Linear Algebra 2

Lecture #19

Determinants

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April 2, 2025

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  6. Laplace expansion

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Definition

The *determinant* of a matrix $A = \left[\ a_{i,j}\ \right]_{n\times n}$ with entries in some field $\mathbb{F}$, denoted by $\det(A)$ or $|A|$, is defined by

$$\begin{aligned}\det(A) &:= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} \\ &= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}.\end{aligned}$$

- **Remark:** Only **square** matrices have determinants. Moreover, the determinant of a matrix in $\mathbb{F}^{n\times n}$ is always a scalar in $\mathbb{F}$.

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- Advertisement:

Theorem 7.4.1

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times n}$. Then A is invertible iff $\det(A) \neq 0$.

- Proof: Later!

- Reminder: $\det(A) := \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$.

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- Let us try to explain this definition.
- Each permutation $\sigma \in S_n$ gives us one way of selecting one entry of A out of each row and each column: we select entries $a_{1,\sigma(1)}, \dots, a_{n,\sigma(n)}$, multiply them together, and then multiply that product by $\operatorname{sgn}(\sigma)$, which yields the product $\operatorname{sgn}(\sigma) a_{1,\sigma(1)} \dots a_{n,\sigma(n)}$.
    - For example, for $n = 4$ and $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} = (134)(2)$, we obtain the product $\operatorname{sgn}(\sigma) a_{1,3} a_{2,2} a_{3,4} a_{4,1} = a_{1,3} a_{2,2} a_{3,4} a_{4,1}$, since $\operatorname{sgn}(\sigma) = 1$.

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \boxed{a_{1,3}} & a_{1,4} \\ a_{2,1} & \boxed{a_{2,2}} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & \boxed{a_{3,4}} \\ \boxed{a_{4,1}} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix},$$

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- We then sum up all products of this type (there are $|S_n| = n!$ many of them), and we obtain the determinant of our matrix.

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Definition

The *characteristic* of a field $\mathbb{F}$ is the smallest positive integer n (if it exists) s.t. in the field $\mathbb{F}$, we have that

$$\underbrace{1 + \cdots + 1}_{n} = 0,$$

where the 1's and the 0 are understood to be in the field $\mathbb{F}$. If no such n exists, then $\text{char}(\mathbb{F}) := 0$.

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- For all prime numbers p, we have that $\text{char}(\mathbb{Z}_p) = p$.

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Theorem 2.4.5

The characteristic of any field is either 0 or a prime number.

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- Note that if the entries of our square matrix belong to a field of characteristic 2 (i.e. a field in which $1+1=0$, such as the field $\mathbb{Z}_2$), then $1=-1$, and so $\operatorname{sgn}(\sigma)$ can be ignored (because it is always equal to 1).

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- Note that if the entries of our square matrix belong to a field of characteristic 2 (i.e. a field in which $1 + 1 = 0$, such as the field $\mathbb{Z}_2$), then $1 = -1$, and so $\operatorname{sgn}(\sigma)$ can be ignored (because it is always equal to 1).
- However, if our field is of characteristic other than 2 (i.e. if $1 + 1 \neq 0$ in our field, and consequently, $1 \neq -1$), then we must keep track of $\operatorname{sgn}(\sigma)$ in each summand from the definition of a determinant.

- **Notation:** We typically write

$$\begin{vmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{vmatrix}$$

instead of

$$\det\left(\begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{bmatrix} \right).$$

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- For 1×1 matrices, this can unfortunately lead to confusion (because of absolute values).
  - To avoid this issue, we can always write $\det\left(\begin{bmatrix} a_{1,1} \end{bmatrix} \right)$ instead of $\begin{vmatrix} a_{1,1} \end{vmatrix}$.

Proposition 7.1.1

Let n be a positive integer, and let $\pi \in S_n$, and consider the matrix P_π of the permutation π (where the 0's and 1's in P_π can be considered as belonging to an arbitrary field $\mathbb{F}$). Then

$$\det(P_\pi) \quad = \quad \mathrm{sgn}(\pi).$$

Proof.

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Proof. Set $P_\pi = \left[\ p_{i,j}\ \right]_{n\times n}$, so that

$$p_{i,j} \quad = \quad \begin{cases} 1 & \text{if} \quad j = \pi(i) \\ 0 & \text{if} \quad j \neq \pi(i) \end{cases}$$

for all $i, j \in \{1, \ldots, n\}$.

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The only permutation $\sigma \in S_n$ for which none of $p_{1,\sigma(1)}, p_{2,\sigma(2)}, \ldots, p_{n,\sigma(n)}$ is 0 is the permutation $\sigma = \pi$.

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The only permutation $\sigma \in S_n$ for which none of $p_{1,\sigma(1)}, p_{2,\sigma(2)}, \dots, p_{n,\sigma(n)}$ is 0 is the permutation $\sigma = \pi$. So,

$$\det(P_\pi) \quad = \quad \operatorname{sgn}(\pi) p_{1,\pi(1)} p_{2,\pi(2)} \cdots p_{n,\pi(n)} \quad \overset{(*)}{=} \quad \operatorname{sgn}(\pi),$$

where (*) follows from the fact that $p_{i,\pi(i)} = 1$ for all $i \in \{1, \dots, n\}$. $\square$

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$$\det(P_\pi) \quad = \quad \mathrm{sgn}(\pi).$$

- Note that the identity matrix I_n is the matrix of the identity permutation 1 in S_n.
- Since $\mathrm{sgn}(1) = 1$, Proposition 7.1.1 guarantees that $\det(I_n) = 1$.

Proposition 7.1.2

We have the following formulas for the determinants of 1×1, 2×2, and 3×3 matrices (with entries in some field $\mathbb{F}$):

(a) $\begin{vmatrix} a_{1,1} \end{vmatrix} = a_{1,1}$;[a]

(b) $\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$;

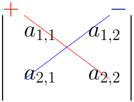
(c) $\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = \begin{cases} a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} \\ -a_{1,3}a_{2,2}a_{3,1} - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3}. \end{cases}$

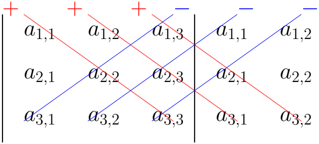
---

[a] Be careful not to confuse this with the absolute value! (The notation is admittedly somewhat unfortunate/ambiguous.) If there is any danger of confusion, it is always possible to write $\det\left(\begin{bmatrix} a_{1,1} \end{bmatrix}\right)$ instead of $\begin{vmatrix} a_{1,1} \end{vmatrix}$.

Proof (outline). This follows straight from the definition, where we simply have to list all the permutations in S_n (for $n = 1, 2, 3$) and keep track of their signs. (Details: Lecture Notes.) $\square$

- Determinants of 2×2 and 3×3 matrices can be represented schematically, as shown below.





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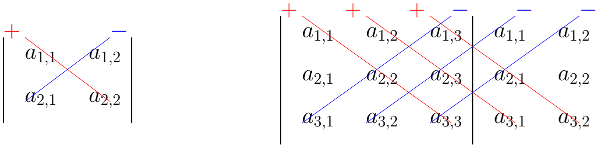

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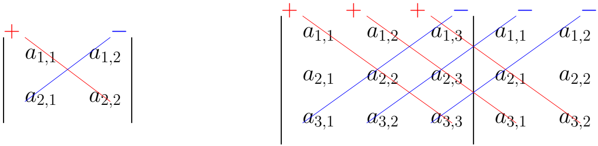
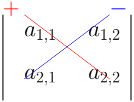
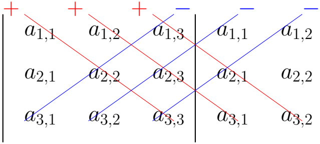


- We multiply the entries along each of the red lines and add them up, and then we multiply the entries along each of the blue lines and subtract them.
- In each case, the result we get is precisely the formula from Proposition 7.1.2.

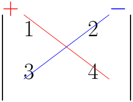




- For example, we can compute the determinant of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

in $\mathbb{R}^{2\times 2}$ by forming the diagram



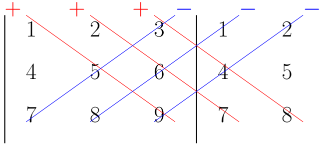
and the computing

$$\det(A) = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \color{red}{1 \cdot 4} - \color{blue}{2 \cdot 3} = -2.$$

- Similarly, we can compute the determinant of the matrix

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

in $\mathbb{R}^{3\times 3}$ by forming the diagram

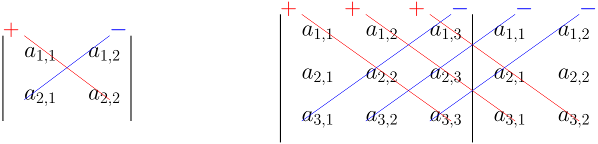


and then computing

$$\det(B) = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

$$= 1\cdot 5\cdot 9 + 2\cdot 6\cdot 7 + 3\cdot 4\cdot 8 - 3\cdot 5\cdot 7 - 1\cdot 6\cdot 8 - 2\cdot 4\cdot 9$$

$$= 0.$$



- **Warning:** Do not try this with matrices of larger size!

Theorem 7.1.3

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Proof. We set $A = \begin{bmatrix} a_{i,j} \end{bmatrix}_{n\times n}$ and $A^T = \begin{bmatrix} a^T_{i,j} \end{bmatrix}_{n\times n}$. So, for all $i, j \in \{1, \dots, n\}$, we have $a^T_{i,j} = a_{j,i}$.

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$$\begin{aligned}
\det(A^T) &= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a^T_{i,\sigma(i)} \\
&= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(i),i} \\
&= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{j=1}^{n} a_{j,\sigma^{-1}(j)} \\
&\overset{(*)}{=} \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma^{-1}) \prod_{j=1}^{n} a_{j,\sigma^{-1}(j)} \\
&= \sum_{\pi\in S_n} \operatorname{sgn}(\pi) \prod_{j=1}^{n} a_{j,\pi(j)} \\
&= \det(A),
\end{aligned}$$

where (*) follows from Proposition 2.3.2. □

Proposition 7.1.4

Let $\mathbb{F}$ be a field, and let $A = \left[\ a_{i,j}\ \right]_{n \times n}$ be a matrix in $\mathbb{F}^{n \times n}$. If A has a zero row or a zero column,[a] then $\det(A) = 0$.

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[a] A *zero row* is a row in which all entries are zero. Similarly, a *zero column* is a column in which all entries are zero.

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Proof. In view of Theorem 7.1.3, it suffices to consider the case when A has a zero row.

Suppose that that the p-th row of A is a zero row. Then for all $\sigma \in S_n$, we have that $a_{p,\sigma(p)} = 0$.

Proposition 7.1.4

Let $\mathbb{F}$ be a field, and let $A = \left[\ a_{i,j}\ \right]_{n \times n}$ be a matrix in $\mathbb{F}^{n \times n}$. If A has a zero row or a zero column,[a] then $\det(A) = 0$.

[a] A *zero row* is a row in which all entries are zero. Similarly, a *zero column* is a column in which all entries are zero.

Proof. In view of Theorem 7.1.3, it suffices to consider the case when A has a zero row.

Suppose that that the p-th row of A is a zero row. Then for all $\sigma \in S_n$, we have that $a_{p,\sigma(p)} = 0$. Consequently,

$$\det(A) \quad = \quad \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{1,\sigma(1)} \dots \underbrace{a_{p,\sigma(p)}}_{=0} \dots a_{n,\sigma(n)} \quad = \quad 0,$$

which is what we needed to show. □

Proposition 7.1.5

Let $\mathbb{F}$ be a field, and let $A = \left[\; a_{i,j} \; \right]_{n \times n}$ be a matrix in $\mathbb{F}^{n \times n}$. If A has two identical rows or two identical columns, then $\det(A) = 0$.

Proof.

Proposition 7.1.5

Let $\mathbb{F}$ be a field, and let $A = \left[\ a_{i,j}\ \right]_{n \times n}$ be a matrix in $\mathbb{F}^{n \times n}$. If A has two identical rows or two identical columns, then $\det(A) = 0$.

Proof. In view of Theorem 7.1.3, it suffices to consider the case when A has two identical rows.

