasis* of  $\mathbb{F}^n$  associated with  $A$  is a basis  $\mathcal{B}$  of  $\mathbb{F}^n$  s.t. all vectors in  $\mathcal{B}$  are eigenvectors of  $A$ .

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## ① Eigenvectors and linear independence. Eigenbases

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- Eigenbases do not always exist, and one of our goals in this section is to determine when they do and do not exist.
- As we shall see (later!), eigenbases play a crucial role in matrix “diagonalization.”



### Proposition 8.4.1

Let  $V$  be a vector space over a field  $\mathbb{F}$ , let  $f : V \rightarrow V$  be a linear function, let  $\lambda_1, \dots, \lambda_k \in \mathbb{F}$  be pairwise distinct eigenvalues of  $f$ , associated with eigenvectors  $\mathbf{v}_1, \dots, \mathbf{v}_k$ , respectively. Then  $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  is a linearly independent set.

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Now, fix an index  $i \in \{0, \dots, k-1\}$ , and assume inductively that the set  $\{\mathbf{v}_1, \dots, \mathbf{v}_i\}$  is linearly independent. We must show that  $\{\mathbf{v}_1, \dots, \mathbf{v}_i, \mathbf{v}_{i+1}\}$  is linearly independent.

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$$\alpha_1 \mathbf{v}_1 + \dots + \alpha_i \mathbf{v}_i + \alpha_{i+1} \mathbf{v}_{i+1} = \mathbf{0}.$$

WTS  $\alpha_1 = \dots = \alpha_i = \alpha_{i+1} = 0$ .

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*Proof (continued).* Reminder:  $\{\mathbf{v}_1, \dots, \mathbf{v}_i\}$  is linearly independent;  
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If we multiply both sides of the **equation** above by  $\lambda_{i+1}$ , we obtain

$$\textcircled{1} \quad \lambda_{i+1} \alpha_1 \mathbf{v}_1 + \dots + \lambda_{i+1} \alpha_i \mathbf{v}_i + \lambda_{i+1} \alpha_{i+1} \mathbf{v}_{i+1} = \mathbf{0}.$$

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If, on the other hand, we apply the function  $f$  to **both sides** and also use the fact that  $f(\mathbf{0}) = \mathbf{0}$ , then we obtain

$$\textcircled{2} \quad f(\alpha_1 \mathbf{v}_1 + \dots + \alpha_i \mathbf{v}_i + \alpha_{i+1} \mathbf{v}_{i+1}) = \mathbf{0}.$$



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*Proof (continued).* We now compute:

$$\begin{aligned} 0 &\stackrel{(2)}{=} f(\alpha_1 \mathbf{v}_1 + \dots + \alpha_i \mathbf{v}_i + \alpha_{i+1} \mathbf{v}_{i+1}) \\ &\stackrel{(*)}{=} \alpha_1 f(\mathbf{v}_1) + \dots + \alpha_i f(\mathbf{v}_i) + \alpha_{i+1} f(\mathbf{v}_{i+1}) \\ &\stackrel{(**)}{=} \alpha_1 \lambda_1 \mathbf{v}_1 + \dots + \alpha_i \lambda_i \mathbf{v}_i + \alpha_{i+1} \lambda_{i+1} \mathbf{v}_{i+1}, \end{aligned}$$

where  $(*)$  follows from the linearity of  $f$ , and  $(**)$  follows from the fact that  $\mathbf{v}_1, \dots, \mathbf{v}_i, \mathbf{v}_{i+1}$  are eigenvectors of  $f$  associated with eigenvalues  $\lambda_1, \dots, \lambda_i, \lambda_{i+1}$ , respectively.

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*Proof (continued).* Reminder:

- ①  $\lambda_{i+1}\alpha_1\mathbf{v}_1 + \dots + \lambda_{i+1}\alpha_i\mathbf{v}_i + \lambda_{i+1}\alpha_{i+1}\mathbf{v}_{i+1} = \mathbf{0};$
- ②  $f(\alpha_1\mathbf{v}_1 + \dots + \alpha_i\mathbf{v}_i + \alpha_{i+1}\mathbf{v}_{i+1}) = \mathbf{0};$
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Combining (1) and (3), we obtain:

$$\begin{aligned} & \alpha_1\lambda_1\mathbf{v}_1 + \dots + \alpha_i\lambda_i\mathbf{v}_i + \alpha_{i+1}\lambda_{i+1}\mathbf{v}_{i+1} \\ = & \lambda_{i+1}\alpha_1\mathbf{v}_1 + \dots + \lambda_{i+1}\alpha_i\mathbf{v}_i + \lambda_{i+1}\alpha_{i+1}\mathbf{v}_{i+1}. \end{aligned}$$

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By subtracting one side from the other and factoring, we get

$$\alpha_1(\lambda_1 - \lambda_{i+1})\mathbf{v}_1 + \dots + \alpha_i(\lambda_i - \lambda_{i+1})\mathbf{v}_i = \mathbf{0}.$$

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$$\alpha_{i+1}\mathbf{v}_{i+1} = \mathbf{0}.$$

But  $\mathbf{v}_{i+1}$  is an eigenvector of  $f$ , and so by definition,  $\mathbf{v}_{i+1} \neq \mathbf{0}$ . So,  $\alpha_{i+1} = 0$ . Thus,  $\alpha_1 = \dots = \alpha_i = \alpha_{i+1} = 0$ .

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Let  $V$  be a vector space over a field  $\mathbb{F}$ , let  $f : V \rightarrow V$  be a linear function, let  $\lambda_1, \dots, \lambda_k \in \mathbb{F}$  be pairwise distinct eigenvalues of  $f$ , associated with eigenvectors  $\mathbf{v}_1, \dots, \mathbf{v}_k$ , respectively. Then  $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  is a linearly independent set.

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Let  $V$  be a vector space over a field  $\mathbb{F}$ , let  $f : V \rightarrow V$  be a linear function, let  $\lambda_1, \dots, \lambda_k \in \mathbb{F}$  be pairwise distinct eigenvalues of  $f$ , associated with eigenvectors  $\mathbf{v}_1, \dots, \mathbf{v}_k$ , respectively. Then  $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  is a linearly independent set.

### Proposition 8.4.2

Let  $V$  be a vector space over a field  $\mathbb{F}$ , let  $f : V \rightarrow V$  be a linear function, and let  $\lambda_1, \lambda_2, \dots, \lambda_k \in \mathbb{F}$  be pairwise distinct eigenvalues of  $f$ . For each  $i \in \{1, \dots, k\}$ , let  $\mathbf{v}_{i,1}, \dots, \mathbf{v}_{i,t_i}$  be linearly independent eigenvectors of  $f$  associated with the eigenvalue  $\lambda_i$ . Then the eigenvectors

$\mathbf{v}_{1,1}, \dots, \mathbf{v}_{1,t_1}, \mathbf{v}_{2,1}, \dots, \mathbf{v}_{2,t_2}, \dots, \mathbf{v}_{k,1}, \dots, \mathbf{v}_{k,t_k}$   
are linearly independent.

*Proof.*

*Proof.* Fix  $\alpha_{1,1}, \dots, \alpha_{1,t_1}, \alpha_{2,1}, \dots, \alpha_{2,t_2}, \dots, \alpha_{k,1}, \dots, \alpha_{k,t_k} \in \mathbb{F}$   
s.t.

$$\sum_{i=1}^k \left( \alpha_{i,1} \mathbf{v}_{i,1} + \dots + \alpha_{i,t_i} \mathbf{v}_{i,t_i} \right) = \mathbf{0}.$$



*Proof.* Fix  $\alpha_{1,1}, \dots, \alpha_{1,t_1}, \alpha_{2,1}, \dots, \alpha_{2,t_2}, \dots, \alpha_{k,1}, \dots, \alpha_{k,t_k} \in \mathbb{F}$   
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$$\sum_{i=1}^k \left( \alpha_{i,1} \mathbf{v}_{i,1} + \dots + \alpha_{i,t_i} \mathbf{v}_{i,t_i} \right) = \mathbf{0}.$$

$\forall i \in \{1, \dots, k\}$ : set  $\mathbf{v}_i := \alpha_{i,1} \mathbf{v}_{i,1} + \dots + \alpha_{i,t_i} \mathbf{v}_{i,t_i}$ , that is

