# Linear Algebra 2

## Lecture #18

## Permutation matrices. Orthogonal matrices

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- This lecture has three parts:

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  1. Permutation matrices

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  2. Orthogonal matrices

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  1. Permutation matrices
  2. Orthogonal matrices
  3. Scalar product, coordinate vectors, and matrices of linear functions

1. Permutation matrices

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**Definition**

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- Examples:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

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- The 0's and 1's in permutation matrices may belong to any field $\mathbb{F}$ of our choice.
  - In our study of permutation matrices, we will never need to add two non-zero numbers, and whenever we multiply two numbers, at least one of the two numbers will be 0 or 1.
  - So, it does not matter which particular field we are working in, and therefore, we will not emphasize this.

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- Moreover, $n \times n$ permutation matrices are precisely the matrices that can be obtained from the identity matrix $I_n$ by reordering (i.e. permuting) rows, or alternatively, by reordering (i.e. permuting) columns.
- So, the columns of an $n \times n$ permutation matrix are the standard basis vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$ (appearing in some order in that matrix), whereas the rows are $\mathbf{e}_1^T, \dots, \mathbf{e}_n^T$ (again, appearing in some order in that matrix).

## Definition

For a positive integer $n$ and a permutation $\pi \in S_n$, we define the *matrix of the permutation* $\pi$, denoted by $P_\pi$, to be the $n \times n$ matrix that has 1 in the $\big(i, \pi(i)\big)$-th entry for each each index $i \in \{1, \ldots, n\}$, and has 0 in all other entries. In other words, for each index $i \in \{1, \ldots, n\}$, the $i$-th row of the matrix $P_\pi$ is $\mathbf{e}^T_{\pi(i)}$.

- For example, for the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 6 & 5 & 3 \end{pmatrix},$$

in $S_6$, we obtain the $6 \times 6$ permutation matrix

$$P_\pi = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

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- Obviously, for a positive integer $n$, the matrix of the identity permutation $1_n$ in $S_n$ is precisely the identity matrix $I_n$, i.e. $P_{1_n} = I_n$.

## Proposition 2.3.10

Let $n$ be a positive integer, and let $\pi \in S_n$. Then $P_\pi$ is a permutation matrix.

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  - The answer to this question is "yes," and it follows from a simple counting argument, as follows.
  - Let $n$ be a positive integer.
  - The $n \times n$ permutation matrices are precisely those $n \times n$ matrices whose columns are the standard basis vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$, appearing in some order. There are $n!$ many ways to order the vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$, and consequently, there are $n!$ many $n \times n$ permutation matrices.

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  - On the other hand, $|S_n| = n!$, and consequently, there are $n!$ many matrices of permutations in $S_n$.
    - We are using the fact that different permutations have different matrices.
  - So, the number of $n \times n$ permutation matrices is the same as the number of matrices of permutations in $S_n$.
  - It now follows from Proposition 2.3.10 that $n \times n$ permutation matrices are precisely the matrices of permutations in $S_n$.

- Reminder:

**Proposition 1.4.5**

Let $\mathbb{F}$ be a field, and let $A = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix}$ be a matrix in $\mathbb{F}^{n\times m}$. Then for all indices $i \in \{1, \dots, m\}$, we have that $A\mathbf{e}_i^m = \mathbf{a}_i$.

**Proposition 1.8.2**

Let $\mathbb{F}$ be a field, and let

$$A = \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_n \end{bmatrix}$$

be a matrix in $\mathbb{F}^{n\times m}$. Then for all $i \in \{1, \dots, n\}$, we have that

$$\mathbf{e}_i^T A = \mathbf{r}_i,$$

where $\mathbf{e}_i$ is the $i$-th standard basis vector of $\mathbb{F}^n$.

## Proposition 2.3.11

Let $n$ be a positive integer, and let $\pi \in S_n$ be a permutation. Then both the following hold:

(a) $\forall i \in \{1, \dots, n\}$: $\mathbf{e}_i^T P_\pi = \mathbf{e}_{\pi(i)}$, i.e. the $i$-th row of $P_\pi$ is $\mathbf{e}_{\pi(i)}^T$;

(b) $\forall j \in \{1, \dots, n\}$: $P_\pi \mathbf{e}_j = \mathbf{e}_{\pi^{-1}(j)}$, i.e. the $j$-th column of $P_\pi$ is $\mathbf{e}_{\pi^{-1}(j)}$.

Consequently, in terms of its rows and columns, $P_\pi$ can be written as follows:

$$P_\pi = \begin{bmatrix} \mathbf{e}_{\pi(1)}^T \\ \vdots \\ \mathbf{e}_{\pi(n)}^T \end{bmatrix} = \begin{bmatrix} \mathbf{e}_{\pi^{-1}(1)} & \dots & \mathbf{e}_{\pi^{-1}(n)} \end{bmatrix}.$$

*Proof.* The last statement of the proposition follows immediately from (a) and (b). So, it is enough to prove (a) and (b).

## Proposition 2.3.11

- (a) $\forall i \in \{1, \dots, n\}$: $\mathbf{e}_i^T P_\pi = \mathbf{e}_{\pi(i)}$, i.e. the $i$-th row of $P_\pi$ is $\mathbf{e}_{\pi(i)}^T$;
- (b) $\forall j \in \{1, \dots, n\}$: $P_\pi \mathbf{e}_j = \mathbf{e}_{\pi^{-1}(j)}$, i.e. the $j$-th column of $P_\pi$ is $\mathbf{e}_{\pi^{-1}(j)}$.

*Proof (continued).* (a) Fix an index $i \in \{1, \dots, n\}$. By Proposition 1.8.2, $\mathbf{e}_i^T P_\pi$ is precisely the $i$-th row of the matrix $P_\pi$, and by the definition of the matrix $P_\pi$, its $i$-th row is precisely $\mathbf{e}_{\pi(i)}$.

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(b) Fix an index $j \in \{1, \dots, n\}$. By Proposition 1.4.5, $P_\pi \mathbf{e}_j$ is precisely the $j$-th column of the matrix $P_\pi$.

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(b) Fix an index $j \in \{1, \dots, n\}$. By Proposition 1.4.5, $P_\pi \mathbf{e}_j$ is precisely the $j$-th column of the matrix $P_\pi$. Set $i := \pi^{-1}(j)$, so that $j = \pi(i)$.