Proposition 7.1.5

Let $\mathbb{F}$ be a field, and let $A = \begin{bmatrix} a_{i,j} \end{bmatrix}_{n \times n}$ be a matrix in $\mathbb{F}^{n \times n}$. If A has two identical rows or two identical columns, then $\det(A) = 0$.

Proof. In view of Theorem 7.1.3, it suffices to consider the case when A has two identical rows.

So, suppose that for some distinct $p, q \in \{1, \dots, n\}$, the p-th and q-th row of A are the same. (In particular, $n \geq 2$.)

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So, suppose that for some distinct $p, q \in \{1, \dots, n\}$, the p-th and q-th row of A are the same. (In particular, $n \geq 2$.)

Now, let A_n be the alternating group of degree n, i.e. the group of all even permutations in S_n, and let O_n be the set of all odd permutations in S_n.

Proposition 7.1.5

Let $\mathbb{F}$ be a field, and let $A = \left[\ a_{i,j}\ \right]_{n\times n}$ be a matrix in $\mathbb{F}^{n\times n}$. If A has two identical rows or two identical columns, then $\det(A) = 0$.

Proof. In view of Theorem 7.1.3, it suffices to consider the case when A has two identical rows.

So, suppose that for some distinct $p, q \in \{1, \dots, n\}$, the p-th and q-th row of A are the same. (In particular, $n \geq 2$.)

Now, let A_n be the alternating group of degree n, i.e. the group of all even permutations in S_n, and let O_n be the set of all odd permutations in S_n. Obviously, $S_n = A_n \cup O_n$ and $A_n \cap O_n = \emptyset$.

Next, consider the transposition $\tau = (pq)$. By Proposition 2.3.2, for all $\sigma \in S_n$, we have that $\text{sgn}(\sigma \circ \tau) = -\text{sgn}(\sigma)$; it then readily follows that $O_n = \{\sigma \circ \tau \mid \sigma \in A_n\}$, and obviously, for all distinct $\sigma_1, \sigma_2 \in A_n$, we have that $\sigma_1 \circ \tau \neq \sigma_2 \circ \tau$.

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Let $\mathbb{F}$ be a field, and let $A = \left[\ a_{i,j}\ \right]_{n \times n}$ be a matrix in $\mathbb{F}^{n \times n}$. If A has two identical rows or two identical columns, then $\det(A) = 0$.

Proof (continued). Reminder: $\tau = (pq)$.

Claim. $\forall \sigma \in S_n$: $\prod_{i=1}^{n} a_{i,\sigma(i)} = \prod_{i=1}^{n} a_{i,\sigma \circ \tau(i)}$.

Proof of the Claim.

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Claim. *$\forall \sigma \in S_n$: $\prod_{i=1}^{n} a_{i,\sigma(i)} = \prod_{i=1}^{n} a_{i,\sigma\circ\tau(i)}$.*

Proof of the Claim. Fix $\sigma \in S_n$.

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Proof of the Claim. Fix $\sigma \in S_n$. First, note that

- $a_{p,\sigma(p)} = a_{p,\sigma\circ\tau(q)} \overset{(*)}{=} a_{q,\sigma\circ\tau(q)}$,
- $a_{q,\sigma(q)} = a_{q,\sigma\circ\tau(p)} \overset{(*)}{=} a_{p,\sigma\circ\tau(p)}$,

where both instances of (*) follow from the fact that the p-th and q-th row of A are the same.

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where both instances of (*) follow from the fact that the p-th and q-th row of A are the same. So, $a_{p,\sigma(p)}a_{q,\sigma(q)} = a_{p,\sigma\circ\tau(p)}a_{q,\sigma\circ\tau(q)}$.

Proposition 7.1.5

Let $\mathbb{F}$ be a field, and let $A = \left[\ a_{i,j}\ \right]_{n \times n}$ be a matrix in $\mathbb{F}^{n \times n}$. If A has two identical rows or two identical columns, then $\det(A) = 0$.

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Claim. $\forall \sigma \in S_n$: $\prod_{i=1}^{n} a_{i,\sigma(i)} = \prod_{i=1}^{n} a_{i,\sigma \circ \tau(i)}$.

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where both instances of (*) follow from the fact that the p-th and q-th row of A are the same. So, $a_{p,\sigma(p)} a_{q,\sigma(q)} = a_{p,\sigma\circ\tau(p)} a_{q,\sigma\circ\tau(q)}$. On the other hand, it is clear that for all $i \in \{1, \dots, n\} \setminus \{p, q\}$, we have that $a_{i,\sigma(i)} = a_{i,\sigma\circ\tau(i)}$.

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Let $\mathbb{F}$ be a field, and let $A = \left[\ a_{i,j}\ \right]_{n \times n}$ be a matrix in $\mathbb{F}^{n \times n}$. If A has two identical rows or two identical columns, then $\det(A) = 0$.

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where both instances of (*) follow from the fact that the p-th and q-th row of A are the same. So, $a_{p,\sigma(p)} a_{q,\sigma(q)} = a_{p,\sigma \circ \tau(p)} a_{q,\sigma \circ \tau(q)}$. On the other hand, it is clear that for all $i \in \{1, \dots, n\} \setminus \{p, q\}$, we have that $a_{i,\sigma(i)} = a_{i,\sigma \circ \tau(i)}$. It follows that $\prod_{i=1}^{n} a_{i,\sigma(i)} = \prod_{i=1}^{n} a_{i,\sigma \circ \tau(i)}$, which is what we needed to show. ♦

Proposition 7.1.5

Let $\mathbb{F}$ be a field, and let $A = \begin{bmatrix} a_{i,j} \end{bmatrix}_{n \times n}$ be a matrix in $\mathbb{F}^{n \times n}$. If A has two identical rows or two identical columns, then $\det(A) = 0$.

Proof (continued). Reminder: $\tau = (pq)$; $O_n = \{\sigma \circ \tau \mid \sigma \in A_n\}$

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Let $\mathbb{F}$ be a field, and let $A = \begin{bmatrix} a_{i,j} \end{bmatrix}_{n \times n}$ be a matrix in $\mathbb{F}^{n \times n}$. If A has two identical rows or two identical columns, then $\det(A) = 0$.

Proof (continued). Reminder: $\tau = (pq)$; $O_n = \{\sigma \circ \tau \mid \sigma \in A_n\}$

Claim. $\forall \sigma \in S_n$: $\prod_{i=1}^{n} a_{i,\sigma(i)} = \prod_{i=1}^{n} a_{i,\sigma\circ\tau(i)}$.

We now compute:

$$\begin{aligned}
\det(A) &= \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{1,\sigma(1)} \dots a_{n,\sigma(n)} \\
&= \sum_{\sigma \in A_n} \underbrace{\text{sgn}(\sigma)}_{=1} a_{1,\sigma(1)} \dots a_{n,\sigma(n)} + \sum_{\pi \in O_n} \underbrace{\text{sgn}(\pi)}_{=-1} a_{1,\pi(1)} \dots a_{n,\pi(n)} \\
&= \sum_{\sigma \in A_n} a_{1,\sigma(1)} \dots a_{n,\sigma(n)} - \sum_{\pi \in O_n} a_{1,\pi(1)} \dots a_{n,\pi(n)} \\
&= \sum_{\sigma \in A_n} a_{1,\sigma(1)} \dots a_{n,\sigma(n)} - \sum_{\sigma \in A_n} a_{1,\sigma\circ\tau(1)} \dots a_{n,\sigma\circ\tau(n)} \overset{(*)}{=} 0,
\end{aligned}$$

where (*) follows from the Claim. □

2. The linearity of determinants in one row or one column

2. The linearity of determinants in one row or one column

- In general, for matrices $A, B \in \mathbb{F}^{n \times n}$ (where $\mathbb{F}$ is some field) and a scalar $\alpha \in \mathbb{F}$, we have that

$$\det(A+B) \;\neq\; \det(A)+\det(B) \qquad \text{and} \qquad \det(\alpha A) \;\neq\; \alpha\det(A).$$

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$$\det(A+B) \;\;\not\asymp\;\; \det(A)+\det(B) \qquad \text{and} \qquad \det(\alpha A) \;\;\not\asymp\;\; \alpha\det(A).$$

- We do, however, have the following proposition (next slide).
  - We first state the proposition, then we give an examples to illustrate how it can be used, and then we prove the proposition.

Proposition 7.2.1

Let $\mathbb{F}$ be a field, and let $\mathbf{a}_1, \dots, \mathbf{a}_{p-1}, \mathbf{a}_{p+1}, \dots, \mathbf{a}_n \in \mathbb{F}^n$. Then:

(a) the function $f_{C_p} : \mathbb{F}^n \to \mathbb{F}$ given by

$$f_{C_p}(\mathbf{x}) \quad = \quad \det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{p-1} & \mathbf{x} & \mathbf{a}_{p+1} & \dots & \mathbf{a}_n \end{bmatrix}\right)$$

for all $\mathbf{x} \in \mathbb{F}^n$ is linear;

(b) the function $f_{R_p} : \mathbb{F}^n \to \mathbb{F}$ given by

$$f_{R_p}(\mathbf{x}) \quad = \quad \det\left(\begin{bmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_{p-1}^T \\ \mathbf{x}^T \\ \mathbf{a}_{p+1}^T \\ \vdots \\ \mathbf{a}_n^T \end{bmatrix}\right)$$

for all $\mathbf{x} \in \mathbb{F}^n$ is linear.

Example 7.2.2

By Proposition 7.2.1, we have the following (entries are understood to be in $\mathbb{R}$, and the row/column being manipulated is in red to facilitate reading):

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- $$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 0 & 1 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 4 \\ 0 & -2 & 5 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 0 & 3 & 5 \end{vmatrix};$$

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- $\begin{vmatrix} 3 & 2 & 4 \\ 6 & -1 & 0 \\ -3 & 0 & 5 \end{vmatrix} = 3 \begin{vmatrix} 1 & 2 & 4 \\ 2 & -1 & 0 \\ -1 & 0 & 5 \end{vmatrix}$;

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- $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 7 & 3 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 4 & 4 & -2 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & -1 & 0 \end{vmatrix}$;

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- $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 7 & 3 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 4 & 4 & -2 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & -1 & 0 \end{vmatrix}$;
- $\begin{vmatrix} 2 & -2 & 4 \\ 1 & 0 & -2 \\ 2 & 1 & 4 \end{vmatrix} = 2 \begin{vmatrix} 1 & -1 & 2 \\ 1 & 0 & -2 \\ 2 & 1 & 4 \end{vmatrix}$.

Proposition 7.2.1

Let $\mathbb{F}$ be a field, and let $\mathbf{a}_1, \dots, \mathbf{a}_{p-1}, \mathbf{a}_{p+1}, \dots, \mathbf{a}_n \in \mathbb{F}^n$. Then:

(a) the function $f_{C_p} : \mathbb{F}^n \to \mathbb{F}$ given by

$$f_{C_p}(\mathbf{x}) = \det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{p-1} & \mathbf{x} & \mathbf{a}_{p+1} & \dots & \mathbf{a}_n \end{bmatrix} \right)$$

for all $\mathbf{x} \in \mathbb{F}^n$ is linear;

(b) the function $f_{R_p} : \mathbb{F}^n \to \mathbb{F}$ given by

$$f_{R_p}(\mathbf{x}) = \det\left(\begin{bmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_{p-1}^T \\ \mathbf{x}^T \\ \mathbf{a}_{p+1}^T \\ \vdots \\ \mathbf{a}_n^T \end{bmatrix} \right)$$

for all $\mathbf{x} \in \mathbb{F}^n$ is linear.

Proof.

Proof. Clearly, (b) and Theorem 7.1.3 imply (a). So, it suffices to prove (b).

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We first set up some notation. For each index $i \in \{1, \dots, n\} \setminus \{p\}$, we set $\mathbf{a}_i = \begin{bmatrix} a_{i,1} & \dots & a_{i,n} \end{bmatrix}^T$, so that $\mathbf{a}_i^T = \begin{bmatrix} a_{i,1} & \dots & a_{i,n} \end{bmatrix}$.

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Proof. Clearly, (b) and Theorem 7.1.3 imply (a). So, it suffices to prove (b).

We first set up some notation. For each index $i \in \{1, \dots, n\} \setminus \{p\}$, we set $\mathbf{a}_i = \begin{bmatrix} a_{i,1} & \dots & a_{i,n} \end{bmatrix}^T$, so that $\mathbf{a}_i^T = \begin{bmatrix} a_{i,1} & \dots & a_{i,n} \end{bmatrix}$. Now, let us prove that f_{R_p} is linear.