- $\mathbf{v}_1 := \alpha_{1,1} \mathbf{v}_{1,1} + \dots + \alpha_{1,t_1} \mathbf{v}_{1,t_1};$
- $\mathbf{v}_2 := \alpha_{2,1} \mathbf{v}_{2,1} + \dots + \alpha_{2,t_2} \mathbf{v}_{2,t_2};$
- $\vdots$
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*Proof.* Fix  $\alpha_{1,1}, \dots, \alpha_{1,t_1}, \alpha_{2,1}, \dots, \alpha_{2,t_2}, \dots, \alpha_{k,1}, \dots, \alpha_{k,t_k} \in \mathbb{F}$   
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So,  $\mathbf{v}_1 + \mathbf{v}_2 + \dots + \mathbf{v}_k = \mathbf{0}$ . Now, note that for each  $i \in \{1, \dots, k\}$ , the vector  $\mathbf{v}_i$  is a linear combination of vectors in  $E_{\lambda_i}(f)$ ;

*Proof.* Fix  $\alpha_{1,1}, \dots, \alpha_{1,t_1}, \alpha_{2,1}, \dots, \alpha_{2,t_2}, \dots, \alpha_{k,1}, \dots, \alpha_{k,t_k} \in \mathbb{F}$   
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So,  $\mathbf{v}_1 + \mathbf{v}_2 + \dots + \mathbf{v}_k = \mathbf{0}$ . Now, note that for each  $i \in \{1, \dots, k\}$ , the vector  $\mathbf{v}_i$  is a linear combination of vectors in  $E_{\lambda_i}(f)$ ; since  $E_{\lambda_i}(f)$  is a subspace of  $V$ , it follows that  $\mathbf{v}_i \in E_{\lambda_i}(f)$ .

*Proof.* Fix  $\alpha_{1,1}, \dots, \alpha_{1,t_1}, \alpha_{2,1}, \dots, \alpha_{2,t_2}, \dots, \alpha_{k,1}, \dots, \alpha_{k,t_k} \in \mathbb{F}$   
s.t.

$$\sum_{i=1}^k \left( \alpha_{i,1} \mathbf{v}_{i,1} + \dots + \alpha_{i,t_i} \mathbf{v}_{i,t_i} \right) = \mathbf{0}.$$

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So,  $\mathbf{v}_1 + \mathbf{v}_2 + \dots + \mathbf{v}_k = \mathbf{0}$ . Now, note that for each  $i \in \{1, \dots, k\}$ , the vector  $\mathbf{v}_i$  is a linear combination of vectors in  $E_{\lambda_i}(f)$ ; since  $E_{\lambda_i}(f)$  is a subspace of  $V$ , it follows that  $\mathbf{v}_i \in E_{\lambda_i}(f)$ . Consequently,  $\forall i \in \{1, \dots, k\}$ :  $\mathbf{v}_i$  is either  $\mathbf{0}$  or an eigenvector of  $f$  associated with the eigenvalue  $\lambda_i$ . WTS  $\mathbf{v}_1 = \mathbf{v}_2 = \dots = \mathbf{v}_k = \mathbf{0}$ .

*Proof.* Fix  $\alpha_{1,1}, \dots, \alpha_{1,t_1}, \alpha_{2,1}, \dots, \alpha_{2,t_2}, \dots, \alpha_{k,1}, \dots, \alpha_{k,t_k} \in \mathbb{F}$   
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So,  $\mathbf{v}_1 + \mathbf{v}_2 + \dots + \mathbf{v}_k = \mathbf{0}$ . Now, note that for each  $i \in \{1, \dots, k\}$ , the vector  $\mathbf{v}_i$  is a linear combination of vectors in  $E_{\lambda_i}(f)$ ; since  $E_{\lambda_i}(f)$  is a subspace of  $V$ , it follows that  $\mathbf{v}_i \in E_{\lambda_i}(f)$ . Consequently,  $\forall i \in \{1, \dots, k\}$ :  $\mathbf{v}_i$  is either  $\mathbf{0}$  or an eigenvector of  $f$  associated with the eigenvalue  $\lambda_i$ . WTS  $\mathbf{v}_1 = \mathbf{v}_2 = \dots = \mathbf{v}_k = \mathbf{0}$ . Suppose otherwise.

*Proof.* Fix  $\alpha_{1,1}, \dots, \alpha_{1,t_1}, \alpha_{2,1}, \dots, \alpha_{2,t_2}, \dots, \alpha_{k,1}, \dots, \alpha_{k,t_k} \in \mathbb{F}$   
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$$\sum_{i=1}^k \left( \alpha_{i,1} \mathbf{v}_{i,1} + \dots + \alpha_{i,t_i} \mathbf{v}_{i,t_i} \right) = \mathbf{0}.$$

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- $\mathbf{v}_2 := \alpha_{2,1} \mathbf{v}_{2,1} + \dots + \alpha_{2,t_2} \mathbf{v}_{2,t_2}$ ;
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*Proof (continued).* So,

$$\mathbf{v}_1 + \cdots + \mathbf{v}_\ell = \mathbf{0},$$

and it follows that  $\{\mathbf{v}_1, \dots, \mathbf{v}_\ell\}$  is a linearly dependent set. But this contradicts Proposition 8.4.1.

*Proof (continued).* So,

$$\mathbf{v}_1 + \cdots + \mathbf{v}_\ell = \mathbf{0},$$

and it follows that  $\{\mathbf{v}_1, \dots, \mathbf{v}_\ell\}$  is a linearly dependent set. But this contradicts Proposition 8.4.1.

We have now shown that  $\mathbf{v}_1 = \cdots = \mathbf{v}_k = \mathbf{0}$ .



*Proof (continued).* So,

$$\mathbf{v}_1 + \cdots + \mathbf{v}_\ell = \mathbf{0},$$

and it follows that  $\{\mathbf{v}_1, \dots, \mathbf{v}_\ell\}$  is a linearly dependent set. But this contradicts Proposition 8.4.1.

We have now shown that  $\mathbf{v}_1 = \cdots = \mathbf{v}_k = \mathbf{0}$ . So, for all indices  $i \in \{1, \dots, k\}$ , we have that

$$\alpha_{i,1}\mathbf{v}_{i,1} + \cdots + \alpha_{i,t_i}\mathbf{v}_{i,t_i} = \mathbf{0};$$

*Proof (continued).* So,

$$\mathbf{v}_1 + \cdots + \mathbf{v}_\ell = \mathbf{0},$$

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since vectors  $\mathbf{v}_{i,1}, \dots, \mathbf{v}_{i,t_i}$  are linearly independent, it follows that  $\alpha_{i,1} = \cdots = \alpha_{i,t_i} = 0$ .

*Proof (continued).* So,

$$\mathbf{v}_1 + \cdots + \mathbf{v}_\ell = \mathbf{0},$$

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We have now shown that  $\mathbf{v}_1 = \cdots = \mathbf{v}_k = \mathbf{0}$ . So, for all indices  $i \in \{1, \dots, k\}$ , we have that

$$\alpha_{i,1}\mathbf{v}_{i,1} + \cdots + \alpha_{i,t_i}\mathbf{v}_{i,t_i} = \mathbf{0};$$

since vectors  $\mathbf{v}_{i,1}, \dots, \mathbf{v}_{i,t_i}$  are linearly independent, it follows that  $\alpha_{i,1} = \cdots = \alpha_{i,t_i} = 0$ .

Since this holds for all indices  $i \in \{1, \dots, k\}$ , we deduce that the eigenvectors

$$\mathbf{v}_{1,1}, \dots, \mathbf{v}_{1,t_1}, \mathbf{v}_{2,1}, \dots, \mathbf{v}_{2,t_2}, \dots, \mathbf{v}_{k,1}, \dots, \mathbf{v}_{k,t_k}$$

are linearly independent, which is what we needed to show.  $\square$

### Proposition 8.4.1

Let  $V$  be a vector space over a field  $\mathbb{F}$ , let  $f : V \rightarrow V$  be a linear function, let  $\lambda_1, \dots, \lambda_k \in \mathbb{F}$  be pairwise distinct eigenvalues of  $f$ , associated with eigenvectors  $\mathbf{v}_1, \dots, \mathbf{v}_k$ , respectively. Then  $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  is a linearly independent set.