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(b) Fix an index $j \in \{1, \dots, n\}$. By Proposition 1.4.5, $P_\pi \mathbf{e}_j$ is precisely the $j$-th column of the matrix $P_\pi$. Set $i := \pi^{-1}(j)$, so that $j = \pi(i)$. By (a), the $i$-th row of $P_\pi$ is the row vector $\mathbf{e}_{\pi(i)}^T = \mathbf{e}_j^T$. So, $P_\pi$ has 1 in its $(i, j)$-th entry. Since $P_\pi$ is a permutation matrix (by Proposition 2.3.10), and therefore has exactly one 1 in each column, it follows that the $j$-th column of $P_\pi$ is $\mathbf{e}_i = \mathbf{e}_{\pi^{-1}(j)}$. $\square$

## Proposition 2.3.11

Let $n$ be a positive integer, and let $\pi \in S_n$ be a permutation. Then both the following hold:

(a) $\forall i \in \{1, \ldots, n\}$: $\mathbf{e}_i^T P_\pi = \mathbf{e}_{\pi(i)}$, i.e. the $i$-th row of $P_\pi$ is $\mathbf{e}_{\pi(i)}^T$;

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*Proof.*

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$$P_{\pi^{-1}} \quad = \quad P_\pi^T.$$

*Proof.* We have that

$$P_\pi^T \quad \overset{(*)}{=} \quad \left( \begin{bmatrix} \mathbf{e}_{\pi^{-1}(1)} & \dots & \mathbf{e}_{\pi^{-1}(n)} \end{bmatrix} \right)^T \quad = \quad \begin{bmatrix} \mathbf{e}_{\pi^{-1}(1)}^T \\ \vdots \\ \mathbf{e}_{\pi^{-1}(n)}^T \end{bmatrix} \quad \overset{(*)}{=} \quad P_{\pi^{-1}},$$

where both instances of (*) follow from Proposition 2.3.11. □

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## Proposition 2.3.13

Let $n$ be a positive integer, and let $\sigma$ and $\pi$ be permutations in $S_n$. Then $P_{\sigma \circ \pi} = P_\pi P_\sigma$.

*Proof.*

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Let $n$ be a positive integer, and let $\sigma$ and $\pi$ be permutations in $S_n$. Then $P_{\sigma\circ\pi} = P_\pi P_\sigma$.

*Proof.* It suffices to show that matrices $P_{\sigma\circ\pi}$ and $P_\pi P_\sigma$ have the same corresponding rows.

**Proposition 2.3.11**

(a) $\forall i \in \{1, \dots, n\}$: $\mathbf{e}_i^T P_\pi = \mathbf{e}_{\pi(i)}$, i.e. the $i$-th row of $P_\pi$ is $\mathbf{e}_{\pi(i)}^T$;

**Proposition 2.3.13**

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*Proof.* It suffices to show that matrices $P_{\sigma \circ \pi}$ and $P_\pi P_\sigma$ have the same corresponding rows. Fix an index $i \in \{1, \dots, n\}$.

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$$\begin{aligned} \mathbf{e}_i^T(P_\pi P_\sigma) &= (\mathbf{e}_i^T P_\pi)P_\sigma \overset{(*)}{=} \mathbf{e}^T_{\pi(i)} P_\sigma \overset{(*)}{=} \mathbf{e}_{\sigma\left(\pi(i)\right)} \\ &= \mathbf{e}^T_{(\sigma\circ\pi)(i)} \overset{(*)}{=} \mathbf{e}_i^T P_{\sigma\circ\pi}, \end{aligned}$$

where all three instances of (*) follow from Prop. 2.3.11(a). □

**Theorem 2.3.14**

Let $n$ be a positive integer, and let $\pi \in S_n$. Then $P_\pi$ is invertible, and moreover,

$$P_\pi^{-1} \;=\; P_{\pi^{-1}} \;=\; P_\pi^T.$$

*Proof.*

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$$P_\pi P_{\pi^{-1}} \stackrel{(*)}{=} P_{\pi^{-1} \circ \pi} = P_{1_n} = I_n,$$

where (*) follows immediately from Proposition 2.3.13.

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**Theorem 2.3.14**

Let $n$ be a positive integer, and let $\pi \in S_n$. Then $P_\pi$ is invertible, and moreover,

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- **Remark:** A matrix $Q \in \mathbb{R}^{n\times n}$ is *orthogonal* if it satisfies $Q^T Q = I_n$.
  - Theorem 2.3.14 guarantees that permutation matrices are orthogonal (as long as we consider the 0's and 1's in those matrices as belonging to $\mathbb{R}$, rather than to some other field).

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- As our next theorem (Theorem 2.3.15, next slide) shows, multiplying a matrix by a permutation matrix on the left permutes the rows of the original matrix.
- On the other hand, multiplying a matrix by a permutation matrix on the right permutes the columns of the original matrix.

## Theorem 2.3.15

Let $A = \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix}$ be an $n \times m$ matrix with entries in some field $\mathbb{F}$. Then all the following hold:

(a) for all $\pi \in S_n$, we have that

$$P_\pi A = \begin{bmatrix} \mathbf{r}_{\pi(1)} \\ \vdots \\ \mathbf{r}_{\pi(n)} \end{bmatrix};$$

(b) for all $\pi \in S_m$, we have that

$$AP_\pi = \begin{bmatrix} \mathbf{a}_{\pi^{-1}(1)} & \dots & \mathbf{a}_{\pi^{-1}(m)} \end{bmatrix};$$

(c) for all $\pi \in S_m$, we have that

$$AP_\pi^T = \begin{bmatrix} \mathbf{a}_{\pi(1)} & \dots & \mathbf{a}_{\pi(m)} \end{bmatrix}.$$

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*Proof.* We prove (b). Parts (a) and (c) are in the Lecture Notes.