1. Fix $\mathbf{x}, \mathbf{y} \in \mathbb{F}^n$, and set $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$ and $\mathbf{y} = \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix}^T$. We compute (next slide):

Proof (continued).

$$f_{R_p}(\mathbf{x}+\mathbf{y}) = \det\left(\begin{bmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_{p-1}^T \\ (\mathbf{x}+\mathbf{y})^T \\ \mathbf{a}_{p+1}^T \\ \vdots \\ \mathbf{a}_n^T \end{bmatrix}\right) = \begin{vmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{p-1,1} & \cdots & a_{p-1,n} \\ x_1+y_1 & \cdots & x_n+y_n \\ a_{p+1,1} & \cdots & a_{p+1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{vmatrix}$$

$$= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} \cdots a_{p-1,\sigma(p-1)} \left(x_{\sigma(p)} + y_{\sigma(p)}\right) a_{p+1,\sigma(p+1)} \cdots a_{n,\sigma(n)}$$

$$= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} \cdots a_{p-1,\sigma(p-1)} x_{\sigma(p)} a_{p+1,\sigma(p+1)} \cdots a_{n,\sigma(n)} + \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} \cdots a_{p-1,\sigma(p-1)} y_{\sigma(p)} a_{p+1,\sigma(p+1)} \cdots a_{n,\sigma(n)}$$

$$= \begin{vmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{p-1,1} & \cdots & a_{p-1,n} \\ x_1 & \cdots & x_n \\ a_{p+1,1} & \cdots & a_{p+1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{vmatrix} + \begin{vmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{p-1,1} & \cdots & a_{p-1,n} \\ y_1 & \cdots & y_n \\ a_{p+1,1} & \cdots & a_{p+1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{vmatrix} = f_{R_p}(\mathbf{x}) + f_{R_p}(\mathbf{y}).$$

Proof (continued). 2. Fix $\mathbf{x} \in \mathbb{F}^n$ and $\alpha \in \mathbb{F}$, and set $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$. We compute:

$$
\begin{aligned}
f_{R_p}(\alpha \mathbf{x}) &= \det\left(\begin{bmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_{p-1}^T \\ \alpha \mathbf{x}^T \\ \mathbf{a}_{p+1}^T \\ \vdots \\ \mathbf{a}_n^T \end{bmatrix}\right) = \begin{vmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{p-1,1} & \dots & a_{p-1,n} \\ \alpha x_1 & \dots & \alpha x_n \\ a_{p+1,1} & \dots & a_{p+1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{vmatrix} \\
&= \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{1,\sigma(1)} \dots a_{p-1,\sigma(p-1)} \left(\alpha x_{\sigma(p)}\right) a_{p+1,\sigma(p+1)} \dots a_{n,\sigma(n)} \\
&= \alpha \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{1,\sigma(1)} \dots a_{p-1,\sigma(p-1)} x_{\sigma(p)} a_{p+1,\sigma(p+1)} \dots a_{n,\sigma(n)} \\
&= \alpha \begin{vmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{p-1,1} & \dots & a_{p-1,n} \\ x_1 & \dots & x_n \\ a_{p+1,1} & \dots & a_{p+1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{vmatrix} = \alpha \det\left(\begin{bmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_{p-1}^T \\ \mathbf{x}^T \\ \mathbf{a}_{p+1}^T \\ \vdots \\ \mathbf{a}_n^T \end{bmatrix}\right) = \alpha f_{R_p}(\mathbf{x}).
\end{aligned}
$$

□

Proposition 7.2.1

Let $\mathbb{F}$ be a field, and let $\mathbf{a}_1, \dots, \mathbf{a}_{p-1}, \mathbf{a}_{p+1}, \dots, \mathbf{a}_n \in \mathbb{F}^n$. Then:

(a) the function $f_{C_p} : \mathbb{F}^n \to \mathbb{F}$ given by

$$f_{C_p}(\mathbf{x}) = \det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{p-1} & \mathbf{x} & \mathbf{a}_{p+1} & \dots & \mathbf{a}_n \end{bmatrix}\right)$$

for all $\mathbf{x} \in \mathbb{F}^n$ is linear;

(b) the function $f_{R_p} : \mathbb{F}^n \to \mathbb{F}$ given by

$$f_{R_p}(\mathbf{x}) = \det\left(\begin{bmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_{p-1}^T \\ \mathbf{x}^T \\ \mathbf{a}_{p+1}^T \\ \vdots \\ \mathbf{a}_n^T \end{bmatrix}\right)$$

for all $\mathbf{x} \in \mathbb{F}^n$ is linear.

Proposition 7.2.3

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n \times n}$, and let $\alpha \in \mathbb{F}$. Then

$$\det(\alpha A) = \alpha^n \det(A).$$

Proof.

Proposition 7.2.3

Let $\mathbb{F}$ be a field, let $A \in \mathbb{F}^{n \times n}$, and let $\alpha \in \mathbb{F}$. Then

$$\det(\alpha A) \quad = \quad \alpha^n \det(A).$$

Proof. We apply Proposition 7.2.1 n times, once to each row (or alternatively, once to each column) of αA, and the result follows. $\square$

3. Computing determinants via elementary row and column operations

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- Our goal is to examine how performing elementary row and column operations affects the value of the determinant, and how we can use these operations to compute the determinant of a square matrix.
  - We studied elementary row operations last semester (see chapter 1 of the Lecture Notes).
  - Elementary column operations are defined completely analogously, only for columns instead of rows.

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  - We studied elementary row operations last semester (see chapter 1 of the Lecture Notes).
  - Elementary column operations are defined completely analogously, only for columns instead of rows.
- Elementary column operations should **not** be used for solving linear systems.

3. Computing determinants via elementary row and column operations

- Our goal is to examine how performing elementary row and column operations affects the value of the determinant, and how we can use these operations to compute the determinant of a square matrix.
  - We studied elementary row operations last semester (see chapter 1 of the Lecture Notes).
  - Elementary column operations are defined completely analogously, only for columns instead of rows.
- Elementary column operations should **not** be used for solving linear systems.
- However, it turns out that both elementary row operations and elementary column operations behave well with respect to determinants, i.e. they change the value of the determinant in a way that we can describe precisely, as we shall see.

Definition

Given a square matrix $A = \left[\ a_{i,j}\ \right]_{n\times n}$ in $\mathbb{F}^{n\times n}$ (where $\mathbb{F}$ is some field), we say that

- A is *upper triangular* if all entries of A below the main diagonal are zero, i.e. if $\forall i, j \in \{1, \dots, n\}$ s.t. $i > j$, we have that $a_{i,j} = 0$;
- A is *lower triangular* if all entries of A above the main diagonal are zero, i.e. if $\forall i, j \in \{1, \dots, n\}$ s.t. $i < j$, we have that $a_{i,j} = 0$;
- A is *triangular* if it is upper triangular or lower triangular.

$$\begin{bmatrix} * & * & * & \dots & * & * \\ 0 & * & * & \dots & * & * \\ 0 & 0 & * & \dots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & * & * \\ 0 & 0 & 0 & \dots & 0 & * \end{bmatrix}$$

upper triangular matrix

$$\begin{bmatrix} * & 0 & 0 & \dots & 0 & 0 \\ * & * & 0 & \dots & 0 & 0 \\ * & * & * & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \dots & * & 0 \\ * & * & * & \dots & * & * \end{bmatrix}$$

lower triangular matrix

$$\begin{bmatrix} * & * & * & \cdots & * & * \\ 0 & * & * & \cdots & * & * \\ 0 & 0 & * & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & * & * \\ 0 & 0 & 0 & \cdots & 0 & * \end{bmatrix} \qquad \begin{bmatrix} * & 0 & 0 & \cdots & 0 & 0 \\ * & * & 0 & \cdots & 0 & 0 \\ * & * & * & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \cdots & * & 0 \\ * & * & * & \cdots & * & * \end{bmatrix}$$

upper triangular matrix — lower triangular matrix

- Note that any square matrix in row echelon form is in fact an upper triangular matrix.
  - However, not all upper triangular matrices are in row echelon form.

$$\begin{bmatrix} * & * & * & \cdots & * & * \\ 0 & * & * & \cdots & * & * \\ 0 & 0 & * & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & * & * \\ 0 & 0 & 0 & \cdots & 0 & * \end{bmatrix} \qquad \begin{bmatrix} * & 0 & 0 & \cdots & 0 & 0 \\ * & * & 0 & \cdots & 0 & 0 \\ * & * & * & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \cdots & * & 0 \\ * & * & * & \cdots & * & * \end{bmatrix}$$

upper triangular matrix — lower triangular matrix

- Note that any square matrix in row echelon form is in fact an upper triangular matrix.
  - However, not all upper triangular matrices are in row echelon form.
- So, the row reduction algorithm performed on a square matrix will, in particular, yield an upper triangular matrix.

$$\begin{bmatrix} * & * & * & \cdots & * & * \\ 0 & * & * & \cdots & * & * \\ 0 & 0 & * & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & * & * \\ 0 & 0 & 0 & \cdots & 0 & * \end{bmatrix}$$

upper triangular matrix

$$\begin{bmatrix} * & 0 & 0 & \cdots & 0 & 0 \\ * & * & 0 & \cdots & 0 & 0 \\ * & * & * & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \cdots & * & 0 \\ * & * & * & \cdots & * & * \end{bmatrix}$$

lower triangular matrix

- Note that any square matrix in row echelon form is in fact an upper triangular matrix.
  - However, not all upper triangular matrices are in row echelon form.
- So, the row reduction algorithm performed on a square matrix will, in particular, yield an upper triangular matrix.
- It turns out that the determinant of any triangular matrix is particularly easy to compute, as we now show (next slide).

Proposition 7.3.1

Let $\mathbb{F}$ be a field, and let $A = \left[\ a_{i,j}\ \right]_{n\times n}$ be a triangular matrix in $\mathbb{F}^{n\times n}$. Then

$$\det(A) \quad = \quad \prod_{i=1}^{n} a_{i,i} \quad = \quad a_{1,1}a_{2,2}\dots a_{n,n},$$

that is, $\det(A)$ is equal to the product of entries on the main diagonal of A.

- For example, we can compute the determinants of the following matrices in $\mathbb{R}^{3\times 3}$ as follows:
  - $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix} = 1\cdot 4\cdot 6 = 24;$
  - $\begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{vmatrix} = 1\cdot 3\cdot 6 = 18.$

Proposition 7.3.1

Let $\mathbb{F}$ be a field, and let $A = \left[\ a_{i,j}\ \right]_{n\times n}$ be a triangular matrix in $\mathbb{F}^{n\times n}$. Then

$$\det(A) \quad = \quad \prod_{i=1}^{n} a_{i,i} \quad = \quad a_{1,1}a_{2,2}\dots a_{n,n},$$

that is, $\det(A)$ is equal to the product of entries on the main diagonal of A.

Proof.

Proposition 7.3.1

Let $\mathbb{F}$ be a field, and let $A = \left[\ a_{i,j}\ \right]_{n \times n}$ be a triangular matrix in $\mathbb{F}^{n \times n}$. Then

$$\det(A) \;=\; \prod_{i=1}^{n} a_{i,i} \;=\; a_{1,1}a_{2,2}\dots a_{n,n},$$

that is, $\det(A)$ is equal to the product of entries on the main diagonal of A.

Proof. Note that the transpose of an upper triangular matrix is a lower triangular matrix, and moreover, the main diagonal remains unchanged when we take the transpose of a square matrix. So, in view of Theorem 7.1.3, it suffices to prove the result for the case when A is lower triangular.

$$\begin{bmatrix} * & * & * & \cdots & * & * \\ 0 & * & * & \cdots & * & * \\ 0 & 0 & * & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & * & * \\ 0 & 0 & 0 & \cdots & 0 & * \end{bmatrix} \qquad \begin{bmatrix} * & 0 & 0 & \cdots & 0 & 0 \\ * & * & 0 & \cdots & 0 & 0 \\ * & * & * & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \cdots & * & 0 \\ * & * & * & \cdots & * & * \end{bmatrix}$$

upper triangular matrix   lower triangular matrix

Proof (continued). Reminder: A is lower triangular; WTS $\det(A) = a_{1,1}a_{2,2}\dots a_{n,n}$.