### Proposition 8.4.2

Let  $V$  be a vector space over a field  $\mathbb{F}$ , let  $f : V \rightarrow V$  be a linear function, and let  $\lambda_1, \lambda_2, \dots, \lambda_k \in \mathbb{F}$  be pairwise distinct eigenvalues of  $f$ . For each  $i \in \{1, \dots, k\}$ , let  $\mathbf{v}_{i,1}, \dots, \mathbf{v}_{i,t_i}$  be linearly independent eigenvectors of  $f$  associated with the eigenvalue  $\lambda_i$ . Then the eigenvectors

$\mathbf{v}_{1,1}, \dots, \mathbf{v}_{1,t_1}, \mathbf{v}_{2,1}, \dots, \mathbf{v}_{2,t_2}, \dots, \mathbf{v}_{k,1}, \dots, \mathbf{v}_{k,t_k}$   
are linearly independent.

### Theorem 8.4.3

Let  $V$  be a non-trivial, finite-dimensional vector space over a field  $\mathbb{F}$ , and set  $n := \dim(V)$ . Let  $f : V \rightarrow V$  be a linear function, let  $\lambda_1, \dots, \lambda_k$  be all (distinct) the eigenvalues of  $f$ , and let  $\mathcal{B}_1, \dots, \mathcal{B}_k$  be bases of the associated eigenspaces  $E_{\lambda_1}(f), \dots, E_{\lambda_k}(f)$ , respectively. Set  $\mathcal{B} := \mathcal{B}_1 \cup \dots \cup \mathcal{B}_k$ . Then all the following hold:

- Ⓐ  $\mathcal{B}$  is a linearly independent set of eigenvectors of  $f$ ;
- Ⓑ  $\dim(E_{\lambda_1}(f)) + \dots + \dim(E_{\lambda_k}(f)) \leq n$ , i.e. the sum of geometric multiplicities of the eigenvalues of  $f$  is at most  $n$ ;
- Ⓒ  $V$  has an eigenbasis associated with  $f$  iff the sum of geometric multiplicities of the eigenvalues of  $f$  is  $n$ , and in this case,  $\mathcal{B}$  is such an eigenbasis;
- Ⓓ  $V$  has an eigenbasis associated with  $f$  iff the sum of algebraic multiplicities of the eigenvalues of  $f$  is  $n$ , and the geometric multiplicity of each eigenvalue is equal to its algebraic multiplicity; in this case,  $\mathcal{B}$  is an eigenbasis of  $V$  associated with the linear function  $f$ .

### Theorem 8.4.3

Let  $V$  be a non-trivial, finite-dimensional vector space over a field  $\mathbb{F}$ , and set  $n := \dim(V)$ . Let  $f : V \rightarrow V$  be a linear function, let  $\lambda_1, \dots, \lambda_k$  be all (distinct) the eigenvalues of  $f$ , and let  $\mathcal{B}_1, \dots, \mathcal{B}_k$  be bases of the associated eigenspaces  $E_{\lambda_1}(f), \dots, E_{\lambda_k}(f)$ , respectively. Set  $\mathcal{B} := \mathcal{B}_1 \cup \dots \cup \mathcal{B}_k$ . Then all the following hold:

- Ⓐ  $\mathcal{B}$  is a linearly independent set of eigenvectors of  $f$ ;
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*Proof.* Part (a) follows immediately from Proposition 8.4.2.

### Theorem 8.4.3

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- Ⓐ  $\mathcal{B}$  is a linearly independent set of eigenvectors of  $f$ ;
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*Proof.* Part (a) follows immediately from Proposition 8.4.2.

Part (b) follows from (a) and from the fact that, by Theorem 3.2.17(a), any linearly independent set of vectors in an  $n$ -dimensional vector space contains at most  $n$  vectors.

### Theorem 8.4.3

- Ⓒ  $V$  has an eigenbasis associated with  $f$  iff the sum of geometric multiplicities of the eigenvalues of  $f$  is  $n$ , and in this case,  $\mathcal{B}$  is such an eigenbasis;

*Proof (continued).* Let us prove (c).



### Theorem 8.4.3

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*Proof (continued).* Let us prove (c). Suppose first that the sum of geometric multiplicities of the eigenvalues of  $f$  is equal to  $n$ .

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*Proof (continued).* Let us prove (c). Suppose first that the sum of geometric multiplicities of the eigenvalues of  $f$  is equal to  $n$ . Then  $\mathcal{B}$  is a linearly independent set of size  $n$  in the  $n$ -dimensional vector space  $V$ .

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*Proof (continued).* Let us prove (c). Suppose first that the sum of geometric multiplicities of the eigenvalues of  $f$  is equal to  $n$ . Then  $\mathcal{B}$  is a linearly independent set of size  $n$  in the  $n$ -dimensional vector space  $V$ . So, by Corollary 3.2.20(a),  $\mathcal{B}$  is a basis of  $V$ .

### Theorem 8.4.3

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*Proof (continued).* Let us prove (c). Suppose first that the sum of geometric multiplicities of the eigenvalues of  $f$  is equal to  $n$ . Then  $\mathcal{B}$  is a linearly independent set of size  $n$  in the  $n$ -dimensional vector space  $V$ . So, by Corollary 3.2.20(a),  $\mathcal{B}$  is a basis of  $V$ . Since all vectors in  $\mathcal{B}$  are eigenvectors of  $f$ , it follows that  $\mathcal{B}$  is an eigenbasis of  $V$  associated with  $f$ .

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*Proof (continued).* Suppose, conversely, that  $V$  has an eigenbasis  $\mathcal{C}$  associated with  $f$ ;

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*Proof (continued).* Suppose, conversely, that  $V$  has an eigenbasis  $\mathcal{C}$  associated with  $f$ ; since  $\dim(V) = n$ , we see that  $|\mathcal{C}| = n$ .

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*Proof (continued).* Suppose, conversely, that  $V$  has an eigenbasis  $\mathcal{C}$  associated with  $f$ ; since  $\dim(V) = n$ , we see that  $|\mathcal{C}| = n$ . Since all vectors in  $\mathcal{C}$  are eigenvectors of  $f$ , we see that they all belong to  $E_{\lambda_1}(f) \cup \cdots \cup E_{\lambda_k}(f)$ . But since the basis  $\mathcal{C}$  of  $V$  is, in particular, linearly independent, we see that it cannot contain more than  $\dim(E_{\lambda_i}(f))$  many vectors from  $E_{\lambda_i}(f)$  for any index  $i \in \{1, \dots, k\}$ .

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### Theorem 8.4.3

- Ⓢ  $V$  has an eigenbasis associated with  $f$  iff the sum of geometric multiplicities of the eigenvalues of  $f$  is  $n$ , and in this case,  $\mathcal{B}$  is such an eigenbasis;

*Proof (continued).* Suppose, conversely, that  $V$  has an eigenbasis  $\mathcal{C}$  associated with  $f$ ; since  $\dim(V) = n$ , we see that  $|\mathcal{C}| = n$ . Since all vectors in  $\mathcal{C}$  are eigenvectors of  $f$ , we see that they all belong to  $E_{\lambda_1}(f) \cup \cdots \cup E_{\lambda_k}(f)$ . But since the basis  $\mathcal{C}$  of  $V$  is, in particular, linearly independent, we see that it cannot contain more than  $\dim(E_{\lambda_i}(f))$  many vectors from  $E_{\lambda_i}(f)$  for any index  $i \in \{1, \dots, k\}$ . So,  $|\mathcal{C}| \leq \dim(E_{\lambda_1}(f)) + \cdots + \dim(E_{\lambda_k}(f))$ . But now we have that

$$n = |\mathcal{C}| \leq \dim(E_{\lambda_1}(f)) + \cdots + \dim(E_{\lambda_k}(f)) \stackrel{(b)}{\leq} n,$$

### Theorem 8.4.3

- Ⓢ  $V$  has an eigenbasis associated with  $f$  iff the sum of geometric multiplicities of the eigenvalues of  $f$  is  $n$ , and in this case,  $\mathcal{B}$  is such an eigenbasis;

*Proof (continued).* Suppose, conversely, that  $V$  has an eigenbasis  $\mathcal{C}$  associated with  $f$ ; since  $\dim(V) = n$ , we see that  $|\mathcal{C}| = n$ . Since all vectors in  $\mathcal{C}$  are eigenvectors of  $f$ , we see that they all belong to  $E_{\lambda_1}(f) \cup \cdots \cup E_{\lambda_k}(f)$ . But since the basis  $\mathcal{C}$  of  $V$  is, in particular, linearly independent, we see that it cannot contain more than  $\dim(E_{\lambda_i}(f))$  many vectors from  $E_{\lambda_i}(f)$  for any index  $i \in \{1, \dots, k\}$ . So,  $|\mathcal{C}| \leq \dim(E_{\lambda_1}(f)) + \cdots + \dim(E_{\lambda_k}(f))$ . But now we have that

$$n = |\mathcal{C}| \leq \dim(E_{\lambda_1}(f)) + \cdots + \dim(E_{\lambda_k}(f)) \stackrel{(b)}{\leq} n,$$

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and it follows that  $\dim(E_{\lambda_1}(f)) + \cdots + \dim(E_{\lambda_k}(f)) = n$ , i.e. the sum of geometric multiplicities of the eigenvalues of  $f$  is  $n$ . This proves (c).