## Theorem 2.3.15

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*Proof of (b).*

## Theorem 2.3.15

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- (b) for all $\pi \in S_m$, we have that

$$AP_\pi = \begin{bmatrix} \mathbf{a}_{\pi^{-1}(1)} & \dots & \mathbf{a}_{\pi^{-1}(m)} \end{bmatrix};$$

*Proof of (b).* Fix any permutation $\pi \in S_m$. In what follows, $\mathbf{e}_1, \dots, \mathbf{e}_m$ are the standard basis vectors of $\mathbb{F}^m$. We compute:

$$\begin{aligned} AP_\pi &= A\begin{bmatrix} \mathbf{e}_{\pi^{-1}(1)} & \dots & \mathbf{e}_{\pi^{-1}(m)} \end{bmatrix} && \text{by Proposition 2.3.11} \\ &= \begin{bmatrix} A\mathbf{e}_{\pi^{-1}(1)} & \dots & A\mathbf{e}_{\pi^{-1}(m)} \end{bmatrix} && \text{by the definition of matrix multiplication} \\ &= \begin{bmatrix} \mathbf{a}_{\pi^{-1}(1)} & \dots & \mathbf{a}_{\pi^{-1}(m)} \end{bmatrix} && \text{by Proposition 1.4.5.} \end{aligned}$$

This proves (b). □

## Theorem 2.3.15

Let $A = \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_m \end{bmatrix}$ be an $n \times m$ matrix with entries in some field $\mathbb{F}$. Then all the following hold:

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2. Orthogonal matrices

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**Theorem 2.3.14**

Let $n$ be a positive integer, and let $\pi \in S_n$. Then $P_\pi$ is invertible, and moreover,

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### Theorem 2.3.14

Let $n$ be a positive integer, and let $\pi \in S_n$. Then $P_\pi$ is invertible, and moreover,

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- The matrices mentioned so far all have entries only $-1, 0, 1$. However, there are many other orthogonal matrices, and we will see a couple of examples later.

- Reminder:

**Corollary 3.3.18**

Let $\mathbb{F}$ be field, and let $A, B \in \mathbb{F}^{n \times n}$ be such that $AB = I_n$ or $BA = I_n$. Then $AB = BA = I_n$, i.e. $A$ and $B$ are both invertible and are each other's inverses.

### Theorem 6.8.1

Let $Q \in \mathbb{R}^{n \times n}$. Then the following are equivalent:

(a) $Q$ is orthogonal (i.e. satisfies $Q^T Q = I_n$);

(b) $Q$ is invertible and satisfies $Q^{-1} = Q^T$;

(c) $QQ^T = I_n$;

(d) $Q^T$ is orthogonal;

(e) $Q$ is invertible and $Q^{-1}$ is orthogonal;

(f) the columns of $Q$ form an orthonormal basis of $\mathbb{R}^n$;

(g) the columns of $Q^T$ form an orthonormal basis of $\mathbb{R}^n$.

*Proof.*

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*Proof.* By Corollary 3.3.18, we have that (a), (b), and (c) are equivalent. Moreover, since $(Q^T)^T = Q$, we have that (c) and (d) are equivalent. This proves that (a), (b), (c), and (d) are equivalent.

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*Proof (continued).* Next, (b) and (d) together imply (e).

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*Proof (continued).* Next, (b) and (d) together imply (e).

Suppose now that (e) holds. Then by applying "(a) $\Longrightarrow$ (b)" to the matrix $Q^{-1}$, we see that $Q^{-1}$ is invertible and satisfies $(Q^{-1})^{-1} = (Q^{-1})^T$. Consequently, $Q^{-1} = Q^T$, and it follows that (b) holds.

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*Proof (continued).* So far, we have established that (a), (b), (c), (d), and (e) are equivalent.

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*Proof (continued).* So far, we have established that (a), (b), (c), (d), and (e) are equivalent.

Let us now show that (a) and (f) are equivalent.

## Theorem 6.8.1

- (a) $Q$ is orthogonal (i.e. satisfies $Q^T Q = I_n$);
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*Proof (continued).*

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*Proof (continued).* Set $Q = \begin{bmatrix} \mathbf{q}_1 & \dots & \mathbf{q}_n \end{bmatrix}$. Then

$$
\begin{aligned}
Q^T Q &= \begin{bmatrix} \mathbf{q}_1^T \\ \mathbf{q}_2^T \\ \vdots \\ \mathbf{q}_n^T \end{bmatrix} \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \dots & \mathbf{q}_n \end{bmatrix} \\
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Q^T Q &= \begin{bmatrix} \mathbf{q}_1^T \\ \mathbf{q}_2^T \\ \vdots \\ \mathbf{q}_n^T \end{bmatrix} \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \dots & \mathbf{q}_n \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{q}_1 \cdot \mathbf{q}_1 & \mathbf{q}_1 \cdot \mathbf{q}_2 & \dots & \mathbf{q}_1 \cdot \mathbf{q}_n \\ \mathbf{q}_2 \cdot \mathbf{q}_1 & \mathbf{q}_2 \cdot \mathbf{q}_2 & \dots & \mathbf{q}_2 \cdot \mathbf{q}_n \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{q}_n \cdot \mathbf{q}_1 & \mathbf{q}_n \cdot \mathbf{q}_2 & \dots & \mathbf{q}_n \cdot \mathbf{q}_n \end{bmatrix}.
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So, $Q^T Q = I_n$ iff $\{\mathbf{q}_1, \dots, \mathbf{q}_n\}$ is an orthonormal set.

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\end{aligned}
$$

So, $Q^TQ = I_n$ iff $\{\mathbf{q}_1, \dots, \mathbf{q}_n\}$ is an orthonormal set. But by Proposition 6.3.4(b), any orthonormal set of $n$ vectors in $\mathbb{R}^n$ is in fact an orthonormal basis of $\mathbb{R}^n$. So, (a) and (f) are equivalent.

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Let $Q \in \mathbb{R}^{n \times n}$. Then the following are equivalent:

- (a) $Q$ is orthogonal (i.e. satisfies $Q^T Q = I_n$);
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- (e) $Q$ is invertible and $Q^{-1}$ is orthogonal;
- (f) the columns of $Q$ form an orthonormal basis of $\mathbb{R}^n$;
- (g) the columns of $Q^T$ form an orthonormal basis of $\mathbb{R}^n$.

*Proof (continued).* Analogously to "(a) $\iff$ (f)," we have that (d) and (g) are equivalent. $\square$

**Theorem 6.8.1**

Let $Q \in \mathbb{R}^{n \times n}$. Then the following are equivalent:

(a) $Q$ is orthogonal (i.e. satisfies $Q^T Q = I_n$);

(b) $Q$ is invertible and satisfies $Q^{-1} = Q^T$;

(c) $QQ^T = I_n$;

(d) $Q^T$ is orthogonal;

(e) $Q$ is invertible and $Q^{-1}$ is orthogonal;

(f) the columns of $Q$ form an orthonormal basis of $\mathbb{R}^n$;

(g) the columns of $Q^T$ form an orthonormal basis of $\mathbb{R}^n$.