Now, note that for all $\sigma \in S_n \setminus \{1\}$, there exists some index $i \in \{1, \dots, n\}$ s.t. $i < \sigma(i)$,

$$\begin{bmatrix} * & * & * & \dots & * & * \\ 0 & * & * & \dots & * & * \\ 0 & 0 & * & \dots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & * & * \\ 0 & 0 & 0 & \dots & 0 & * \end{bmatrix} \qquad \begin{bmatrix} * & 0 & 0 & \dots & 0 & 0 \\ * & * & 0 & \dots & 0 & 0 \\ * & * & * & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \dots & * & 0 \\ * & * & * & \dots & * & * \end{bmatrix}$$

upper triangular matrix — lower triangular matrix

Proof (continued). Reminder: A is lower triangular; WTS $\det(A) = a_{1,1}a_{2,2}\dots a_{n,n}$.

Now, note that for all $\sigma \in S_n \setminus \{1\}$, there exists some index $i \in \{1,\dots,n\}$ s.t. $i < \sigma(i)$, and consequently, $a_{i,\sigma(i)} = 0$ (since A is lower triangular).

$$\begin{bmatrix} * & * & * & \dots & * & * \\ 0 & * & * & \dots & * & * \\ 0 & 0 & * & \dots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & * & * \\ 0 & 0 & 0 & \dots & 0 & * \end{bmatrix}$$

upper triangular matrix

$$\begin{bmatrix} * & 0 & 0 & \dots & 0 & 0 \\ * & * & 0 & \dots & 0 & 0 \\ * & * & * & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \dots & * & 0 \\ * & * & * & \dots & * & * \end{bmatrix}$$

lower triangular matrix

Proof (continued). Reminder: A is lower triangular; WTS $\det(A) = a_{1,1}a_{2,2}\dots a_{n,n}$.

Now, note that for all $\sigma \in S_n \setminus \{1\}$, there exists some index $i \in \{1, \dots, n\}$ s.t. $i < \sigma(i)$, and consequently, $a_{i,\sigma(i)} = 0$ (since A is lower triangular).

It follows that for all $\sigma \in S_n \setminus \{1\}$, we have that $a_{1,\sigma(1)}a_{2,\sigma(2)}\dots a_{n,\sigma(n)} = 0$,

$$\begin{bmatrix} * & * & * & \dots & * & * \\ 0 & * & * & \dots & * & * \\ 0 & 0 & * & \dots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & * & * \\ 0 & 0 & 0 & \dots & 0 & * \end{bmatrix}$$

upper triangular matrix

$$\begin{bmatrix} * & 0 & 0 & \dots & 0 & 0 \\ * & * & 0 & \dots & 0 & 0 \\ * & * & * & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \dots & * & 0 \\ * & * & * & \dots & * & * \end{bmatrix}$$

lower triangular matrix

Proof (continued). Reminder: A is lower triangular; WTS $\det(A) = a_{1,1}a_{2,2}\dots a_{n,n}$.

Now, note that for all $\sigma \in S_n \setminus \{1\}$, there exists some index $i \in \{1, \dots, n\}$ s.t. $i < \sigma(i)$, and consequently, $a_{i,\sigma(i)} = 0$ (since A is lower triangular).

It follows that for all $\sigma \in S_n \setminus \{1\}$, we have that $a_{1,\sigma(1)}a_{2,\sigma(2)}\dots a_{n,\sigma(n)} = 0$, and consequently,

$$\begin{aligned} \det(A) &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1,\sigma(1)}a_{2,\sigma(2)}\dots a_{n,\sigma(n)} \\ &= \operatorname{sgn}(1) a_{1,1}a_{2,2}\dots a_{n,n} \\ &= a_{1,1}a_{2,2}\dots a_{n,n}. \end{aligned}$$

□

Proposition 7.3.1

Let $\mathbb{F}$ be a field, and let $A = \left[\ a_{i,j}\ \right]_{n\times n}$ be a triangular matrix in $\mathbb{F}^{n\times n}$. Then

$$\det(A) \quad = \quad \prod_{i=1}^{n} a_{i,i} \quad = \quad a_{1,1}a_{2,2}\ldots a_{n,n},$$

that is, $\det(A)$ is equal to the product of entries on the main diagonal of A.

$$\begin{bmatrix} * & * & * & \ldots & * & * \\ 0 & * & * & \ldots & * & * \\ 0 & 0 & * & \ldots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & * & * \\ 0 & 0 & 0 & \ldots & 0 & * \end{bmatrix}$$

upper triangular matrix

$$\begin{bmatrix} * & 0 & 0 & \ldots & 0 & 0 \\ * & * & 0 & \ldots & 0 & 0 \\ * & * & * & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \ldots & * & 0 \\ * & * & * & \ldots & * & * \end{bmatrix}$$

lower triangular matrix

Theorem 7.3.2

Let $\mathbb{F}$ be a field, and let $A = \left[\ a_{i,j}\ \right]_{n \times n}$ be a matrix in $\mathbb{F}^{n \times n}$. Then all the following hold:

(a) if a matrix B is obtained by swapping two rows or swapping two columns of A, then
$$\det(B) \quad = \quad -\det(A);$$

(b) if a matrix B is obtained by multiplying some row or some column of A by a scalar $\alpha \in \mathbb{F} \setminus \{0\}$, then
$$\det(B) = \alpha \det(A) \qquad \text{and} \qquad \det(A) = \alpha^{-1} \det(B);$$

(c) if a matrix B is obtained from A by adding a scalar multiple of one row (resp. column) of A to another row (resp. column) of A, then
$$\det(B) \quad = \quad \det(A).$$

- First an example, then a proof.

Example 7.3.3

Compute the determinant of the matrix below (with entries understood to be in $\mathbb{R}$).

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 2 & 4 & 4 \\ 3 & 3 & 7 \end{bmatrix}$$

Solution.

Example 7.3.3

Compute the determinant of the matrix below (with entries understood to be in $\mathbb{R}$).

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 2 & 4 & 4 \\ 3 & 3 & 7 \end{bmatrix}$$

Solution. We perform elementary row operations on A (keeping track of the way that this changes the value of the determinant, as per Theorem 7.3.2) until we transform A into a matrix in row echelon form. Square matrices in row echelon form are upper triangular, and so by Proposition 7.3.1, we can obtain their determinant by multiplying the entries on the main diagonal.

Example 7.3.3

Compute the determinant of the matrix below (with entries understood to be in $\mathbb{R}$).

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Proof (continued).

$$\det(A) = \begin{vmatrix} 2 & 4 & 6 \\ 2 & 4 & 4 \\ 3 & 3 & 7 \end{vmatrix} \overset{R_2 \to R_2 - R_1}{=} \begin{vmatrix} 2 & 4 & 6 \\ 0 & 0 & -2 \\ 3 & 3 & 7 \end{vmatrix}$$

$$\overset{R_2 \leftrightarrow R_3}{=} -\begin{vmatrix} 2 & 4 & 6 \\ 3 & 3 & 7 \\ 0 & 0 & -2 \end{vmatrix} \overset{R_1 \to \frac{1}{2} R_1}{=} -2\begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 7 \\ 0 & 0 & -2 \end{vmatrix}$$

$$\overset{R_2 \to R_1 - 3R_1}{=} -2\begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & 0 & -2 \end{vmatrix}$$

$$\overset{(*)}{=} (-2)1(-3)(-2) = -12,$$

where (*) follows by taking the determinant of an upper triangular matrix. □

Theorem 7.3.2

Let $\mathbb{F}$ be a field, and let $A = \left[\ a_{i,j}\ \right]_{n \times n}$ be a matrix in $\mathbb{F}^{n \times n}$. Then all the following hold:

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Proof.

Theorem 7.3.2

Let $\mathbb{F}$ be a field, and let $A = \left[\begin{array}{c} a_{i,j} \end{array}\right]_{n \times n}$ be a matrix in $\mathbb{F}^{n \times n}$. Then all the following hold:

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(c) if a matrix B is obtained from A by adding a scalar multiple of one row (resp. column) of A to another row (resp. column) of A, then

$$\det(B) = \det(A).$$

Proof. In view of Theorem 7.1.3, it suffices to prove the result for row operations only.

Proof (continued). (a) Fix distinct indices $p, q \in \{1, \ldots, n\}$, and suppose that B is obtained by swapping rows p and q of A ("$R_p \leftrightarrow R_q$").

Proof (continued). (a) Fix distinct indices $p, q \in \{1, \dots, n\}$, and suppose that B is obtained by swapping rows p and q of A ("$R_p \leftrightarrow R_q$"). Set $B = \left[\; b_{i,j} \; \right]_{n \times n}$, so that

- for all $j \in \{1, \dots, n\}$, we have that $b_{p,j} = a_{q,j}$ and $b_{q,j} = a_{p,j}$;
- for all $i \in \{1, \dots, n\} \setminus \{p, q\}$ and $j \in \{1, \dots, n\}$, we have that $b_{i,j} = a_{i,j}$.

Proof (continued). (a) Fix distinct indices $p, q \in \{1, \dots, n\}$, and suppose that B is obtained by swapping rows p and q of A ("$R_p \leftrightarrow R_q$"). Set $B = \left[\begin{array}{c} b_{i,j} \end{array}\right]_{n \times n}$, so that

- for all $j \in \{1, \dots, n\}$, we have that $b_{p,j} = a_{q,j}$ and $b_{q,j} = a_{p,j}$;
- for all $i \in \{1, \dots, n\} \setminus \{p, q\}$ and $j \in \{1, \dots, n\}$, we have that $b_{i,j} = a_{i,j}$.

Next, consider the transposition $\tau = (pq)$ in S_n.

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Claim. $\forall \sigma \in S_n$: $\prod_{i=1}^{n} b_{i,\sigma(i)} = \prod_{i=1}^{n} a_{i,\sigma\circ\tau(i)}$.

Proof of the Claim.

Proof (continued). (a) Fix distinct indices $p, q \in \{1, \dots, n\}$, and suppose that B is obtained by swapping rows p and q of A ("$R_p \leftrightarrow R_q$"). Set $B = \left[\ b_{i,j}\ \right]_{n \times n}$, so that

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Proof of the Claim. First, we note that

- $b_{p,\sigma(p)} = a_{q,\sigma(p)} = a_{q,\sigma\circ\tau(q)}$;
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- $b_{q,\sigma(q)} = a_{p,\sigma(q)} = a_{p,\sigma \circ \tau(p)}$.

So, $b_{p,\sigma(p)} b_{q,\sigma(q)} = a_{p,\sigma \circ \tau(p)} a_{q,\sigma \circ \tau(q)}$.

Proof (continued). (a) Fix distinct indices $p, q \in \{1, \dots, n\}$, and suppose that B is obtained by swapping rows p and q of A ("$R_p \leftrightarrow R_q$"). Set $B = \left[\ b_{i,j}\ \right]_{n \times n}$, so that

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So, $b_{p,\sigma(p)} b_{q,\sigma(q)} = a_{p,\sigma\circ\tau(p)} a_{q,\sigma\circ\tau(q)}$. On the other hand, for all $i \in \{1, \dots, n\} \setminus \{p, q\}$, we have that $b_{i,\sigma(i)} = a_{i,\sigma\circ\tau(i)}$.

Proof (continued). (a) Fix distinct indices $p, q \in \{1, \dots, n\}$, and suppose that B is obtained by swapping rows p and q of A ("$R_p \leftrightarrow R_q$"). Set $B = \begin{bmatrix} b_{i,j} \end{bmatrix}_{n \times n}$, so that

- for all $j \in \{1, \dots, n\}$, we have that $b_{p,j} = a_{q,j}$ and $b_{q,j} = a_{p,j}$;
- for all $i \in \{1, \dots, n\} \setminus \{p, q\}$ and $j \in \{1, \dots, n\}$, we have that $b_{i,j} = a_{i,j}$.

Next, consider the transposition $\tau = (pq)$ in S_n.

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- $b_{q,\sigma(q)} = a_{p,\sigma(q)} = a_{p,\sigma\circ\tau(p)}$.