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*Proof (continued).* It remains to prove (d). If the sum of algebraic multiplicities of the eigenvalues of  $f$  is equal to  $n$ , and the geometric multiplicity of each eigenvalue is equal to its algebraic multiplicity, then obviously, the sum of geometric multiplicities of  $f$  is equal to  $n$ ,

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For the converse, assume that  $V$  has an eigenbasis  $\mathcal{C}$  associated with  $f$ . Let  $\lambda_1, \dots, \lambda_k$  be the eigenvalues of  $f$ , with **geometric** multiplicities  $g_1, \dots, g_k$ , respectively, and **algebraic** multiplicities  $a_1, \dots, a_k$ , respectively.

### Theorem 8.4.3

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*Proof (continued).* By (c), we have that  $g_1 + \cdots + g_k = n$ . On the other hand, the characteristic polynomial of  $f$  is of degree  $n$ , we see that the sum of algebraic multiplicities of  $f$  is at most  $n$ , i.e.  $a_1 + \cdots + a_k \leq n$ .

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$$n = g_1 + \cdots + g_k \leq a_1 + \cdots + a_k \leq n,$$

and we deduce that  $a_1 + \cdots + a_k = n$  and that  $g_i = a_i$  for all  $i \in \{1, \dots, k\}$ . This proves (d).  $\square$

### Theorem 8.4.3

Let  $V$  be a non-trivial, finite-dimensional vector space over a field  $\mathbb{F}$ , and set  $n := \dim(V)$ . Let  $f : V \rightarrow V$  be a linear function, let  $\lambda_1, \dots, \lambda_k$  be all (distinct) the eigenvalues of  $f$ , and let  $\mathcal{B}_1, \dots, \mathcal{B}_k$  be bases of the associated eigenspaces  $E_{\lambda_1}(f), \dots, E_{\lambda_k}(f)$ , respectively. Set  $\mathcal{B} := \mathcal{B}_1 \cup \dots \cup \mathcal{B}_k$ . Then all the following hold:

- Ⓐ  $\mathcal{B}$  is a linearly independent set of eigenvectors of  $f$ ;
- Ⓑ  $\dim(E_{\lambda_1}(f)) + \dots + \dim(E_{\lambda_k}(f)) \leq n$ , i.e. the sum of geometric multiplicities of the eigenvalues of  $f$  is at most  $n$ ;
- Ⓒ  $V$  has an eigenbasis associated with  $f$  iff the sum of geometric multiplicities of the eigenvalues of  $f$  is  $n$ , and in this case,  $\mathcal{B}$  is such an eigenbasis;
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### Corollary 8.4.4

Let  $V$  be a non-trivial, finite-dimensional vector space over a field  $\mathbb{F}$ , and set  $n := \dim(V)$ . If a linear function  $f : V \rightarrow V$  has  $n$  distinct eigenvalues, then  $V$  has an eigenbasis associated with  $f$ .

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Thus,  $\dim(E_{\lambda_1}(f)) + \dots + \dim(E_{\lambda_n}(f)) = n$ , and so by Theorem 8.4.3(c),  $V$  has an eigenbasis associated with  $f$ .  $\square$

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- Given a field  $\mathbb{F}$  and a square matrix  $A \in \mathbb{F}^{n \times n}$ , we can define  $f_A : \mathbb{F}^n \rightarrow \mathbb{F}^n$  by setting  $f_A(\mathbf{v}) = A\mathbf{v}$  for all  $\mathbf{v} \in \mathbb{F}^n$ .
  - So,  $f_A$  is linear, and its standard matrix is  $A$ .

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  - So,  $f_A$  is linear, and its standard matrix is  $A$ .
- We can apply Theorem 8.4.3 and Corollary 8.4.4 to the linear function  $f_A$ , and then get the same result for  $A$  “for free.”
  - Next two slides!

### Theorem 8.4.5

Let  $\mathbb{F}$  be a field, and let  $A \in \mathbb{F}^{n \times n}$ . Let  $\lambda_1, \dots, \lambda_k$  be all the (distinct) eigenvalues of  $A$ , and let  $\mathcal{B}_1, \dots, \mathcal{B}_k$  be bases of the associated eigenspaces  $E_{\lambda_1}(A), \dots, E_{\lambda_k}(A)$ , respectively. Set  $\mathcal{B} := \mathcal{B}_1 \cup \dots \cup \mathcal{B}_k$ . Then all the following hold:

- Ⓐ  $\mathcal{B}$  is a linearly independent set of eigenvectors of  $A$ ;
- Ⓑ  $\dim(E_{\lambda_1}(A)) + \dots + \dim(E_{\lambda_k}(A)) \leq n$ , i.e. the sum of geometric multiplicities of the eigenvalues of  $A$  is at most  $n$ ;
- Ⓒ  $\mathbb{F}^n$  has an eigenbasis associated with  $A$  iff the sum of geometric multiplicities of the eigenvalues of  $A$  is  $n$ , and in this case,  $\mathcal{B}$  is such an eigenbasis;
- Ⓓ  $\mathbb{F}^n$  has an eigenbasis associated with  $A$  iff the sum of algebraic multiplicities of the eigenvalues of  $A$  is  $n$ , and the geometric multiplicity of each eigenvalue is equal to its algebraic multiplicity; in this case,  $\mathcal{B}$  is an eigenbasis of  $\mathbb{F}^n$  associated with the matrix  $A$ .



#### Corollary 8.4.6

Let  $\mathbb{F}$  be a field, and let  $A \in \mathbb{F}^{n \times n}$ . If  $A$  has  $n$  distinct eigenvalues, then  $\mathbb{F}^n$  has an eigenbasis associated with  $A$ .

## ② Diagonalization

## 2 Diagonalization

### Definition

For a field  $\mathbb{F}$ , a square matrix  $D \in \mathbb{F}^{n \times n}$  is *diagonal* if all its entries off the main diagonal are zero (the entries on the main diagonal may or may not be zero). For scalars  $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{F}$ ,  $D(\lambda_1, \lambda_2, \dots, \lambda_n)$  is the  $n \times n$  matrix with  $\lambda_1, \lambda_2, \dots, \lambda_n$  on the main diagonal (appearing in that order) and 0's everywhere else, i.e.

$$\begin{aligned} D(\lambda_1, \lambda_2, \dots, \lambda_n) &:= \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 \mathbf{e}_1 & \dots & \lambda_n \mathbf{e}_n \end{bmatrix}, \end{aligned}$$

where as usual,  $\mathbf{e}_1, \dots, \mathbf{e}_n$  are the standard basis vectors of  $\mathbb{F}^n$ .

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- Note that diagonal matrices are, in particular, triangular.

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$$\begin{aligned} D(\lambda_1, \lambda_2, \dots, \lambda_n) &:= \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 \mathbf{e}_1 & \dots & \lambda_n \mathbf{e}_n \end{bmatrix}, \end{aligned}$$

where as usual,  $\mathbf{e}_1, \dots, \mathbf{e}_n$  are the standard basis vectors of  $\mathbb{F}^n$ .

- Note that diagonal matrices are, in particular, triangular.
- So, Propositions 7.3.1 and 8.2.7 (next slide) apply.