- We can make new orthogonal matrices out of old ones, as Propositions 6.8.2, 6.8.3, and 6.8.4 (below and next slide) show.
- The proofs of these propositions are easy and are in the Lecture Notes (we omit them here).

**Proposition 6.8.2**

Let

$$Q = \begin{bmatrix} \mathbf{q}_1 & \dots & \mathbf{q}_n \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1^T \\ \vdots \\ \mathbf{r}_n^T \end{bmatrix}$$

be an orthogonal matrix in $\mathbb{R}^n$. Then all the following hold:

(a) $\forall \alpha_1, \dots, \alpha_n \in \{-1, 1\}$: $\begin{bmatrix} \alpha_1 \mathbf{q}_1 & \dots & \alpha_n \mathbf{q}_n \end{bmatrix}$ is orthogonal;

(b) $\forall \alpha_1, \dots, \alpha_n \in \{-1, 1\}$: $\begin{bmatrix} \alpha_1 \mathbf{r}_1^T \\ \vdots \\ \alpha_n \mathbf{r}_n^T \end{bmatrix}$ is orthogonal;

(c) the matrix $-Q$ is orthogonal.

## Proposition 6.8.3

If $Q_1, Q_2 \in \mathbb{R}^{n\times n}$ are orthogonal, then so is their product $Q_1 Q_2$.

## Proposition 6.8.4

Let $Q_1 \in \mathbb{R}^{m\times m}$ and $Q_2 \in \mathbb{R}^{n\times n}$ be orthogonal matrices. Then the $(m+n)\times(m+n)$ matrix

$$Q = \left[\begin{array}{c:c} Q_1 & O_{m\times n} \\ \hdashline O_{n\times m} & Q_2 \end{array}\right]$$

is an orthogonal matrix in $\mathbb{R}^{(m+n)\times(m+n)}$.

- Next, we discuss two particularly significant orthogonal matrices: the Householder matrix and the Givens matrix.

- Next, we discuss two particularly significant orthogonal matrices: the Householder matrix and the Givens matrix.
- Reminder:

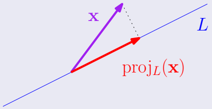
### Corollary 6.6.4

Let $\mathbf{a}$ be a non-zero vector in $\mathbb{R}^n$. Then the standard matrix of orthogonal projection onto the line $L := \text{Span}(\mathbf{a})$ is the matrix

$$\mathbf{a}(\mathbf{a}^T\mathbf{a})^{-1}\mathbf{a}^T \;=\; \mathbf{a}(\mathbf{a}\cdot\mathbf{a})^{-1}\mathbf{a}^T \;=\; \frac{1}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T.$$

Consequently, for every vector $\mathbf{x} \in \mathbb{R}^n$, we have that

$$\mathbf{x}_L \;=\; \text{proj}_L(\mathbf{x}) \;=\; \frac{1}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T\mathbf{x}.$$



## Definition

For a non-zero vector $\mathbf{a}$ in $\mathbb{R}^n$, the *Householder matrix* is the $n \times n$ matrix

$$H(\mathbf{a}) \quad := \quad I_n - \frac{2}{\mathbf{a}^T\mathbf{a}}\mathbf{a}\mathbf{a}^T \quad = \quad I_n - \frac{2}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T.$$

## Definition

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- To see that $H(\mathbf{a})$ is an orthogonal matrix, we compute:

$$\begin{aligned}
H(\mathbf{a})^T\ H(\mathbf{a}) &= (I_n - \tfrac{2}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T)^T\ (I_n - \tfrac{2}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T) \\
&= (I_n^T - \tfrac{2}{\mathbf{a}\cdot\mathbf{a}}(\mathbf{a}\mathbf{a}^T)^T)\ (I_n - \tfrac{2}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T) \\
&= (I_n - \tfrac{2}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T)\ (I_n - \tfrac{2}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T) \\
&= I_n - \tfrac{4}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T + \tfrac{4}{(\mathbf{a}\cdot\mathbf{a})^2}\mathbf{a}\,\underbrace{\mathbf{a}^T\mathbf{a}}_{=\mathbf{a}\cdot\mathbf{a}}\,\mathbf{a}^T \\
&= I_n - \tfrac{4}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T + \tfrac{4}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T \\
&= I_n.
\end{aligned}$$

- Let us now discuss the geometric meaning of this matrix.

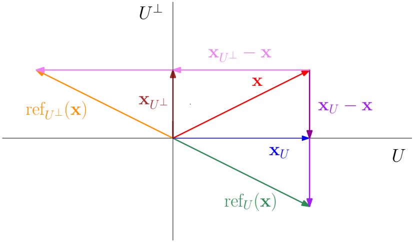
- Let us now discuss the geometric meaning of this matrix.
- First of all, we observe that if $U$ is a subspace of $\mathbb{R}^n$, then for any vector $\mathbf{x} \in \mathbb{R}^n$, the reflection of $\mathbf{x}$ about $U$ is given by

$$\text{ref}_U(\mathbf{x}) \quad := \quad \mathbf{x} + 2(\mathbf{x}_U - \mathbf{x}) \quad = \quad 2\mathbf{x}_U - \mathbf{x},$$

and the reflection of $\mathbf{x}$ about $U^\perp$ is given by

$$\text{ref}_{U^\perp}(\mathbf{x}) \quad := \quad \mathbf{x} + 2(\mathbf{x}_{U^\perp} - \mathbf{x}) \quad \overset{(*)}{=} \quad \mathbf{x} - 2\mathbf{x}_U \quad = \quad -\text{ref}_U(\mathbf{x}),$$

where (*) follows from the fact that $\mathbf{x} = \mathbf{x}_U + \mathbf{x}_{U^\perp}$ (by Corollary 6.5.3); so, either one of $\mathbf{x}_U$ and $\mathbf{x}_{U^\perp}$ can be obtained from the other by reflecting about the origin.



- Reminder: $H(\mathbf{a}) := I_n - \frac{2}{\mathbf{a}^T\mathbf{a}}\mathbf{a}\mathbf{a}^T = I_n - \frac{2}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T$.