So, $b_{p,\sigma(p)} b_{q,\sigma(q)} = a_{p,\sigma\circ\tau(p)} a_{q,\sigma\circ\tau(q)}$. On the other hand, for all $i \in \{1, \dots, n\} \setminus \{p, q\}$, we have that $b_{i,\sigma(i)} = a_{i,\sigma\circ\tau(i)}$. It follows that $\prod_{i=1}^{n} b_{i,\sigma(i)} = \prod_{i=1}^{n} a_{i,\sigma\circ\tau(i)}$, which is what we needed to show. ♦

Proof (continued). Reminder: $\tau = (pq)$.

Claim. $\forall \sigma \in S_n$: $\prod_{i=1}^{n} b_{i,\sigma(i)} = \prod_{i=1}^{n} a_{i,\sigma\circ\tau(i)}$.

Proof (continued). Reminder: $\tau = (pq)$.

Claim. $\forall \sigma \in S_n$: $\prod_{i=1}^{n} b_{i,\sigma(i)} = \prod_{i=1}^{n} a_{i,\sigma\circ\tau(i)}$.

We now compute:

$$
\begin{aligned}
\det(B) &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \textstyle\prod_{i=1}^{n} b_{i,\sigma(i)} \\
&\overset{(*)}{=} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \textstyle\prod_{i=1}^{n} a_{i,\sigma\circ\tau(i)} \\
&\overset{(**)}{=} \sum_{\sigma \in S_n} \Big(-\operatorname{sgn}(\sigma\circ\tau) \Big) \textstyle\prod_{i=1}^{n} a_{i,\sigma\circ\tau(i)} \\
&= -\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma\circ\tau) \textstyle\prod_{i=1}^{n} a_{i,\sigma\circ\tau(i)} \\
&= -\sum_{\pi \in S_n} \operatorname{sgn}(\pi) \textstyle\prod_{i=1}^{n} a_{i,\pi(i)} \\
&= -\det(A),
\end{aligned}
$$

where (*) follows from the Claim, and (**) follows from Proposition 2.3.2. This proves (a).

Proof (continued). (b) Fix an index $p \in \{1, \dots, n\}$ and a scalar $\alpha \in \mathbb{F} \setminus \{0\}$, and suppose that B is obtained by multiplying the p-th row of A by α ("$R_p \to \alpha R_p$").

Proof (continued). (b) Fix an index $p \in \{1, \dots, n\}$ and a scalar $\alpha \in \mathbb{F} \setminus \{0\}$, and suppose that B is obtained by multiplying the p-th row of A by α ("$R_p \to \alpha R_p$"). By Proposition 7.2.1(b), the determinant is linear in the p-th row, and we deduce that $\det(B) = \alpha \det(A)$.

Proof (continued). (b) Fix an index $p \in \{1, \dots, n\}$ and a scalar $\alpha \in \mathbb{F} \setminus \{0\}$, and suppose that B is obtained by multiplying the p-th row of A by α ("$R_p \to \alpha R_p$"). By Proposition 7.2.1(b), the determinant is linear in the p-th row, and we deduce that $\text{det}(B) = \alpha\text{det}(A)$. Since $\alpha \neq 0$, we deduce that $\text{det}(A) = \alpha^{-1}\text{det}(B)$. This proves (b).

Proof (continued). (c) Fix distinct indices $p, q \in \{1, \dots, n\}$ and a scalar $\alpha \in \mathbb{F}$, and suppose that B is obtained by adding α times row p to row q ("$R_q \to R_q + \alpha R_p$"). Set $B = \left[\ b_{i,j}\ \right]_{n \times n}$, so that

- $\forall j \in \{1, \dots, n\}$: $b_{q,j} = a_{q,j} + \alpha a_{p,j}$;
- $\forall i \in \{1, \dots, n\} \setminus \{q\}, j \in \{1, \dots, n\}$: $b_{i,j} = a_{i,j}$.

We now compute (the q-th row is in red for emphasis):

Proof (continued).

$$\det(B) = \begin{vmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{q-1,1} & \cdots & a_{q-1,n} \\ a_{q,1} + \alpha a_{p,1} & \cdots & a_{q,n} + \alpha a_{p,n} \\ a_{q+1,1} & \cdots & a_{q+1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{vmatrix}$$

$$\overset{(*)}{=} \underbrace{\begin{vmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{q-1,1} & \cdots & a_{q-1,n} \\ a_{q,1} & \cdots & a_{q,n} \\ a_{q+1,1} & \cdots & a_{q+1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{vmatrix}}_{=\det(A)} + \alpha \underbrace{\begin{vmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{q-1,1} & \cdots & a_{q-1,n} \\ a_{p,1} & \cdots & a_{p,n} \\ a_{q+1,1} & \cdots & a_{q+1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{vmatrix}}_{\overset{(**)}{=} 0}$$

$$= \det(A),$$

where (*) follows from the fact that the determinant is linear in the q-th row (by Proposition 7.2.1), and (**) follows from the fact that a matrix with two identical rows (in this case, the p-th and q-th row) has determinant zero (by Proposition 7.1.5). □

Theorem 7.3.2

Let $\mathbb{F}$ be a field, and let $A = \left[\ a_{i,j}\ \right]_{n \times n}$ be a matrix in $\mathbb{F}^{n \times n}$. Then all the following hold:

(a) if a matrix B is obtained by swapping two rows or swapping two columns of A, then

$$\det(B) \;=\; -\det(A);$$

(b) if a matrix B is obtained by multiplying some row or some column of A by a scalar $\alpha \in \mathbb{F} \setminus \{0\}$, then

$$\det(B) = \alpha \det(A) \qquad \text{and} \qquad \det(A) = \alpha^{-1} \det(B);$$

(c) if a matrix B is obtained from A by adding a scalar multiple of one row (resp. column) of A to another row (resp. column) of A, then

$$\det(B) \;=\; \det(A).$$

4. Determinants and matrix invertibility

4. Determinants and matrix invertibility

Theorem 7.4.1

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$. Then A is invertible iff $\det(A) \neq 0$.

Proof.

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Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times n}$. Then A is invertible iff $\det(A) \neq 0$.

Proof. We can transform A into a matrix in reduced row echelon form via a sequence of elementary row operations. By Theorem 7.3.2, each elementary row operation has the effect of multiplying the value of the determinant by some non-zero scalar.

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Proof. We can transform A into a matrix in reduced row echelon form via a sequence of elementary row operations. By Theorem 7.3.2, each elementary row operation has the effect of multiplying the value of the determinant by some non-zero scalar. So, there exists some scalar $\alpha \in \mathbb{F} \setminus \{0\}$ s.t. $\det(A) = \alpha \det\big(\text{RREF}(A)\big)$.

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Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$. Then A is invertible iff $\det(A) \neq 0$.

Proof. We can transform A into a matrix in reduced row echelon form via a sequence of elementary row operations. By Theorem 7.3.2, each elementary row operation has the effect of multiplying the value of the determinant by some non-zero scalar. So, there exists some scalar $\alpha \in \mathbb{F} \setminus \{0\}$ s.t. $\det(A) = \alpha \det\big(\text{RREF}(A)\big)$. Therefore, $\det(A) = 0$ iff $\det\big(\text{RREF}(A)\big) = 0$. Moreover, $\text{RREF}(A)$ is an upper triangular matrix, and so (by Proposition 7.3.1) its determinant is zero iff at least one entry on its main diagonal is zero. We now have the following sequence of equivalent statements (next slide):

Theorem 7.4.1

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$. Then A is invertible iff $\det(A) \neq 0$.

Proof (continued).

$$
\begin{aligned}
\det(A) = 0 \quad &\iff \quad \det\big(\mathrm{RREF}(A)\big) = 0 \\
&\iff \quad \text{at least one entry on the main diagonal of } \mathrm{RREF}(A) \text{ is } 0 \\
&\overset{(*)}{\iff} \quad \mathrm{RREF}(A) \neq I_n \\
&\overset{(**)}{\iff} \quad A \text{ is not invertible,}
\end{aligned}
$$

where (*) follows from the fact that RREF(A) is a square matrix in reduced row echelon form, and (**) follows from the Invertible Matrix Theorem (version 1 or version 2).

Theorem 7.4.1

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$. Then A is invertible iff $\det(A) \neq 0$.

Proof (continued).

$$
\begin{aligned}
\det(A) = 0 \quad &\iff \quad \det\big(\mathrm{RREF}(A)\big) = 0 \\
&\iff \quad \text{at least one entry on the main diagonal of } \mathrm{RREF}(A) \text{ is } 0 \\
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\end{aligned}
$$

where (*) follows from the fact that RREF(A) is a square matrix in reduced row echelon form, and (**) follows from the Invertible Matrix Theorem (version 1 or version 2). It now obviously follows that A is invertible iff $\det(A) \neq 0$, and we are done. □

Theorem 7.4.1

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$. Then A is invertible iff $\det(A) \neq 0$.

- We can now expand the previous version of the Invertible Matrix Theorem to include Theorem 7.4.1.

The Invertible Matrix Theorem (version 3)

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times n}$ be a **square** matrix. Further, let $f : \mathbb{F}^n \to \mathbb{F}^n$ be given by $f(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{F}^n$.[a] Then the following are equivalent:

(a) A is invertible (i.e. A has an inverse);

(b) A^T is invertible;

(c) $\text{RREF}(A) = I_n$;

(d) $\text{RREF}(\left[\begin{array}{c:c} A & I_n \end{array}\right]) = \left[\begin{array}{c:c} I_n & B \end{array}\right]$ for some matrix $B \in \mathbb{F}^{n\times n}$;

(e) $\text{rank}(A) = n$;

(f) $\text{rank}(A^T) = n$;

(g) A is a product of elementary matrices;

---

[a] Since f is a matrix transformation, Proposition 1.10.4 guarantees that f is linear. Moreover, A is the standard matrix of f.

The Invertible Matrix Theorem (version 3, continued)

- (h) the homogeneous matrix-vector equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution (i.e. the solution $\mathbf{x} = \mathbf{0}$);
- (i) there exists some vector $\mathbf{b} \in \mathbb{F}^n$ such that the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has a unique solution;
- (j) for all vectors $\mathbf{b} \in \mathbb{F}^n$, the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has a unique solution;
- (k) for all vectors $\mathbf{b} \in \mathbb{F}^n$, the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ has at most one solution;
- (l) for all vectors $\mathbf{b} \in \mathbb{F}^n$, the matrix-vector equation $A\mathbf{x} = \mathbf{b}$ is consistent;
- (m) f is one-to-one;
- (n) f is onto;
- (o) f is an isomorphism;

The Invertible Matrix Theorem (version 3, continued)

- (p) there exists a matrix $B \in \mathbb{F}^{n \times n}$ such that $BA = I_n$ (i.e. A has a left inverse);
- (q) there exists a matrix $C \in \mathbb{F}^{n \times n}$ such that $AC = I_n$ (i.e. A has a right inverse);
- (r) the columns of A are linearly independent;
- (s) the columns of A span $\mathbb{F}^n$ (i.e. $\mathrm{Col}(A) = \mathbb{F}^n$);
- (t) the columns of A form a basis of $\mathbb{F}^n$;
- (u) the rows of A are linearly independent;
- (v) the rows of A span $\mathbb{F}^{1 \times n}$ (i.e. $\mathrm{Row}(A) = \mathbb{F}^{1 \times n}$);
- (w) the rows of A form a basis of $\mathbb{F}^{1 \times n}$;
- (x) $\mathrm{Nul}(A) = \{\mathbf{0}\}$ (i.e. $\dim(\mathrm{Nul}(A)) = 0$);
- (y) $\det(A) \neq 0$.

6. The multiplicative property of determinants

6. The multiplicative property of determinants

- In general, for a field $\mathbb{F}$, matrices $A, B \in \mathbb{F}^{n \times n}$, and a scalar $\alpha \in \mathbb{F}$, we have that
  - $\det(A + B) \not\asymp \det(A) + \det(B)$;
  - $\det(\alpha A) \not\asymp \alpha \det(A)$.

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$$\det(AB) = \det(A)\det(B).$$

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Let $\mathbb{F}$ be a field, and let $A, B \in \mathbb{F}^{n \times n}$. Then

$$\det(AB) = \det(A)\det(B).$$

- To prove Theorem 7.5.2, we first need a technical proposition (next slide).
- Recall that an *elementary matrix* is any matrix obtained by performing one elementary row operation on an identity matrix I_n.
  - Here, it is possible that $E = I_n$. In this case, we can take R to be the multiplication of the first row by the scalar 1.