### Proposition 7.3.1

Let  $\mathbb{F}$  be a field, and let  $A = [a_{i,j}]_{n \times n}$  be a triangular matrix in  $\mathbb{F}^{n \times n}$ . Then  $\det(A) = \prod_{i=1}^n a_{i,i} = a_{1,1}a_{2,2} \dots a_{n,n}$ , that is,  $\det(A)$  is equal to the product of entries on the main diagonal of  $A$ .

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### Proposition 8.2.7

Let  $\mathbb{F}$  be a field, and let  $A = [a_{i,j}]_{n \times n}$  be a triangular matrix in  $\mathbb{F}^{n \times n}$ . Then the characteristic polynomial of  $A$  is

$$p_A(\lambda) = \prod_{i=1}^n (\lambda - a_{i,i}) = (\lambda - a_{1,1})(\lambda - a_{2,2}) \dots (\lambda - a_{n,n}),$$

the eigenvalues of  $A$  are precisely the entries of  $A$  on its main diagonal, and moreover, the algebraic multiplicity of each eigenvalue is precisely the number of times that it appears on the main diagonal of  $A$ . Consequently, the spectrum of  $A$  is  $\{a_{1,1}, a_{2,2}, \dots, a_{n,n}\}$ , i.e. the multiset formed precisely by the main diagonal entries of  $A$ , with each number appearing in the spectrum of  $A$  the same number of times as on the main diagonal of  $A$ .

- Thus, for scalars  $\lambda_1, \dots, \lambda_n \in \mathbb{F}$  (where  $\mathbb{F}$  is a field), and for the diagonal matrix  $D := D(\lambda_1, \dots, \lambda_n)$ , we have the following:
  - $\det(D) = \lambda_1 \dots \lambda_n$ ;
  - $p_D(\lambda) = (\lambda - \lambda_1) \dots (\lambda - \lambda_n)$ .



- Thus, for scalars  $\lambda_1, \dots, \lambda_n \in \mathbb{F}$  (where  $\mathbb{F}$  is a field), and for the diagonal matrix  $D := D(\lambda_1, \dots, \lambda_n)$ , we have the following:
  - $\det(D) = \lambda_1 \dots \lambda_n$ ;
  - $p_D(\lambda) = (\lambda - \lambda_1) \dots (\lambda - \lambda_n)$ .
- We now state three simple propositions about diagonal matrices.
  - The proofs are easy and we omit them here.
  - However, the proofs can be found in the Lecture Notes.

### Proposition 8.5.1

Let  $\mathbb{F}$  be a field, let  $\lambda_1, \dots, \lambda_n \in \mathbb{F}$  ( $n \geq 1$ ) be arbitrary scalars, and set  $D := D(\lambda_1, \dots, \lambda_n)$ . Then both the following hold:

- Ⓐ for all vectors  $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$  in  $\mathbb{F}^n$ , we have that

$$D\mathbf{x} = \begin{bmatrix} \lambda_1 x_1 \\ \vdots \\ \lambda_n x_n \end{bmatrix};$$

- Ⓑ for all matrices  $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix}$  in  $\mathbb{F}^{m \times n}$ , we have that

$$AD = \begin{bmatrix} \lambda_1 \mathbf{a}_1 & \dots & \lambda_n \mathbf{a}_n \end{bmatrix}.$$

- Proof: Lecture Notes (easy!).

### Proposition 8.5.2

Let  $\mathbb{F}$  be a field, and let  $\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_n \in \mathbb{F}$  ( $n \geq 1$ ) be arbitrary scalars. Then

$$D(\lambda_1, \dots, \lambda_n) D(\mu_1, \dots, \mu_n) = D(\lambda_1 \mu_1, \dots, \lambda_n \mu_n).$$

- Proof: Lecture Notes (easy!)

### Proposition 8.5.2

Let  $\mathbb{F}$  be a field, and let  $\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_n \in \mathbb{F}$  ( $n \geq 1$ ) be arbitrary scalars. Then

$$D(\lambda_1, \dots, \lambda_n) D(\mu_1, \dots, \mu_n) = D(\lambda_1 \mu_1, \dots, \lambda_n \mu_n).$$

- Proof: Lecture Notes (easy!)

### Proposition 8.5.3

Let  $\mathbb{F}$  be a field, let  $\lambda_1, \dots, \lambda_n \in \mathbb{F}$  ( $n \geq 1$ ), and set  $D := D(\lambda_1, \dots, \lambda_n)$ . Then both the following hold:

- Ⓐ for all non-negative integers  $m$ , we have that  $D^m = D(\lambda_1^m, \dots, \lambda_n^m)$ ;
- Ⓑ  $D$  is invertible iff  $\lambda_1, \dots, \lambda_n$  are all non-zero, and in this case, we have that  $D^m = D(\lambda_1^m, \dots, \lambda_n^m)$  for all integers  $m$ .

- Proof: Lecture Notes (easy!)

### Theorem 8.5.4

Let  $V$  be a non-trivial, finite-dimensional vector space, let  $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  be a basis of  $V$ , and let  $f : V \rightarrow V$  be a linear function. Then  $\mathcal{B}$  is an eigenbasis of  $V$  associated with  $f$  iff the matrix  ${}_B[f]_B$  is diagonal. Moreover, in this case, we have that

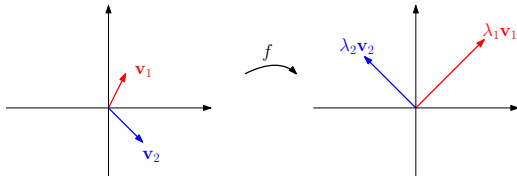
$${}_B[f]_B = D(\lambda_1, \dots, \lambda_n),$$

where  $\lambda_1, \dots, \lambda_n$  are the eigenvalues of  $f$  associated with the eigenvectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$ , respectively.

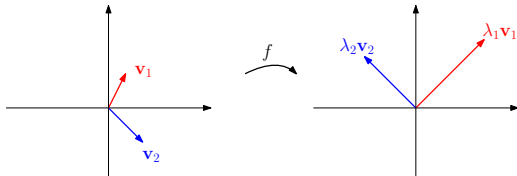
- First a remark (with pictures!), and then a proof.

- **Remark:** Suppose that  $V$  is a non-trivial, finite-dimensional vector space over a field  $\mathbb{F}$ .

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  - By Theorems 4.3.2 and 8.5.4, linear functions from  $V$  to  $V$  that have a diagonal matrix are precisely those that can be defined starting from some basis, and then scaling each of the basis elements.



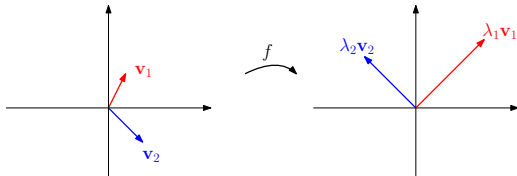
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  - By Theorems 4.3.2 and 8.5.4, linear functions from  $V$  to  $V$  that have a diagonal matrix are precisely those that can be defined starting from some basis, and then scaling each of the basis elements.



- Indeed, suppose that  $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is some basis of  $V$ , and that  $\lambda_1, \dots, \lambda_n \in \mathbb{F}$  are some scalars.

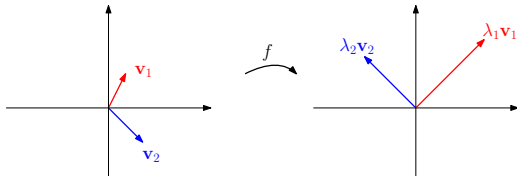


- **Remark:** Suppose that  $V$  is a non-trivial, finite-dimensional vector space over a field  $\mathbb{F}$ .
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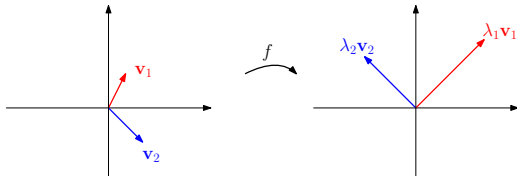
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- By Theorem 4.3.2, there exists a unique linear function  $f : V \rightarrow V$  s.t.  $f(\mathbf{v}_i) = \lambda_i \mathbf{v}_i$ .

- **Remark:** Suppose that  $V$  is a non-trivial, finite-dimensional vector space over a field  $\mathbb{F}$ .
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- But then by Theorem 8.5.4,  ${}_B [ f ]_B = D(\lambda_1, \dots, \lambda_n)$ .