- Reminder: $H(\mathbf{a}) := I_n - \frac{2}{\mathbf{a}^T\mathbf{a}}\mathbf{a}\mathbf{a}^T = I_n - \frac{2}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T$.
- So, for any $\mathbf{x} \in \mathbb{R}^n$, then the projection of $\mathbf{x}$ onto $L$ is given by $\mathbf{x}_L = \frac{1}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T\mathbf{x}$,

- Reminder: $H(\mathbf{a}) := I_n - \frac{2}{\mathbf{a}^T\mathbf{a}}\mathbf{a}\mathbf{a}^T = I_n - \frac{2}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T$.
- So, for any $\mathbf{x} \in \mathbb{R}^n$, then the projection of $\mathbf{x}$ onto $L$ is given by $\mathbf{x}_L = \frac{1}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T\mathbf{x}$, the reflection of $\mathbf{x}$ about the line $L$ is given by

$$
\begin{aligned}
\mathrm{ref}_L(\mathbf{x}) &= 2\mathbf{x}_L - \mathbf{x} = \frac{2}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T\mathbf{x} - I_n\mathbf{x} \\
&= \left(\frac{2}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T - I_n\right)\mathbf{x} = -H(\mathbf{a})\mathbf{x},
\end{aligned}
$$

- Reminder: $H(\mathbf{a}) := I_n - \frac{2}{\mathbf{a}^T\mathbf{a}}\mathbf{a}\mathbf{a}^T = I_n - \frac{2}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T$.
- So, for any $\mathbf{x} \in \mathbb{R}^n$, then the projection of $\mathbf{x}$ onto $L$ is given by $\mathbf{x}_L = \frac{1}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T\mathbf{x}$, the reflection of $\mathbf{x}$ about the line $L$ is given by

$$\begin{aligned} \text{ref}_L(\mathbf{x}) &= 2\mathbf{x}_L - \mathbf{x} = \frac{2}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T\mathbf{x} - I_n\mathbf{x} \\ &= \left(\frac{2}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T - I_n\right)\mathbf{x} = -H(\mathbf{a})\mathbf{x}, \end{aligned}$$

  and the reflection of $\mathbf{x}$ about $L^\perp$ is given by

$$\text{ref}_{L^\perp}(\mathbf{x}) = -\text{ref}_L(\mathbf{x}) = H(\mathbf{a})\mathbf{x}.$$

- Reminder: $H(\mathbf{a}) := I_n - \frac{2}{\mathbf{a}^T\mathbf{a}}\mathbf{a}\mathbf{a}^T = I_n - \frac{2}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T$.
- So, for any $\mathbf{x} \in \mathbb{R}^n$, then the projection of $\mathbf{x}$ onto $L$ is given by $\mathbf{x}_L = \frac{1}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T\mathbf{x}$, the reflection of $\mathbf{x}$ about the line $L$ is given by

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  and the reflection of $\mathbf{x}$ about $L^\perp$ is given by

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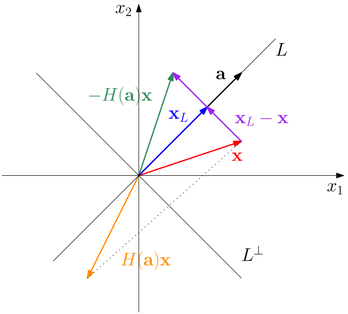
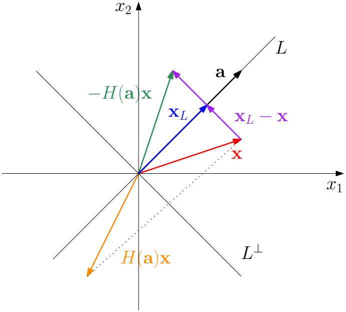
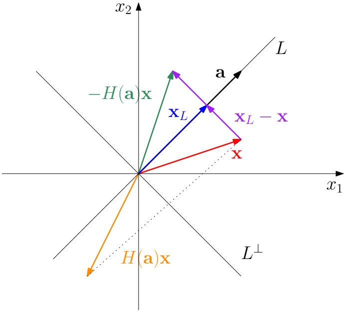


- Reminder: $H(\mathbf{a}) := I_n - \frac{2}{\mathbf{a}^T\mathbf{a}}\mathbf{a}\mathbf{a}^T = I_n - \frac{2}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T$.




- Thus, $-H(\mathbf{a})$ is the standard matrix of reflection about the line $L = \text{Span}(\mathbf{a})$, whereas the Householder matrix $H(\mathbf{a})$ itself is the standard matrix of reflection about $L^\perp$.Thus, $-H(\mathbf{a})$ is the standard matrix of reflection about the Span($\mathbf{a}$) line.

- Reminder: $H(\mathbf{a}) := I_n - \frac{2}{\mathbf{a}^T\mathbf{a}}\mathbf{a}\mathbf{a}^T = I_n - \frac{2}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}\mathbf{a}^T$.




- **Remark:** Suppose that $\mathbf{a}$ is a non-zero vector in $\mathbb{R}^n$.
    - Then the standard matrix of reflection about the line $L := \text{Span}(\mathbf{a})$ in $\mathbb{R}^n$ is an orthogonal matrix.
    - Indeed, as we saw, the Householder matrix $H(\mathbf{a})$ is an orthogonal matrix.
    - By Proposition 6.8.2(c), it follows that $-H(\mathbf{a})$ is also an orthogonal matrix, and as we saw above, $-H(\mathbf{a})$ is the standard matrix of reflection about the line $L = \text{Span}(\mathbf{a})$ in $\mathbb{R}^n$.

- Given an integer $n \geq 2$, indices $i, j \in \{1, \ldots, n\}$ such that $i < j$, and real numbers $c$ and $s$ such that $c^2 + s^2 = 1$, we define the *Givens matrix* $G_{i,j}(c, s)$ as follows:

$$
G_{i,j}(c,s) = \begin{array}{c} \\ \\ \\ \\ \\ i \\ \\ \\ \\ j \\ \\ \\ \\ \end{array}
\begin{array}{cccccccccccc}
 & & & & i & & & & j & & & \\
1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & c & 0 & \ldots & 0 & -s & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & s & 0 & \ldots & 0 & c & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 & & 0 & 0 & 1 & & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & 0 & 0 & & 0 & 0 & 0 & \ldots & 1
\end{array}
$$