Proposition 7.5.1

Let $\mathbb{F}$ be a field, let $A, E \in \mathbb{F}^{n \times n}$, and assume that E is an elementary matrix. Then $\det(EA) = \det(E)\det(A)$.

Proof.

Proposition 7.5.1

Let $\mathbb{F}$ be a field, let $A, E \in \mathbb{F}^{n \times n}$, and assume that E is an elementary matrix. Then $\det(EA) = \det(E)\det(A)$.

Proof. Let R be an elementary row operation that corresponds to the elementary matrix E, i.e. E is the matrix obtained by performing R on I_n.

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By Proposition 1.11.11(a), EA is the matrix obtained by performing R on A.

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Proof. Let R be an elementary row operation that corresponds to the elementary matrix E, i.e. E is the matrix obtained by performing R on I_n.

By Proposition 1.11.11(a), EA is the matrix obtained by performing R on A.

Now, by Theorem 7.3.2, there exists some scalar $\alpha \in \mathbb{F} \setminus \{0\}$ s.t. for any matrix $M \in \mathbb{F}^{n \times n}$, if M_R is the matrix obtained by performing R on M, then $\det(M_R) = \alpha\det(M)$.

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By Proposition 1.11.11(a), EA is the matrix obtained by performing R on A.

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- $\det(E) = \alpha\det(I_n) = \alpha$;
- $\det(EA) = \alpha\det(A)$.

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Now, by Theorem 7.3.2, there exists some scalar $\alpha \in \mathbb{F} \setminus \{0\}$ s.t. for any matrix $M \in \mathbb{F}^{n \times n}$, if M_R is the matrix obtained by performing R on M, then $\det(M_R) = \alpha \det(M)$. So,

- $\det(E) = \alpha \det(I_n) = \alpha$;
- $\det(EA) = \alpha \det(A)$.

It follows that

$$\det(EA) = \alpha \det(A) = \det(E)\det(A).$$

□

Theorem 7.5.2

Let $\mathbb{F}$ be a field, and let $A, B \in \mathbb{F}^{n \times n}$. Then

$$\det(AB) = \det(A)\det(B).$$

Proof.

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Let $\mathbb{F}$ be a field, and let $A, B \in \mathbb{F}^{n\times n}$. Then

$$\det(AB) \quad = \quad \det(A)\det(B).$$

Proof. Suppose first that at least one of A, B is non-invertible.

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Let $\mathbb{F}$ be a field, and let $A, B \in \mathbb{F}^{n\times n}$. Then

$$\det(AB) \;=\; \det(A)\det(B).$$

Proof. Suppose first that at least one of A, B is non-invertible. Then by Corollary 3.3.18, AB is also non-invertible.

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Let $\mathbb{F}$ be a field, and let $A, B \in \mathbb{F}^{n \times n}$. Then

$$\det(AB) \;=\; \det(A)\det(B).$$

Proof. Suppose first that at least one of A, B is non-invertible. Then by Corollary 3.3.18, AB is also non-invertible. But by Theorem 7.4.1, non-invertible matrices have determinant zero, and so $\det(AB) = 0 = \det(A)\det(B)$.

- If A is non-invertible, then $\det(A) = 0$.
- If B is non-invertible, then $\det(B) = 0$.
- In either case, $\det(A)\det(B) = 0$.

Theorem 7.5.2

Let $\mathbb{F}$ be a field, and let $A, B \in \mathbb{F}^{n \times n}$. Then

$$\det(AB) = \det(A)\det(B).$$

Proof (continued). From now on, we assume that A and B are both invertible.

Theorem 7.5.2

Let $\mathbb{F}$ be a field, and let $A, B \in \mathbb{F}^{n\times n}$. Then

$$\det(AB) \;=\; \det(A)\det(B).$$

Proof (continued). From now on, we assume that A and B are both invertible. Therefore, by the Invertible Matrix Theorem, each of them can be written as a product of elementary matrices, say $A = E_1^A \dots E_p^A$ and $B = E_1^B \dots E_q^B$, where $E_1^A, \dots, E_p^A, E_1^B, \dots, E_q^B$ are elementary matrices.

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Let $\mathbb{F}$ be a field, and let $A, B \in \mathbb{F}^{n \times n}$. Then

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- $\det(A) = \det(E_1^A) \dots \det(E_p^A)$;
- $\det(B) = \det(E_1^B) \dots \det(E_q^B)$;
- $\det(AB) = \det(E_1^A) \dots \det(E_p^A)\det(E_1^B) \dots \det(E_q^B)$.

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Let $\mathbb{F}$ be a field, and let $A, B \in \mathbb{F}^{n\times n}$. Then

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Proof (continued). From now on, we assume that A and B are both invertible. Therefore, by the Invertible Matrix Theorem, each of them can be written as a product of elementary matrices, say $A = E_1^A \dots E_p^A$ and $B = E_1^B \dots E_q^B$, where $E_1^A, \dots, E_p^A, E_1^B, \dots, E_q^B$ are elementary matrices. So, $AB = E_1^A \dots E_p^A E_1^B \dots E_q^B$. By repeatedly applying Proposition 7.5.1, we get that

- $\det(A) = \det(E_1^A) \dots \det(E_p^A)$;
- $\det(B) = \det(E_1^B) \dots \det(E_q^B)$;
- $\det(AB) = \det(E_1^A) \dots \det(E_p^A)\det(E_1^B) \dots \det(E_q^B)$.

But now

$$\det(AB) \;=\; \underbrace{\det(E_1^A) \dots \det(E_p^A)}_{=\det(A)}\underbrace{\det(E_1^B) \dots \det(E_q^B)}_{=\det(B)} \;=\; \det(A)\det(B),$$

which is what we needed to show. $\square$

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  - For instance, in $\mathbb{Z}_3$, we have $\frac{1}{1} = 1^{-1} = 1$ and $\frac{1}{2} = 2^{-1} = 2$ (because in $\mathbb{Z}_3$, we have that $2 \cdot 2 = 1$).

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- In a similar vein, for scalars $a, b \in \mathbb{F}$ s.t. $b \neq 0$, we sometimes write $\frac{a}{b}$ instead of $b^{-1}a$.

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- In a similar vein, for scalars $a, b \in \mathbb{F}$ s.t. $b \neq 0$, we sometimes write $\frac{a}{b}$ instead of $b^{-1}a$.
  - For example, in $\mathbb{Z}_5$, we have that $3^{-1} = 2$ (because $3 \cdot 2 = 1$), and so $\frac{4}{3} = 3^{-1} \cdot 4 = 2 \cdot 4 = 3$.

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- In a similar vein, for scalars $a, b \in \mathbb{F}$ s.t. $b \neq 0$, we sometimes write $\frac{a}{b}$ instead of $b^{-1}a$.
  - For example, in $\mathbb{Z}_5$, we have that $3^{-1} = 2$ (because $3 \cdot 2 = 1$), and so $\frac{4}{3} = 3^{-1} \cdot 4 = 2 \cdot 4 = 3$.
- It is sometimes more convenient to use the notation $\frac{1}{a}$ instead of a^{-1}, and $\frac{a}{b}$ instead of $b^{-1}a$.

- Intermission (reminder): Fraction notation in fields (subsection 2.4.3 of the Lecture Notes)

- Let $\mathbb{F}$ be a field. For $a \in \mathbb{F} \setminus \{0\}$, we sometimes use the notation $\frac{1}{a}$ instead of a^{-1} (the multiplicative inverse of a in the field $\mathbb{F}$).
  - For instance, in $\mathbb{Z}_3$, we have $\frac{1}{1} = 1^{-1} = 1$ and $\frac{1}{2} = 2^{-1} = 2$ (because in $\mathbb{Z}_3$, we have that $2 \cdot 2 = 1$).
- In a similar vein, for scalars $a, b \in \mathbb{F}$ s.t. $b \neq 0$, we sometimes write $\frac{a}{b}$ instead of $b^{-1}a$.
  - For example, in $\mathbb{Z}_5$, we have that $3^{-1} = 2$ (because $3 \cdot 2 = 1$), and so $\frac{4}{3} = 3^{-1} \cdot 4 = 2 \cdot 4 = 3$.
- It is sometimes more convenient to use the notation $\frac{1}{a}$ instead of a^{-1}, and $\frac{a}{b}$ instead of $b^{-1}a$.
- However, when working over a finite field such as $\mathbb{Z}_p$ (for a prime number p), we **never** leave a fraction as a final answer, and instead, we always simplify.

Theorem 7.5.2

Let $\mathbb{F}$ be a field, and let $A, B \in \mathbb{F}^{n \times n}$. Then

$$\det(AB) = \det(A)\det(B).$$

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Corollary 7.5.3

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times n}$ be an invertible matrix. Then

$$\det(A^{-1}) \;=\; \frac{1}{\det(A)}.$$

Proof.

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$$\det(A^{-1}) \quad = \quad \tfrac{1}{\det(A)}.$$

Proof. Since $AA^{-1} = I_n$, we see that

$$\det(A)\det(A^{-1}) \quad \overset{\text{Thm. 7.5.2}}{=} \quad \det(AA^{-1}) \quad = \quad \det(I_n) \quad = \quad 1.$$

We now see that $\det(A^{-1}) = \frac{1}{\det(A)}$, which is what we needed to show. $\square$

- Reminder:

Definition

Matrices $A, B \in \mathbb{F}^{n\times n}$ (where $\mathbb{F}$ is a field) are said to be *similar* if there exists an invertible matrix $P \in \mathbb{F}^{n\times n}$ s.t. $B = P^{-1}AP$.

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Corollary 7.5.4

Let $\mathbb{F}$ be a field, and let A and B be similar matrices in $\mathbb{F}^{n \times n}$. Then $\det(A) = \det(B)$.

Proof.

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Corollary 7.5.4

Let $\mathbb{F}$ be a field, and let A and B be similar matrices in $\mathbb{F}^{n\times n}$. Then $\det(A) = \det(B)$.

Proof. Since A and B are similar, there exists an invertible matrix $P \in \mathbb{F}^{n\times n}$ s.t. $B = P^{-1}AP$. We then have that

$$\begin{aligned} \det(B) &= \det(P^{-1}AP) \\ &= \det(P^{-1})\det(A)\det(P) && \text{by Theorem 7.5.2} \\ &= \tfrac{1}{\det(P)}\det(A)\det(P) && \text{by Corollary 7.5.3} \\ &= \det(A). \end{aligned}$$

□

- Reminder:

Theorem 4.5.16

Let $\mathbb{F}$ be a field, let $B, C \in \mathbb{F}^{n \times n}$ be matrices, and let V be an n-dimensional vector space over the field $\mathbb{F}$. Then the following are equivalent:

- (a) B and C are similar;
- (b) for all bases $\mathcal{B}$ of V and linear functions $f : V \rightarrow V$ s.t. $B = {}_{\mathcal{B}}[\ f\ ]_{\mathcal{B}}$, there exists a basis $\mathcal{C}$ of V s.t. $C = {}_{\mathcal{C}}[\ f\ ]_{\mathcal{C}}$;
- (c) for all bases $\mathcal{C}$ of V and linear functions $f : V \rightarrow V$ s.t. $C = {}_{\mathcal{C}}[\ f\ ]_{\mathcal{C}}$, there exists a basis $\mathcal{B}$ of V s.t. $B = {}_{\mathcal{B}}[\ f\ ]_{\mathcal{B}}$;
- (d) there exist bases $\mathcal{B}$ and $\mathcal{C}$ of V and a linear function $f : V \rightarrow V$ s.t. $B = {}_{\mathcal{B}}[\ f\ ]_{\mathcal{B}}$ and $C = {}_{\mathcal{C}}[\ f\ ]_{\mathcal{C}}$.

Definition

Suppose that V is a non-trivial, finite-dimensional vector space over a field $\mathbb{F}$, and that $f : V \to V$ is a linear function. Then we define the determinant of f to be

$$\det(f) \;\; := \;\; \det\Big({}_{\mathcal{B}}\big[\ f\ \big]_{\mathcal{B}} \Big),$$

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- Let us explain why this is well-defined, that is, why the value of $\det(f)$ that we get depends only on f, and not on the particular choice of the basis $\mathcal{B}$.