- **Remark:** Suppose that  $V$  is a non-trivial, finite-dimensional vector space over a field  $\mathbb{F}$ .
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- Indeed, suppose that  $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is some basis of  $V$ , and that  $\lambda_1, \dots, \lambda_n \in \mathbb{F}$  are some scalars.
- By Theorem 4.3.2, there exists a unique linear function  $f : V \rightarrow V$  s.t.  $f(\mathbf{v}_i) = \lambda_i \mathbf{v}_i$ .
- But then by Theorem 8.5.4,  ${}_B[f]_B = D(\lambda_1, \dots, \lambda_n)$ .
- By Theorem 8.5.4, the converse also holds.

### Theorem 8.5.4

Let  $V$  be a non-trivial, finite-dimensional vector space, let  $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  be a basis of  $V$ , and let  $f : V \rightarrow V$  be a linear function. Then  $\mathcal{B}$  is an eigenbasis of  $V$  associated with  $f$  iff the matrix  ${}_B[f]_B$  is diagonal. Moreover, in this case, we have that

$${}_B[f]_B = D(\lambda_1, \dots, \lambda_n),$$

where  $\lambda_1, \dots, \lambda_n$  are the eigenvalues of  $f$  associated with the eigenvectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$ , respectively.

*Proof.*

### Theorem 8.5.4

Let  $V$  be a non-trivial, finite-dimensional vector space, let  $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  be a basis of  $V$ , and let  $f : V \rightarrow V$  be a linear function. Then  $\mathcal{B}$  is an eigenbasis of  $V$  associated with  $f$  iff the matrix  ${}_B[f]_B$  is diagonal. Moreover, in this case, we have that

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*Proof.* Suppose first that  $\mathcal{B}$  is an eigenbasis of  $V$  associated with  $f$ .

### Theorem 8.5.4

Let  $V$  be a non-trivial, finite-dimensional vector space, let  $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  be a basis of  $V$ , and let  $f : V \rightarrow V$  be a linear function. Then  $\mathcal{B}$  is an eigenbasis of  $V$  associated with  $f$  iff the matrix  ${}_B[f]_B$  is diagonal. Moreover, in this case, we have that

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*Proof.* Suppose first that  $\mathcal{B}$  is an eigenbasis of  $V$  associated with  $f$ . Then, by definition, vectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$  are eigenvectors of  $f$ , and we let  $\lambda_1, \dots, \lambda_n$ , respectively, be the associated eigenvalues.

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### Theorem 8.5.4

Let  $V$  be a non-trivial, finite-dimensional vector space, let  $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  be a basis of  $V$ , and let  $f : V \rightarrow V$  be a linear function. Then  $\mathcal{B}$  is an eigenbasis of  $V$  associated with  $f$  iff the matrix  ${}_B[f]_B$  is diagonal. Moreover, in this case, we have that

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*Proof (continued).*

$$\begin{aligned} {}_B[f]_B &= \begin{bmatrix} [f(\mathbf{v}_1)]_B & \dots & [f(\mathbf{v}_n)]_B \end{bmatrix} && \text{by Theorem 4.5.1} \\ &= \begin{bmatrix} [\lambda_1 \mathbf{v}_1]_B & \dots & [\lambda_n \mathbf{v}_n]_B \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 \mathbf{e}_1 & \dots & \lambda_n \mathbf{e}_n \end{bmatrix} \\ &= D(\lambda_1, \dots, \lambda_n). \end{aligned}$$



### Theorem 8.5.4

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*Proof (continued).* Conversely, suppose that the matrix  ${}_B[f]_B$  is diagonal, and let  $\lambda_1, \dots, \lambda_n$  be the entries of this matrix on the main diagonal, so that

$${}_B[f]_B = D(\lambda_1, \dots, \lambda_n) = \begin{bmatrix} \lambda_1 \mathbf{e}_1 & \dots & \lambda_n \mathbf{e}_n \end{bmatrix}.$$

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### Theorem 8.5.4

Let  $V$  be a non-trivial, finite-dimensional vector space, let  $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  be a basis of  $V$ , and let  $f : V \rightarrow V$  be a linear function. Then  $\mathcal{B}$  is an eigenbasis of  $V$  associated with  $f$  iff the matrix  ${}_B[f]_B$  is diagonal. Moreover, in this case, we have that

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We will show that the basis vectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$  are eigenvectors of  $f$  with associated eigenvalues  $\lambda_1, \dots, \lambda_n$ , respectively. Fix any index  $i \in \{1, \dots, n\}$ ; **WTS**  $f(\mathbf{v}_i) = \lambda_i \mathbf{v}_i$ . Since  $\mathbf{v}_i$  is the  $i$ -th basis vector of  $\mathcal{B}$ , we have that  $[\mathbf{v}_i]_B = \mathbf{e}_i$ . We now compute (next slide):

### Theorem 8.5.4

Let  $V$  be a non-trivial, finite-dimensional vector space, let  $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  be a basis of  $V$ , and let  $f : V \rightarrow V$  be a linear function. Then  $\mathcal{B}$  is an eigenbasis of  $V$  associated with  $f$  iff the matrix  ${}_B[f]_B$  is diagonal. Moreover, in this case, we have that

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where  $\lambda_1, \dots, \lambda_n$  are the eigenvalues of  $f$  associated with the eigenvectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$ , respectively.

*Proof (continued).*

$$\begin{aligned} [f(\mathbf{v}_i)]_B &= {}_B[f]_B [\mathbf{v}_i]_B = [\lambda_1 \mathbf{e}_1 \ \dots \ \lambda_n \mathbf{e}_n] \mathbf{e}_i \\ &\stackrel{(*)}{=} \lambda_i \mathbf{e}_i = \lambda_i [\mathbf{v}_i]_B \stackrel{(**)}{=} [\lambda_i \mathbf{v}_i]_B, \end{aligned}$$

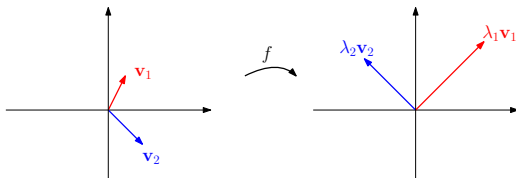
where  $(*)$  follows from Proposition 1.4.4, and  $(**)$  follows from the linearity of  $[\cdot]_B$ . Since  $[\cdot]_B$  is an isomorphism (and in particular, one-to-one), it follows that  $f(\mathbf{v}_i) = \lambda_i \mathbf{v}_i$ , which is what we needed to show.  $\square$

### Theorem 8.5.4

Let  $V$  be a non-trivial, finite-dimensional vector space, let  $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  be a basis of  $V$ , and let  $f : V \rightarrow V$  be a linear function. Then  $\mathcal{B}$  is an eigenbasis of  $V$  associated with  $f$  iff the matrix  ${}_B[f]_B$  is diagonal. Moreover, in this case, we have that

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where  $\lambda_1, \dots, \lambda_n$  are the eigenvalues of  $f$  associated with the eigenvectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$ , respectively.



### Definition

A matrix  $A \in \mathbb{F}^{n \times n}$  (where  $\mathbb{F}$  is a field) is *diagonalizable* if it is similar to a diagonal matrix. To *diagonalize* a diagonalizable matrix  $A$  means to compute a diagonal matrix  $D$  and an invertible matrix  $P$  s.t.  $D = P^{-1}AP$  (equivalently:  $A = PDP^{-1}$ ).



### Theorem 8.5.6

Let  $\mathbb{F}$  be a field, and let  $A \in \mathbb{F}^{n \times n}$  be a matrix. Then  $A$  is diagonalizable iff  $\mathbb{F}^n$  has an eigenbasis associated with  $A$ . Moreover, if  $\mathcal{P} = \{\mathbf{p}_1, \dots, \mathbf{p}_n\}$  is any eigenbasis of  $\mathbb{F}^n$  associated with  $A$ , and  $\lambda_1, \dots, \lambda_n$  are the eigenvalues of  $A$  associated with the eigenvectors  $\mathbf{p}_1, \dots, \mathbf{p}_n$ , respectively, then

$$D = P^{-1}AP \quad \text{and} \quad A = PDP^{-1},$$

where  $D = D(\lambda_1, \dots, \lambda_n)$  and  $P = [\mathbf{p}_1 \ \dots \ \mathbf{p}_n]$ .

- Proof: Lecture Notes.