- Given an integer $n \geq 2$, indices $i, j \in \{1, \dots, n\}$ such that $i < j$, and real numbers $c$ and $s$ such that $c^2 + s^2 = 1$, we define the *Givens matrix* $G_{i,j}(c, s)$ as follows:

$$
G_{i,j}(c,s) = \begin{array}{c} \\ \\ \\ \\ i \\ \\ \\ \\ j \\ \\ \\ \\ \end{array}
\overset{\hspace{5em} i \hspace{6em} j}{
\begin{bmatrix}
1 & 0 & \dots 0 & 0 & 0 & \dots 0 & 0 & 0 & \dots 0 \\
0 & 1 & \dots 0 & 0 & 0 & \dots 0 & 0 & 0 & \dots 0 \\
\vdots & \vdots & \ddots \vdots & \vdots & \vdots & \ddots \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots 1 & 0 & 0 & \dots 0 & 0 & 0 & \dots 0 \\
0 & 0 & \dots 0 & c & 0 & \dots 0 & -s & 0 & \dots 0 \\
0 & 0 & \dots 0 & 0 & 1 & \dots 0 & 0 & 0 & \dots 0 \\
\vdots & \vdots & \dots \vdots & \vdots & \vdots & \ddots \vdots & \vdots & \vdots & 0 \\
0 & 0 & \dots 0 & 0 & 0 & \dots 1 & 0 & 0 & \dots 0 \\
0 & 0 & \dots 0 & s & 0 & \dots 0 & c & 0 & \dots 0 \\
0 & 0 & \dots 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \vdots \\
0 & 0 & \dots 0 & 0 & 0 & 0 & 0 & 0 & \dots 1
\end{bmatrix}}
$$

- It is not hard to check that the columns of $G_{i,j}(c, s)$ form an orthonormal set of vectors in $\mathbb{R}^n$, and therefore (by Proposition 6.3.4) an orthonormal basis of $\mathbb{R}^n$.
- So, by Theorem 6.8.1, $G_{i,j}(c, s)$ is orthogonal.

- Let us now give a geometric interpretation of this matrix.

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- Since $c^2 + s^2 = 1$, we see that there exists a real number (angle in radians) $\theta$ such that $c = \cos\theta$ and $s = \sin\theta$.

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- With this set-up, we see that $G_{i,j}(c, s)$ represents rotation about the origin by angle $\theta$ in the $x_i x_j$-plane.

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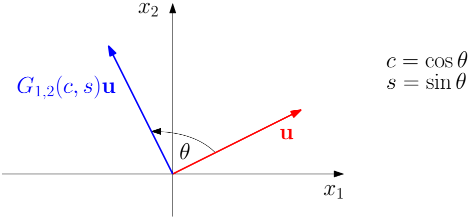
- Since $c^2 + s^2 = 1$, we see that there exists a real number (angle in radians) $\theta$ such that $c = \cos\theta$ and $s = \sin\theta$.
- With this set-up, we see that $G_{i,j}(c,s)$ represents rotation about the origin by angle $\theta$ in the $x_i x_j$-plane.
- This is particularly easy to see in the case when $n = 2$. In that case, we have that

$$G_{1,2}(c,s) = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

which is precisely the standard matrix of counterclockwise rotation about the origin by angle $\theta$.



## Theorem 6.8.5

Let $Q = \left[\ q_{i,j}\ \right]_{n\times n}$ be an orthogonal matrix in $\mathbb{R}^{n\times n}$. Then:

1. (a) for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $(Q\mathbf{x}) \cdot (Q\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$;
2. (b) for all $\mathbf{x} \in \mathbb{R}^n$, $||Q\mathbf{x}|| = ||\mathbf{x}||$;
3. (c) for all $i, j \in \{1, \dots, n\}$, $|q_{i,j}| \leq 1$.

- Proof: next slide.

## Theorem 6.8.5

Let $Q = \left[\ q_{i,j}\ \right]_{n \times n}$ be an orthogonal matrix in $\mathbb{R}^{n \times n}$. Then:

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- Proof: next slide.
- **Remark:** By Theorem 6.8.5(b), multiplication by an orthogonal matrix (on the left) preserves vector length.

## Theorem 6.8.5

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(c) for all $i, j \in \{1, \dots, n\}$, $|q_{i,j}| \leq 1$.

- Proof: next slide.
- **Remark:** By Theorem 6.8.5(b), multiplication by an orthogonal matrix (on the left) preserves vector length.
  - On the other hand, recall that for non-zero vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have that $\mathbf{x} \cdot \mathbf{y} = ||\mathbf{x}|| \; ||\mathbf{y}|| \cos\theta$, where $\theta$ is the angle between $\mathbf{x}$ and $\mathbf{y}$.

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- **Remark:** By Theorem 6.8.5(b), multiplication by an orthogonal matrix (on the left) preserves vector length.
  - On the other hand, recall that for non-zero vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have that $\mathbf{x} \cdot \mathbf{y} = ||\mathbf{x}|| \, ||\mathbf{y}|| \cos\theta$, where $\theta$ is the angle between $\mathbf{x}$ and $\mathbf{y}$.
  - So, Theorem 6.8.5(a-b) implies that multiplication (on the left) by an orthogonal matrix preserves angles between non-zero vectors.

## Theorem 6.8.5

Let $Q = \begin{bmatrix} q_{i,j} \end{bmatrix}_{n \times n}$ be an orthogonal matrix in $\mathbb{R}^{n \times n}$. Then:

(a) for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $(Q\mathbf{x}) \cdot (Q\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$;

(b) for all $\mathbf{x} \in \mathbb{R}^n$, $||Q\mathbf{x}|| = ||\mathbf{x}||$;

(c) for all $i, j \in \{1, \dots, n\}$, $|q_{i,j}| \leq 1$.

*Proof.*

## Theorem 6.8.5

Let $Q = \left[ \begin{array}{c} q_{i,j} \end{array} \right]_{n\times n}$ be an orthogonal matrix in $\mathbb{R}^{n\times n}$. Then:

- (a) for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $(Q\mathbf{x}) \cdot (Q\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$;
- (b) for all $\mathbf{x} \in \mathbb{R}^n$, $||Q\mathbf{x}|| = ||\mathbf{x}||$;
- (c) for all $i, j \in \{1, \ldots, n\}$, $|q_{i,j}| \leq 1$.