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- Suppose that $\mathcal{C}$ is any basis of V.
- Then by Theorem 4.5.16, matrices ${}_{\mathcal{B}}\big[\ f\ \big]_{\mathcal{B}}$ and ${}_{\mathcal{C}}\big[\ f\ \big]_{\mathcal{C}}$ are similar, and consequently (by Corollary 7.5.4), they have the same determinant.

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- Then by Theorem 4.5.16, matrices ${}_{\mathcal{B}}\big[\; f \;\big]_{\mathcal{B}}$ and ${}_{\mathcal{C}}\big[\; f \;\big]_{\mathcal{C}}$ are similar, and consequently (by Corollary 7.5.4), they have the same determinant.
- So, $\det(f)$ is well-defined.

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$$\det(f) \quad := \quad \det\Big({}_{\mathcal{B}}\big[\ f\ \big]_{\mathcal{B}} \Big),$$

where $\mathcal{B}$ is any basis of V.

- **Remark:** Note that we defined determinants only for linear functions whose domain and codomain are one and the same, and moreover, are finite-dimensional and non-null.

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Let $\mathbb{F}$ be a field, and let $A, B \in \mathbb{F}^{n\times n}$. Then

$$\det(AB) = \det(A)\det(B).$$

Corollary 7.5.5

Let A be an orthogonal matrix in $\mathbb{R}^{n\times n}$. Then $\det(A) = \pm 1$ (i.e. $\det(A)$ is either $+1$ or -1).

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$$1 = \det(I_n) = \det(A^T A) \overset{(*)}{=} \det(A^T)\det(A) \overset{(**)}{=} \det(A)^2,$$

where (*) follows from Theorem 7.5.2, and (**) follows from Theorem 7.1.3.

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Let A be an orthogonal matrix in $\mathbb{R}^{n \times n}$. Then $\det(A) = \pm 1$ (i.e. $\det(A)$ is either $+1$ or -1).

- **Warning:** The converse of Corollary 7.5.5 is false, i.e. matrices whose determinant is ± 1 need not be orthogonal.

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  - Then by Theorem 7.4.1, we have that $\det(A) \neq 0$.

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  - We now form the matrix B by multiplying one row or one column of A by $\frac{1}{\det(A)}$, and we see that $\det(B) = 1$.

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  - Then by Theorem 7.4.1, we have that $\det(A) \neq 0$.
  - We now form the matrix B by multiplying one row or one column of A by $\frac{1}{\det(A)}$, and we see that $\det(B) = 1$.
  - However, B need not be orthogonal.

6. Laplace expansion

Definition

For a matrix $A = \left[\ a_{i,j}\ \right]_{n \times n}$ (where $n \geq 2$) with entries in some field $\mathbb{F}$, and for indices $p, q \in \{1, \ldots, n\}$, $A_{p,q}$ is the $(n-1) \times (n-1)$ matrix obtained from A by deleting the p-th row and q-th column.

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,q-1} & a_{1,q} & a_{1,q+1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{p-1,1} & \cdots & a_{p-1,q-1} & a_{p-1,q} & a_{p-1,q+1} & \cdots & a_{p-1,n} \\ a_{p,1} & \cdots & a_{p,q-1} & a_{p,q} & a_{p,q+1} & \cdots & a_{p,n} \\ a_{p+1,1} & \cdots & a_{p+1,q-1} & a_{p+1,q} & a_{p+1,q+1} & \cdots & a_{p+1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,q-1} & a_{n,q} & a_{n,q+1} & \cdots & a_{n,n} \end{bmatrix}$$

Definition

For a matrix $A = \left[\ a_{i,j}\ \right]_{n\times n}$ (where $n \geq 2$) with entries in some field $\mathbb{F}$, and for indices $p, q \in \{1, \ldots, n\}$, $A_{p,q}$ is the $(n-1)\times(n-1)$ matrix obtained from A by deleting the p-th row and q-th column.

- **Terminology:** The determinants

$$\det(A_{i,j}), \qquad \text{with } i, j \in \{1, \ldots, n\}$$

are referred to as the *first minors* of A, whereas numbers

$$C_{i,j} \;\; := \;\; (-1)^{i+j}\det(A_{i,j}) \qquad \text{with } i, j \in \{1, \ldots, n\}$$

are referred to as the *cofactors* of A.

Laplace expansion

Let $\mathbb{F}$ be a field, and let $A = \left[\ a_{i,j}\ \right]_{n\times n}$ (where $n \geq 2$) be a matrix in $\mathbb{F}^{n\times n}$. Then both the following hold:

(a) **[expansion along the i-th row]** for all $i \in \{1, \dots, n\}$:

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j}\ a_{i,j}\ \det(A_{i,j});$$

(b) **[expansion along the j-th column]** for all $j \in \{1, \dots, n\}$:

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j}\ a_{i,j}\ \det(A_{i,j}).$$

- **Remark:** If we write $C_{i,j} := (-1)^{i+j}\det(A_{i,j})$ for all $i, j \in \{1, \dots, n\}$ (so, the $C_{i,j}$'s are the cofactors of A), then the formula from (a) becomes $\det(A) = \sum_{j=1}^{n} a_{i,j} C_{i,j}$, and the formula from (b) becomes $\det(A) = \sum_{i=1}^{n} a_{i,j} C_{i,j}$.
- This is why Laplace expansion is also referred to as "cofactor expansion."

Laplace expansion

Let $\mathbb{F}$ be a field, and let $A = \left[\ a_{i,j}\ \right]_{n\times n}$ (where $n \geq 2$) be a matrix in $\mathbb{F}^{n\times n}$. Then both the following hold:

(a) **[expansion along the *i*-th row]** for all $i \in \{1, \dots, n\}$:

$$\det(A) \;=\; \sum_{j=1}^{n} (-1)^{i+j}\, a_{i,j} \det(A_{i,j});$$

(b) **[expansion along the *j*-th column]** for all $j \in \{1, \dots, n\}$:

$$\det(A) \;=\; \sum_{i=1}^{n} (-1)^{i+j}\, a_{i,j} \det(A_{i,j}).$$

- First an example, then a proof.

Example 7.6.3

Consider the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 4 & 5 \\ 7 & 0 & 8 \end{bmatrix},$$

with entries understood to be in $\mathbb{R}$. Compute $\det(A)$ in two ways:

(a) via Laplace expansion along the third row;

(b) via Laplace expansion along the second column.

Solution. (a) Laplace expansion along the third row:

$$\begin{aligned}\det(A) &= \begin{vmatrix} 2 & 0 & 1 \\ 3 & 4 & 5 \\ 7 & 0 & 8 \end{vmatrix} \\ &= (-1)^{3+1}\, 7 \begin{vmatrix} 0 & 1 \\ 4 & 5 \end{vmatrix} + (-1)^{3+2}\, 0 \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} + (-1)^{3+3}\, 8 \begin{vmatrix} 2 & 0 \\ 3 & 4 \end{vmatrix} \\ &= 7 \underbrace{\begin{vmatrix} 0 & 1 \\ 4 & 5 \end{vmatrix}}_{=-4} + 8 \underbrace{\begin{vmatrix} 2 & 0 \\ 3 & 4 \end{vmatrix}}_{=8} \quad = \quad 36.\end{aligned}$$

Solution (continued). (b) Laplace expansion along the second column:

$$\begin{aligned}\det(A) &= \begin{vmatrix} 2 & 0 & 1 \\ 3 & 4 & 5 \\ 7 & 0 & 8 \end{vmatrix} \\ &= (-1)^{1+2}\, 0 \begin{vmatrix} 3 & 5 \\ 7 & 8 \end{vmatrix} + (-1)^{2+2}\, 4 \begin{vmatrix} 2 & 1 \\ 7 & 8 \end{vmatrix} + (-1)^{3+2}\, 0 \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} \\ &= 4 \underbrace{\begin{vmatrix} 2 & 1 \\ 7 & 8 \end{vmatrix}}_{=9} = 36.\end{aligned}$$

□

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with entries understood to be in $\mathbb{R}$. Compute $\det(A)$ in two ways:

(a) via Laplace expansion along the third row;

(b) via Laplace expansion along the second column.

- As a general rule, it is best to expand along a row or column that has a lot of zeros (if such a row or column exists), since that reduces the amount of calculation that we need to perform.
  - So, in Example 7.6.3, it was easier to expand along the second column.

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- As a general rule, it is best to expand along a row or column that has a lot of zeros (if such a row or column exists), since that reduces the amount of calculation that we need to perform.
    - So, in Example 7.6.3, it was easier to expand along the second column.
- See the Lecture Notes for another example (with a larger matrix).

Laplace expansion

Let $\mathbb{F}$ be a field, and let $A = \left[\ a_{i,j}\ \right]_{n \times n}$ (where $n \geq 2$) be a matrix in $\mathbb{F}^{n \times n}$. Then both the following hold:

(a) **[expansion along the *i*-th row]** for all $i \in \{1, \dots, n\}$:

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j}\, a_{i,j} \det(A_{i,j});$$

(b) **[expansion along the *j*-th column]** for all $j \in \{1, \dots, n\}$:

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j}\, a_{i,j} \det(A_{i,j}).$$

- Let's prove this!
- We begin with a technical proposition.

Proposition 7.6.1

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{(n-1)\times(n-1)}$ (where $n \geq 2$) and $\mathbf{a} \in \mathbb{F}^{n-1}$. Then

$$\det\left(\left[\begin{array}{c:c} A & \mathbf{0} \\ \hdashline \mathbf{a}^T & 1 \end{array} \right]_{n\times n} \right) = \det(A).$$

Proof.

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Proof. First, set $\left[\begin{array}{c:c} A & \mathbf{0} \\ \hdashline \mathbf{a}^T & 1 \end{array} \right]_{n\times n} = \left[\begin{array}{c} a_{i,j} \end{array} \right]_{n\times n}$, so that all the following hold:

- $A = \left[\begin{array}{c} a_{i,j} \end{array} \right]_{(n-1)\times(n-1)}$;
- $a_{n,n} = 1$;
- for all $i \in \{1, \dots, n-1\}$, $a_{i,n} = 0$;
- for all $j \in \{1, \dots, n-1\}$, $a_{n,j}$ is the j-th entry of the vector $\mathbf{a}$.

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Proof (continued). Next, for all $\sigma \in S_{n-1}$, let $\sigma^* \in S_n$ be given by

- $\sigma^*(i) = \sigma(i)$ for all $i \in \{1, \dots, n-1\}$,
- $\sigma^*(n) = n$.

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So, for any $\sigma \in S_{n-1}$, the disjoint cycle decomposition of σ^* is obtained by adding the one-element cycle (n) to the disjoint cycle decomposition of σ, and consequently, $\text{sgn}(\sigma) = \text{sgn}(\sigma^*)$.

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Set

$$S_n^* \;:=\; \{\sigma^* \mid \sigma \in S_{n-1}\} \;=\; \{\pi \in S_n \mid \pi(n) = n\}.$$

Proposition 7.6.1

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{(n-1)\times(n-1)}$ (where $n \geq 2$) and $\mathbf{a} \in \mathbb{F}^{n-1}$. Then

$$\det\left(\begin{bmatrix} A & \mathbf{0} \\ \mathbf{a}^T & 1 \end{bmatrix}_{n\times n}\right) = \det(A).$$

Proof (continued). Next, for all $\sigma \in S_{n-1}$, let $\sigma^* \in S_n$ be given by

- $\sigma^*(i) = \sigma(i)$ for all $i \in \{1, \dots, n-1\}$,
- $\sigma^*(n) = n$.

So, for any $\sigma \in S_{n-1}$, the disjoint cycle decomposition of σ^* is obtained by adding the one-element cycle (n) to the disjoint cycle decomposition of σ, and consequently, $\text{sgn}(\sigma) = \text{sgn}(\sigma^*)$.