### Theorem 8.5.6

Let  $\mathbb{F}$  be a field, and let  $A \in \mathbb{F}^{n \times n}$  be a matrix. Then  $A$  is diagonalizable iff  $\mathbb{F}^n$  has an eigenbasis associated with  $A$ . Moreover, if  $\mathcal{P} = \{\mathbf{p}_1, \dots, \mathbf{p}_n\}$  is any eigenbasis of  $\mathbb{F}^n$  associated with  $A$ , and  $\lambda_1, \dots, \lambda_n$  are the eigenvalues of  $A$  associated with the eigenvectors  $\mathbf{p}_1, \dots, \mathbf{p}_n$ , respectively, then

$$D = P^{-1}AP \quad \text{and} \quad A = PDP^{-1},$$

where  $D = D(\lambda_1, \dots, \lambda_n)$  and  $P = [\mathbf{p}_1 \ \dots \ \mathbf{p}_n]$ .

- Proof: Lecture Notes.
  - Theorem 8.5.6 can be obtained as a corollary of Theorem 8.5.4 (try it!).
  - However, in the Lecture Notes, there is a proof “from scratch” (i.e. one that uses matrices only).

### Theorem 8.5.6

Let  $\mathbb{F}$  be a field, and let  $A \in \mathbb{F}^{n \times n}$  be a matrix. Then  $A$  is diagonalizable iff  $\mathbb{F}^n$  has an eigenbasis associated with  $A$ . Moreover, if  $\mathcal{P} = \{\mathbf{p}_1, \dots, \mathbf{p}_n\}$  is any eigenbasis of  $\mathbb{F}^n$  associated with  $A$ , and  $\lambda_1, \dots, \lambda_n$  are the eigenvalues of  $A$  associated with the eigenvectors  $\mathbf{p}_1, \dots, \mathbf{p}_n$ , respectively, then

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### Corollary 8.5.7

Let  $\mathbb{F}$  be a field, and let  $A \in \mathbb{F}^{n \times n}$ . If  $A$  has  $n$  distinct eigenvalues, then  $A$  is diagonalizable.

*Proof.*

### Theorem 8.5.6

Let  $\mathbb{F}$  be a field, and let  $A \in \mathbb{F}^{n \times n}$  be a matrix. Then  $A$  is diagonalizable iff  $\mathbb{F}^n$  has an eigenbasis associated with  $A$ . Moreover, if  $\mathcal{P} = \{\mathbf{p}_1, \dots, \mathbf{p}_n\}$  is any eigenbasis of  $\mathbb{F}^n$  associated with  $A$ , and  $\lambda_1, \dots, \lambda_n$  are the eigenvalues of  $A$  associated with the eigenvectors  $\mathbf{p}_1, \dots, \mathbf{p}_n$ , respectively, then

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### Corollary 8.5.7

Let  $\mathbb{F}$  be a field, and let  $A \in \mathbb{F}^{n \times n}$ . If  $A$  has  $n$  distinct eigenvalues, then  $A$  is diagonalizable.

*Proof.* Assume that  $A$  has  $n$  distinct eigenvalues. By Corollary 8.4.6,  $\mathbb{F}^n$  has an eigenbasis associated with  $A$ . So, by Theorem 8.5.6,  $A$  is diagonalizable.  $\square$

- Theorems 8.4.5 and 8.5.6 together give us a recipe for determining whether a matrix  $A \in \mathbb{F}^{n \times n}$  is diagonalizable, and if so, for diagonalizing it (i.e. for finding a diagonal matrix  $D$  and an invertible matrix  $P$ , both in  $\mathbb{F}^{n \times n}$ , s.t.  $D = P^{-1}AP$ ).
- We proceed as follows (next two slides).

- ① We compute the characteristic polynomial  $p_A(\lambda)$  and its roots. By Theorem 8.2.2, the roots of  $p_A(\lambda)$  are the eigenvalues of  $A$ , and we can read off the algebraic multiplicities of those eigenvalues from the polynomial  $p_A(\lambda)$ .
  - Computing the roots of  $p_A(\lambda)$  is the computationally tricky part, since there is no formula for computing the roots of a high-degree polynomial. If we cannot figure out how to compute the roots of  $p_A(\lambda)$ , then we are stuck: the matrix  $A$  may or may not be diagonalizable, but computationally, we cannot diagonalize it.

- ① We compute the characteristic polynomial  $p_A(\lambda)$  and its roots. By Theorem 8.2.2, the roots of  $p_A(\lambda)$  are the eigenvalues of  $A$ , and we can read off the algebraic multiplicities of those eigenvalues from the polynomial  $p_A(\lambda)$ .
  - Computing the roots of  $p_A(\lambda)$  is the computationally tricky part, since there is no formula for computing the roots of a high-degree polynomial. If we cannot figure out how to compute the roots of  $p_A(\lambda)$ , then we are stuck: the matrix  $A$  may or may not be diagonalizable, but computationally, we cannot diagonalize it.
- ② If the sum of algebraic multiplicities of the eigenvalues of  $A$  is less than  $n$ , then by Theorem 8.4.5,  $\mathbb{F}^n$  does not have an eigenbasis associated with  $A$ , and so by Theorem 8.5.6,  $A$  is not diagonalizable.

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- ③ From now on, we assume that the sum of algebraic multiplicities of the eigenvalues of  $A$ , call them  $\lambda_1, \dots, \lambda_k$ , is  $n$ . We then compute a basis  $\mathcal{B}_i$  for each eigenspace  $E_{\lambda_i}(A)$ , which allows us to compute the geometric multiplicities of all the eigenvalues of  $A$ .



- 4 If the geometric multiplicity of some eigenvalue of  $A$  is smaller than its algebraic multiplicity, then by Theorem 8.4.5,  $\mathbb{F}^n$  does not have an eigenbasis associated with  $A$ , and so by Theorem 8.5.6,  $A$  is not diagonalizable.

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- ⑤ From now on, we assume that the geometric multiplicity of each eigenvalue of  $A$  is equal to its algebraic multiplicity. Theorem 8.4.5 then guarantees that  $\mathbb{F}^n$  has an eigenbasis associated with  $A$ , and moreover, that  $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k$  is one such eigenbasis.

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- ⑥ By Theorem 8.5.6,  $A$  is diagonalizable. We now follow the recipe from Theorem 8.5.6 to actually diagonalize  $A$ .
- ⑦ We form the matrix  $P$  whose columns are precisely the vectors in the eigenbasis  $\mathcal{B}$ . We form the diagonal matrix  $D$ , where on the main diagonal we place the eigenvalues of  $A$ , taking care that, for each  $i \in \{1, \dots, n\}$ , the  $i$ -th entry on the main diagonal of  $D$  is the eigenvalue associated with the  $i$ -th column of  $P$  (which is, by construction, an eigenvector of  $A$ ). Now  $D = P^{-1}AP$ .

### Example 8.5.8.

Consider the following matrix in  $\mathbb{C}^{3 \times 3}$ :

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}.$$

Determine whether  $A$  is diagonalizable, and if so, diagonalize it.

*Solution.*

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Determine whether  $A$  is diagonalizable, and if so, diagonalize it.

*Solution.* The matrix  $A$  is precisely the matrix from Example 8.2.4. In that example, we determined that  $A$  has two eigenvalues:

- $\lambda_1 = 4$  (with algebraic multiplicity 1 and geometric multiplicity 1);
- $\lambda_2 = 5$  (with algebraic multiplicity 2 and geometric multiplicity 2).

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- $\lambda_1 = 4$  (with algebraic multiplicity 1 and geometric multiplicity 1);
- $\lambda_2 = 5$  (with algebraic multiplicity 2 and geometric multiplicity 2).

Since the sum of algebraic multiplicities of the eigenvalues of  $A$  is 3, and since the geometric multiplicity of each eigenvalue of  $A$  is equal to its algebraic multiplicity, we see that the  $3 \times 3$  matrix  $A$  is indeed diagonalizable.

*Solution (continued).* Reminder:  $\lambda_1 = 4$ ,  $\lambda_2 = 5$ .

In Example 8.2.4, we saw that:

- $\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$  is a basis of the eigenspace  $E_{\lambda_1}(A)$ ;
- $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$  is a basis of the eigenspace  $E_{\lambda_2}(A)$ .



*Solution (continued).* Reminder:  $\lambda_1 = 4$ ,  $\lambda_2 = 5$ .