*Proof.* (a) For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have the following:

$$(Q\mathbf{x}) \cdot (Q\mathbf{y}) = (Q\mathbf{x})^T(Q\mathbf{x}) = \mathbf{x}^T \underbrace{Q^T Q}_{=I_n} \mathbf{y} = \mathbf{x}^T\mathbf{y} = \mathbf{x} \cdot \mathbf{y}.$$

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(b) For $\mathbf{x} \in \mathbb{R}^n$, we have the following:

$$||Q\mathbf{x}|| = \sqrt{(Q\mathbf{x}) \cdot (Q\mathbf{x})} \overset{(a)}{=} \sqrt{\mathbf{x} \cdot \mathbf{x}} = ||\mathbf{x}||.$$

## Theorem 6.8.5

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*Proof.* (a) For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have the following:

$$(Q\mathbf{x}) \cdot (Q\mathbf{y}) \quad = \quad (Q\mathbf{x})^T (Q\mathbf{x}) \quad = \quad \mathbf{x}^T \underbrace{Q^T Q}_{= I_n} \mathbf{y} \quad = \quad \mathbf{x}^T \mathbf{y} \quad = \quad \mathbf{x} \cdot \mathbf{y}.$$

(b) For $\mathbf{x} \in \mathbb{R}^n$, we have the following:

$$||Q\mathbf{x}|| \quad = \quad \sqrt{(Q\mathbf{x}) \cdot (Q\mathbf{x})} \quad \stackrel{(a)}{=} \quad \sqrt{\mathbf{x} \cdot \mathbf{x}} \quad = \quad ||\mathbf{x}||.$$

(c) By Theorem 6.8.1, the columns of $Q$ form an orthonormal basis. In particular, all columns of $Q$ are unit vectors, and it follows that all entries of $Q$ have absolute value at most 1. □

3. Scalar product, coordinate vectors, and matrices of linear functions

3. Scalar product, coordinate vectors, and matrices of linear functions

**Proposition 6.9.1**

Let $V$ be a real or complex vector space, equipped with the scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $|| \cdot ||$, and let $\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be an **orthonormal** basis of $V$. Let $\cdot$ be the standard scalar product in $\mathbb{R}^n$ or $\mathbb{C}^n$ (depending on whether the vector space $V$ is real or complex). Then for all $\mathbf{x}, \mathbf{y} \in V$, we have that

$$\langle \mathbf{x}, \mathbf{y} \rangle \;=\; \left[\, \mathbf{x} \,\right]_{\mathcal{B}} \cdot \left[\, \mathbf{y} \,\right]_{\mathcal{B}}.$$

- Proof: Lecture Notes.

## Theorem 6.9.2

Let $U$ and $V$ be non-trivial, finite-dimensional **real** vector spaces. Assume that $U$ is equipped with a scalar product $\langle \cdot, \cdot \rangle_U$ and the induced norm $|| \cdot ||_U$, and that $V$ is equipped with a scalar product $\langle \cdot, \cdot \rangle_V$ and the induced norm $|| \cdot ||_V$. Let $\mathcal{B}_U = \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ and $\mathcal{B}_V = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be **orthonormal** bases of $U$ and $V$, respectively, and let $f : U \to V$ be a linear function. Then the following two statements are equivalent:

1. the columns of the $n \times m$ matrix ${}_{\mathcal{B}_V}\left[\ f\ \right]_{\mathcal{B}_U}$ form an orthonormal set of vectors in $\mathbb{R}^n$ (with respect to the standard scalar product $\cdot$ and the induced norm $|| \cdot ||$);[a]
2. $f$ preserves the scalar product, that is, for all vectors $\mathbf{x}, \mathbf{y} \in U$, we have that $\langle f(\mathbf{x}), f(\mathbf{y}) \rangle_V = \langle \mathbf{x}, \mathbf{y} \rangle_U$.

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[a] However, despite Theorem 6.8.1, this does not necessarily mean that the matrix ${}_{\mathcal{B}_V}\left[\ f\ \right]_{\mathcal{B}_U}$ is orthogonal. This is because ${}_{\mathcal{B}_V}\left[\ f\ \right]_{\mathcal{B}_U}$ is an $n \times m$ matrix, and it is possible that $m \neq n$, in which case ${}_{\mathcal{B}_V}\left[\ f\ \right]_{\mathcal{B}_U}$ is not a square matrix. Only square matrices can be orthogonal!

*Proof.* Set ${}_{\mathcal{B}_V}[\ f\ ]_{\mathcal{B}_U} = [\ \mathbf{c}_1 \quad \dots \quad \mathbf{c}_m\ ]$.

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$$
\begin{aligned}
({}_{\mathcal{B}_V}\big[\ f\ \big]_{\mathcal{B}_U})^T\ {}_{\mathcal{B}_V}\big[\ f\ \big]_{\mathcal{B}_U} &= \begin{bmatrix} \mathbf{c}_1^T \\ \mathbf{c}_2^T \\ \vdots \\ \mathbf{c}_m^T \end{bmatrix} \big[\ \mathbf{c}_1 \quad \mathbf{c}_2 \quad \dots \quad \mathbf{c}_m\ \big] \\
&= \begin{bmatrix} \mathbf{c}_1 \cdot \mathbf{c}_1 & \mathbf{c}_1 \cdot \mathbf{c}_2 & \dots & \mathbf{c}_1 \cdot \mathbf{c}_m \\ \mathbf{c}_2 \cdot \mathbf{c}_1 & \mathbf{c}_2 \cdot \mathbf{c}_2 & \dots & \mathbf{c}_2 \cdot \mathbf{c}_m \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{c}_m \cdot \mathbf{c}_1 & \mathbf{c}_m \cdot \mathbf{c}_2 & \dots & \mathbf{c}_m \cdot \mathbf{c}_m \end{bmatrix}.
\end{aligned}
$$

So, we see that (i) holds iff $({}_{\mathcal{B}_V}\big[\ f\ \big]_{\mathcal{B}_U})^T\ {}_{\mathcal{B}_V}\big[\ f\ \big]_{\mathcal{B}_U} = I_m$.

*Proof (cont.).* Reminder: (i) holds iff $({}_{\mathcal{B}_V}[\ f\ ]_{\mathcal{B}_U})^T\ {}_{\mathcal{B}_V}[\ f\ ]_{\mathcal{B}_U} = I_m$.

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Next, by Proposition 6.9.1, the following hold for all $\mathbf{x}, \mathbf{y} \in U$:

(1) $\langle \mathbf{x}, \mathbf{y} \rangle_U = [\ \mathbf{x}\ ]_{\mathcal{B}_U} \cdot [\ \mathbf{y}\ ]_{\mathcal{B}_U}$;

(2) $\big\langle f(\mathbf{x}), f(\mathbf{y}) \big\rangle_V = [\ f(\mathbf{x})\ ]_{\mathcal{B}_V} \cdot [\ f(\mathbf{y})\ ]_{\mathcal{B}_V}$.