Set

$$S_n^* := \{\sigma^* \mid \sigma \in S_{n-1}\} = \{\pi \in S_n \mid \pi(n) = n\}.$$

We then have the following (next slide):

Proof (continued).

$$
\begin{aligned}
\det(A) &= \sum_{\sigma \in S_{n-1}} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} \dots a_{n-1,\sigma(n-1)} \\
&= \sum_{\sigma \in S_{n-1}} \operatorname{sgn}(\sigma^*) a_{1,\sigma^*(1)} \dots a_{n-1,\sigma^*(n-1)} \underbrace{a_{n,\sigma^*(n)}}_{=1} \\
&= \sum_{\pi \in S_n^*} \operatorname{sgn}(\pi) a_{1,\pi(1)} \dots a_{n-1,\pi(n-1)} a_{n,\pi(n)} \\
&\overset{(*)}{=} \sum_{\pi \in S_n} \operatorname{sgn}(\pi) a_{1,\pi(1)} \dots a_{n-1,\pi(n-1)} a_{n,\pi(n)} \\
&= \det\Big(\left[\begin{array}{c|c} A & \mathbf{0} \\ \hline \mathbf{a}^T & 1 \end{array} \right]_{n \times n} \Big),
\end{aligned}
$$

where (*) follows from the fact that for all $\pi \in S_n \setminus S_n^*$, we have that $i := \pi^{-1}(n) \neq n$ (because $\pi(n) \neq n$), and so $a_{i,\pi(i)} = a_{i,n} = 0$. $\square$

Proposition 7.6.1

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{(n-1)\times(n-1)}$ (where $n \geq 2$) and $\mathbf{a} \in \mathbb{F}^{n-1}$. Then

$$\det\left(\left[\begin{array}{c|c} A & \mathbf{0} \\ \hline \mathbf{a}^T & 1 \end{array} \right]_{n\times n} \right) \quad = \quad \det(A).$$

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$$\det\left(\left[\begin{array}{c|c} A & \mathbf{0} \\ \hline \mathbf{a}^T & 1 \end{array} \right]_{n\times n} \right) \;=\; \det(A).$$

- Reminder:

Theorem 7.1.3

Let $\mathbb{F}$ be a field. For all $A \in \mathbb{F}^{n\times n}$, we have that $\det(A^T) = \det(A)$.

Laplace expansion

Let $\mathbb{F}$ be a field, and let $A = \left[\ a_{i,j}\ \right]_{n \times n}$ (where $n \geq 2$) be a matrix in $\mathbb{F}^{n \times n}$. Then both the following hold:

(a) **[expansion along the i-th row]** for all $i \in \{1, \dots, n\}$:

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j}\, a_{i,j} \det(A_{i,j});$$

(b) **[expansion along the j-th column]** for all $j \in \{1, \dots, n\}$:

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j}\, a_{i,j} \det(A_{i,j}).$$

Proof.

Laplace expansion

Let $\mathbb{F}$ be a field, and let $A = \left[\ a_{i,j}\ \right]_{n \times n}$ (where $n \geq 2$) be a matrix in $\mathbb{F}^{n \times n}$. Then both the following hold:

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Proof. In view of Theorem 7.1.3, it is enough to prove (b).

Laplace expansion

Let $\mathbb{F}$ be a field, and let $A = \left[\ a_{i,j}\ \right]_{n \times n}$ (where $n \geq 2$) be a matrix in $\mathbb{F}^{n \times n}$. Then both the following hold:

(a) **[expansion along the i-th row]** for all $i \in \{1, \dots, n\}$:

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(b) **[expansion along the j-th column]** for all $j \in \{1, \dots, n\}$:

$$\det(A) \;=\; \sum_{i=1}^{n} (-1)^{i+j}\, a_{i,j} \det(A_{i,j}).$$

Proof. In view of Theorem 7.1.3, it is enough to prove (b).

Fix $j \in \{1, \dots, n\}$. We must show that

$$\det(A) \;=\; \sum_{i=1}^{n} (-1)^{i+j}\, a_{i,j} \det(A_{i,j}).$$

Proof (cont.). Reminder: WTS $\det(A) = \sum_{i=1}^{n} (-1)^{i+j}\ a_{i,j} \det(A_{i,j})$.

Proof (cont.). Reminder: WTS $\det(A) = \sum_{i=1}^{n} (-1)^{i+j} \, a_{i,j} \, \det(A_{i,j})$.

First, set $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix}$.

Proof (cont.). Reminder: WTS $\det(A) = \sum_{i=1}^{n} (-1)^{i+j}\ a_{i,j} \det(A_{i,j})$.

First, set $A = \left[\ \mathbf{a}_1 \quad \ldots \quad \mathbf{a}_n\ \right]$. Then $\mathbf{a}_j = \sum_{i=1}^{n} a_{i,j}\mathbf{e}_i$, *Proof (cont.).* Reminder: WTS $\det(A) = \sum_{i=1}^{n} (-1)^{i+j} \; a_{i,j} \; \det(A_{i,j})$.

First, set $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix}$. Then $\mathbf{a}_j = \sum_{i=1}^{n} a_{i,j}\mathbf{e}_i$, and so

$$\begin{aligned} \det(A) &= \det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{a}_j & \mathbf{a}_{j+1} & \dots & \mathbf{a}_n \end{bmatrix} \right) \\ &= \det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \sum_{i=1}^{n} a_{i,j}\mathbf{e}_i & \mathbf{a}_{j+1} & \dots & \mathbf{a}_n \end{bmatrix} \right) \\ &\overset{(*)}{=} \sum_{i=1}^{n} a_{i,j} \det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{e}_i & \mathbf{a}_{j+1} & \dots & \mathbf{a}_n \end{bmatrix} \right), \end{aligned}$$

where (*) follows from Proposition 7.2.1(a).

Proof (cont.). Reminder: WTS $\det(A) = \sum_{i=1}^{n} (-1)^{i+j}\, a_{i,j} \det(A_{i,j})$.

First, set $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix}$. Then $\mathbf{a}_j = \sum_{i=1}^{n} a_{i,j}\mathbf{e}_i$, and so

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\det(A) &= \det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{a}_j & \mathbf{a}_{j+1} & \dots & \mathbf{a}_n \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \sum_{i=1}^{n} a_{i,j}\mathbf{e}_i & \mathbf{a}_{j+1} & \dots & \mathbf{a}_n \end{bmatrix}\right) \\
&\overset{(*)}{=} \sum_{i=1}^{n} a_{i,j} \det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{e}_i & \mathbf{a}_{j+1} & \dots & \mathbf{a}_n \end{bmatrix}\right),
\end{aligned}$$

where (*) follows from Proposition 7.2.1(a).

Fix an arbitrary index $i \in \{1, \dots, n\}$.

Proof (cont.). Reminder: WTS $\det(A) = \sum_{i=1}^{n} (-1)^{i+j}\, a_{i,j} \det(A_{i,j})$.

First, set $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_n \end{bmatrix}$. Then $\mathbf{a}_j = \sum_{i=1}^{n} a_{i,j}\mathbf{e}_i$, and so

$$\begin{aligned}
\det(A) &= \det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{a}_j & \mathbf{a}_{j+1} & \dots & \mathbf{a}_n \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \sum_{i=1}^{n} a_{i,j}\mathbf{e}_i & \mathbf{a}_{j+1} & \dots & \mathbf{a}_n \end{bmatrix}\right) \\
&\overset{(*)}{=} \sum_{i=1}^{n} a_{i,j} \det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{e}_i & \mathbf{a}_{j+1} & \dots & \mathbf{a}_n \end{bmatrix}\right),
\end{aligned}$$

where (*) follows from Proposition 7.2.1(a).

Fix an arbitrary index $i \in \{1, \dots, n\}$. To complete the proof, it now suffices to show that

$$\det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{e}_i & \mathbf{a}_{j+1} & \dots & \mathbf{a}_n \end{bmatrix}\right) = (-1)^{i+j} \det(A_{i,j}).$$

Proof (continued). Reminder: WTS

$$\det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{e}_i & \mathbf{a}_{j+1} & \dots & \mathbf{a}_n \end{bmatrix}\right) = (-1)^{i+j} \det(A_{i,j}).$$

Proof (continued). Reminder: WTS

$$\det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{e}_i & \mathbf{a}_{j+1} & \dots & \mathbf{a}_n \end{bmatrix} \right) = (-1)^{i+j} \det(A_{i,j}).$$

By iteratively performing $n - j$ column swaps on the matrix

$$B_i := \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{e}_i & \mathbf{a}_{j+1} & \dots & \mathbf{a}_n \end{bmatrix},$$

we can obtain the matrix

$$C_i := \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{a}_{j+1} & \dots & \mathbf{a}_n & \mathbf{e}_i \end{bmatrix}.$$

Proof (continued). Reminder: WTS

$$\det\left(\begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{e}_i & \mathbf{a}_{j+1} & \dots & \mathbf{a}_n \end{bmatrix} \right) = (-1)^{i+j} \det(A_{i,j}).$$

By iteratively performing $n-j$ column swaps on the matrix

$$B_i \quad := \quad \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{e}_i & \mathbf{a}_{j+1} & \dots & \mathbf{a}_n \end{bmatrix},$$

we can obtain the matrix

$$C_i := \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_{j-1} & \mathbf{a}_{j+1} & \dots & \mathbf{a}_n & \mathbf{e}_i \end{bmatrix}.$$

By iteratively performing $n-i$ row swaps on the matrix C_i, we can obtain the matrix

$$\left[\begin{array}{c|c} A_{i,j} & \mathbf{0} \\ \hline \mathbf{a}^T & 1 \end{array}\right],$$

where $\mathbf{a}^T$ is the row vector of length $n-1$ obtained from the i-th row of A by deleting its j-th entry.

Proof (continued). Since swapping two rows or two columns has the effect of changing the sign of the determinant, we see that

$$\begin{aligned}
\det(B_i) &= (-1)^{n-j}\det(C_i) \\
&= (-1)^{n-j}(-1)^{n-i}\det\left(\left[\begin{array}{c:c} A_{i,j} & \mathbf{0} \\ \hdashline \mathbf{a}^T & 1 \end{array}\right]\right) \\
&\overset{(*)}{=} (-1)^{2n-i-j}\det(A_{i,j}) \\
&= (-1)^{i+j}\det(A_{i,j}),
\end{aligned}$$

where (*) follows from Proposition 7.6.1. This completes the argument. □

Laplace expansion

Let $\mathbb{F}$ be a field, and let $A = \left[\ a_{i,j}\ \right]_{n \times n}$ (where $n \geq 2$) be a matrix in $\mathbb{F}^{n \times n}$. Then both the following hold:

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$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j}\, a_{i,j} \det(A_{i,j});$$

(b) **[expansion along the j-th column]** for all $j \in \{1, \dots, n\}$:

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j}\, a_{i,j} \det(A_{i,j}).$$

Theorem 7.6.6

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$ and $B \in \mathbb{F}^{m \times m}$ be square matrices. Then

$$\det\left(\left[\begin{array}{c|c} A & O_{n \times m} \\ \hline O_{m \times n} & B \end{array} \right] \right) \ = \ \det(A)\det(B).$$

Proof (outline).

Theorem 7.6.6

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$ and $B \in \mathbb{F}^{m \times m}$ be square matrices. Then

$$\det\left(\left[\begin{array}{c|c} A & O_{n \times m} \\ \hline O_{m \times n} & B \end{array} \right] \right) \;=\; \det(A)\det(B).$$

Proof (outline). This can be proven (for example) by induction on n, via Laplace expansion along the leftmost column. The details are left as an exercise. $\square$

Theorem 7.6.6

Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n \times n}$ and $B \in \mathbb{F}^{m \times m}$ be square matrices. Then

$$\det\left(\left[\begin{array}{c|c} A & O_{n \times m} \\ \hline O_{m \times n} & B \end{array} \right] \right) \quad = \quad \det(A) \det(B).$$

Corollary 7.6.7

Let $\mathbb{F}$ be a field, and let $A_1 \in \mathbb{F}^{n_1 \times n_1}, A_2 \in \mathbb{F}^{n_2 \times n_2}, \dots, A_k \in \mathbb{F}^{n_k \times n_k}$ be square matrices. Then

$$\det\left(\left[\begin{array}{c|c|c|c} A_1 & O_{n_1 \times n_2} & \dots & O_{n_1 \times n_k} \\ \hline O_{n_2 \times n_1} & A_2 & \dots & O_{n_2 \times n_k} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline O_{n_k \times n_1} & O_{n_k \times n_2} & \dots & A_k \end{array} \right] \right) \quad = \quad \prod_{i=1}^{k} \det(A_i).$$

Proof. This follows from Theorem 7.6.6 via an easy induction on k. $\square$