In Example 8.2.4, we saw that:

- $\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$  is a basis of the eigenspace  $E_{\lambda_1}(A)$ ;
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So, we set

$$D := \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \quad \text{and} \quad P := \begin{bmatrix} -1 & 0 & -2 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and we see that  $D = P^{-1}AP$ .  $\square$

### Example 8.5.9

Consider the following matrix in  $\mathbb{C}^{5 \times 5}$ :

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

Determine whether  $A$  is diagonalizable, and if so, diagonalize it.

*Solution.*

### Example 8.5.9

Consider the following matrix in  $\mathbb{C}^{5 \times 5}$ :

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

Determine whether  $A$  is diagonalizable, and if so, diagonalize it.

*Solution.* The matrix  $A$  is precisely the matrix from Example 8.2.8.

### Example 8.5.9

Consider the following matrix in  $\mathbb{C}^{5 \times 5}$ :

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

Determine whether  $A$  is diagonalizable, and if so, diagonalize it.

*Solution.* The matrix  $A$  is precisely the matrix from Example 8.2.8. In that example, we determined that  $A$  has three eigenvalues:

- $\lambda_1 = 1$  (with alg. mult. 2 and geom. mult. 2);
- $\lambda_2 = 2$  (with alg. mult. 1 and geom. mult. 1);
- $\lambda_3 = 3$  (with alg. mult. 2 and geom. mult. 1).

### Example 8.5.9

Consider the following matrix in  $\mathbb{C}^{5 \times 5}$ :

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

Determine whether  $A$  is diagonalizable, and if so, diagonalize it.

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- $\lambda_1 = 1$  (with alg. mult. 2 and geom. mult. 2);
- $\lambda_2 = 2$  (with alg. mult. 1 and geom. mult. 1);
- $\lambda_3 = 3$  (with alg. mult. 2 and geom. mult. 1).

Since the geometric multiplicity of the eigenvalue  $\lambda_3 = 3$  is strictly smaller than the algebraic multiplicity, we see that  $A$  is not diagonalizable.  $\square$

- Suppose that we have successfully diagonalized a square matrix  $A \in \mathbb{F}^{n \times n}$  (where  $\mathbb{F}$  is a field), that is, that we have computed a diagonal matrix  $D$  and an invertible matrix  $P$ , both in  $\mathbb{F}^{n \times n}$ , s.t.  $D = P^{-1}AP$ .

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- Then we can easily read off the spectrum and a basis of each eigenspace of  $A$ , as Proposition 8.5.12 (next slide) shows.

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- Then we can easily read off the spectrum and a basis of each eigenspace of  $A$ , as Proposition 8.5.12 (next slide) shows.
- This proposition essentially summarizes various facts about diagonalizable matrices that we have proven already, but it is convenient to have them stated in one proposition.



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- This proposition essentially summarizes various facts about diagonalizable matrices that we have proven already, but it is convenient to have them stated in one proposition.
  - The proof is in the Lecture Notes. Here, we omit it.

### Proposition 8.5.12

Let  $\mathbb{F}$  be a field, and let  $A \in \mathbb{F}^{n \times n}$ . Assume that  $D = P^{-1}AP$ , where  $D = D(\lambda_1, \dots, \lambda_n)$  is a diagonal and  $P = [\mathbf{p}_1 \ \dots \ \mathbf{p}_n]$  an invertible matrix, both in  $\mathbb{F}^{n \times n}$ . Then the characteristic polynomial of  $A$  is

$$p_A(\lambda) = \prod_{i=1}^n (\lambda - \lambda_i) = (\lambda - \lambda_1) \dots (\lambda - \lambda_n),$$

and the spectrum of  $A$  is  $\{\lambda_1, \dots, \lambda_n\}$ . Moreover, for each eigenvalue  $\lambda_0$  of  $A$ ,<sup>a</sup> the algebraic and geometric multiplicity of  $\lambda_0$  are both equal to the number of times that  $\lambda_0$  appears on the main diagonal of  $D$ , and moreover, if  $\lambda_0$  appears precisely in positions  $i_1, \dots, i_k$  of the main diagonal of  $D$ , then the corresponding columns of  $P$  (i.e. vectors  $\mathbf{p}_{i_1}, \dots, \mathbf{p}_{i_k}$ ) form a basis of the eigenspace  $E_{\lambda_0}(A)$ . Finally,  $\{\mathbf{p}_1, \dots, \mathbf{p}_n\}$  is an eigenbasis of  $\mathbb{F}^n$  associated with the matrix  $A$ .

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<sup>a</sup>So,  $\lambda_0 \in \{\lambda_1, \dots, \lambda_n\}$ , since  $\{\lambda_1, \dots, \lambda_n\}$  is the spectrum of  $A$ .

### Example 8.5.13

Consider the following matrices in  $\mathbb{C}^{6 \times 6}$  (color coded for emphasis):

$$D = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 3 & 8 & 8 & 3 & 4 \\ 2 & 8 & 0 & 0 & 0 & 2 \\ 5 & 4 & 6 & 4 & 5 & 0 \\ 0 & 5 & 8 & 5 & 4 & 3 \\ 1 & 0 & 8 & 0 & 3 & 0 \\ 0 & 2 & 0 & 3 & 0 & 2 \end{bmatrix}.$$

It can be checked that  $P$  is invertible (for example, we can compute that  $\det(P) = -1020 \neq 0$ , and so by Theorem 7.4.1,  $P$  is invertible). We now set  $A = PDP^{-1}$ , so that  $D = P^{-1}AP$ . Then by Proposition 8.5.12, all the following hold (next three slides):

$$D = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 3 & 8 & 8 & 3 & 4 \\ 2 & 8 & 0 & 0 & 0 & 2 \\ 5 & 4 & 6 & 4 & 5 & 0 \\ 0 & 5 & 8 & 5 & 4 & 3 \\ 1 & 0 & 8 & 0 & 3 & 0 \\ 0 & 2 & 0 & 3 & 0 & 2 \end{bmatrix}.$$

### Example 8.5.13 (continued)

- the characteristic polynomial of  $A$  is

$$p_A(\lambda) = (\lambda - 3)(\lambda - 4)^3(\lambda - 5)^2;$$

- the spectrum of  $A$  is  $\{5, 4, 5, 3, 4, 4\}$ , which we can optionally reorder as  $\{3, 4, 4, 4, 5, 5\}$ ;
- the eigenvalues of  $A$  are 3 (with algebraic and geometric multiplicity 1), 4 (with algebraic and geometric multiplicity 3), and 5 (with algebraic and geometric multiplicity 2);

$$D = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 3 & 8 & 8 & 3 & 4 \\ 2 & 8 & 0 & 0 & 0 & 2 \\ 5 & 4 & 6 & 4 & 5 & 0 \\ 0 & 5 & 8 & 5 & 4 & 3 \\ 1 & 0 & 8 & 0 & 3 & 0 \\ 0 & 2 & 0 & 3 & 0 & 2 \end{bmatrix}.$$

### Example 8.5.13 (continued)

- we can read off bases of the eigenspaces  $E_3(A)$ ,  $E_4(A)$ , and  $E_5(A)$ , as follows:

- a basis of  $E_3(A)$  is  $\left\{ \begin{bmatrix} 8 \\ 0 \\ 4 \\ 5 \\ 0 \\ 3 \end{bmatrix} \right\},$
- a basis of  $E_4(A)$  is  $\left\{ \begin{bmatrix} 3 \\ 8 \\ 4 \\ 5 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 5 \\ 4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 0 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\},$
- a basis of  $E_5(A)$  is  $\left\{ \begin{bmatrix} 1 \\ 2 \\ 5 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 0 \\ 6 \\ 8 \\ 8 \\ 0 \end{bmatrix} \right\};$

$$D = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 3 & 8 & 8 & 3 & 4 \\ 2 & 8 & 0 & 0 & 0 & 2 \\ 5 & 4 & 6 & 4 & 5 & 0 \\ 0 & 5 & 8 & 5 & 4 & 3 \\ 1 & 0 & 8 & 0 & 3 & 0 \\ 0 & 2 & 0 & 3 & 0 & 2 \end{bmatrix}.$$

### Example 8.5.13 (continued)

- the columns of  $P$  form an eigenbasis of  $\mathbb{C}^n$  associated with the matrix  $A$ .