*Proof (cont.).* Reminder: (i) holds iff $({}_{\mathcal{B}_V}[\ f\ ]_{\mathcal{B}_U})^T\ {}_{\mathcal{B}_V}[\ f\ ]_{\mathcal{B}_U} = I_m$.

Next, by Proposition 6.9.1, the following hold for all $\mathbf{x}, \mathbf{y} \in U$:

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(2) $\big\langle f(\mathbf{x}), f(\mathbf{y}) \big\rangle_V = [\ f(\mathbf{x})\ ]_{\mathcal{B}_V} \cdot [\ f(\mathbf{y})\ ]_{\mathcal{B}_V}$.

Now, for all $\mathbf{x}, \mathbf{y} \in U$, we have that

$$\begin{aligned} \big\langle f(\mathbf{x}), f(\mathbf{y}) \big\rangle_V &\stackrel{(2)}{=} [\ f(\mathbf{x})\ ]_{\mathcal{B}_V} \cdot [\ f(\mathbf{y})\ ]_{\mathcal{B}_V} \\ &= \big([\ f(\mathbf{x})\ ]_{\mathcal{B}_V}\big)^T\ [\ f(\mathbf{y})\ ]_{\mathcal{B}_V} \\ &= \Big({}_{\mathcal{B}_V}[\ f\ ]_{\mathcal{B}_U}\ [\ \mathbf{x}\ ]_{\mathcal{B}_U}\Big)^T \Big({}_{\mathcal{B}_V}[\ f\ ]_{\mathcal{B}_U}\ [\ \mathbf{y}\ ]_{\mathcal{B}_U}\Big) \\ &= \big([\ \mathbf{x}\ ]_{\mathcal{B}_U}\big)^T \big({}_{\mathcal{B}_V}[\ f\ ]_{\mathcal{B}_U}\big)^T\ {}_{\mathcal{B}_V}[\ f\ ]_{\mathcal{B}_U}\ [\ \mathbf{y}\ ]_{\mathcal{B}_U}. \end{aligned}$$

*Proof (continued).* Suppose first that (i) holds. Then $({}_{\mathcal{B}_V}[\ f\ ]_{\mathcal{B}_U})^T\ {}_{\mathcal{B}_V}[\ f\ ]_{\mathcal{B}_U} = I_m$, and consequently, for all $\mathbf{x}, \mathbf{y} \in U$, we have that

$$\begin{aligned}
\big\langle f(\mathbf{x}), f(\mathbf{y}) \big\rangle_V &= \big( [\ \mathbf{x}\ ]_{\mathcal{B}_U} \big)^T \underbrace{({}_{\mathcal{B}_V}[\ f\ ]_{\mathcal{B}_U})^T\ {}_{\mathcal{B}_V}[\ f\ ]_{\mathcal{B}_U}}_{=I_m} [\ \mathbf{y}\ ]_{\mathcal{B}_U} \\
&= \big( [\ \mathbf{x}\ ]_{\mathcal{B}_U} \big)^T [\ \mathbf{y}\ ]_{\mathcal{B}_U} \\
&= [\ \mathbf{x}\ ]_{\mathcal{B}_U} \cdot [\ \mathbf{y}\ ]_{\mathcal{B}_U} \\
&\overset{(1)}{=} \langle \mathbf{x}, \mathbf{y} \rangle_U.
\end{aligned}$$

Thus, (ii) holds.

*Proof (continued).* Reminder: ${}_{\mathcal{B}_V}\big[\ f\ \big]_{\mathcal{B}_U} = \big[\ \mathbf{c}_1 \quad \dots \quad \mathbf{c}_m\ \big]$

Suppose now that (ii) holds. Then for all $i, j \in \{1, \dots, m\}$, we have that

$$\begin{aligned}
\mathbf{e}_i^m \cdot \mathbf{e}_j^m &= \big[\ \mathbf{u}_i\ \big]_{\mathcal{B}_U} \cdot \big[\ \mathbf{u}_j\ \big]_{\mathcal{B}_U} \\
&\overset{(1)}{=} \langle \mathbf{u}_i, \mathbf{u}_j \rangle_U \\
&\overset{(ii)}{=} \big\langle f(\mathbf{u}_i), f(\mathbf{u}_j) \big\rangle_V \\
&\overset{(2)}{=} \big[\ f(\mathbf{u}_i)\ \big]_{\mathcal{B}_V} \cdot \big[\ f(\mathbf{u}_j)\ \big]_{\mathcal{B}_V} \\
&= \big({}_{\mathcal{B}_V}\big[\ f\ \big]_{\mathcal{B}_U}\ \big[\ \mathbf{u}_i\ \big]_{\mathcal{B}_U}\big) \cdot \big({}_{\mathcal{B}_V}\big[\ f\ \big]_{\mathcal{B}_U}\ \big[\ \mathbf{u}_j\ \big]_{\mathcal{B}_U}\big) \\
&= \big({}_{\mathcal{B}_V}\big[\ f\ \big]_{\mathcal{B}_U}\ \mathbf{e}_i^m\big) \cdot \big({}_{\mathcal{B}_V}\big[\ f\ \big]_{\mathcal{B}_U}\ \mathbf{e}_j^m\big) \\
&= \mathbf{c}_i \cdot \mathbf{c}_j.
\end{aligned}$$

So, $\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ is an orthonormal set of vectors in $\mathbb{R}^n$, that is, (i) holds. $\square$

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Let $U$ and $V$ be non-trivial, finite-dimensional **real** vector spaces. Assume that $U$ is equipped with a scalar product $\langle\cdot,\cdot\rangle_U$ and the induced norm $||\cdot||_U$, and that $V$ is equipped with a scalar product $\langle\cdot,\cdot\rangle_V$ and the induced norm $||\cdot||_V$. Let $\mathcal{B}_U = \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ and $\mathcal{B}_V = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be **orthonormal** bases of $U$ and $V$, respectively, and let $f : U \to V$ be a linear function. Then the following two statements are equivalent:

1. (i) the columns of the $n \times m$ matrix ${}_{\mathcal{B}_V}\left[\ f\ \right]_{\mathcal{B}_U}$ form an orthonormal set of vectors in $\mathbb{R}^n$ (with respect to the standard scalar product $\cdot$ and the induced norm $||\cdot||$);[a]
2. (ii) $f$ preserves the scalar product, that is, for all vectors $\mathbf{x}, \mathbf{y} \in U$, we have that $\langle f(\mathbf{x}), f(\mathbf{y})\rangle_V = \langle \mathbf{x}, \mathbf{y}\rangle_U$.

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