Linear Algebra 2

Lecture #15

Gram-Schmidt orthogonalization

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  - The first version first produces an orthogonal basis, and then (optionally) produces an orthonormal basis.
  - The second version produces an orthonormal basis directly.
- We first describe the first version, we give a numerical example, and we outline the proof of correctness of the process (the full technical details are in the Lecture Notes).
- Then we describe the second version.
  - The proof of correctness is similar to the proof of the first, and we omit it.
  - A numerical example is given in the Lecture Notes.

Gram-Schmidt orthogonalization process (version 1)

Let V be a real or complex vector space, equipped with a scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $|| \cdot ||$, and let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be linearly independent vectors in V. For all $\ell \in \{1, \dots, k\}$, set

$$\mathbf{u}_\ell \quad := \quad \mathbf{v}_\ell - \sum_{i=1}^{\ell-1} \text{proj}_{\mathbf{u}_i}(\mathbf{v}_\ell) \quad = \quad \mathbf{v}_\ell - \sum_{i=1}^{\ell-1} \frac{\langle \mathbf{v}_\ell, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i.$$

Then $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal basis of $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$, and $\left\{ \frac{\mathbf{u}_1}{||\mathbf{u}_1||}, \dots, \frac{\mathbf{u}_k}{||\mathbf{u}_k||} \right\}$ is an orthonormal basis of $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$.

- The sequence $\mathbf{u}_1, \dots, \mathbf{u}_k$ is obtained (recursively) as follows:
  - $\mathbf{u}_1 := \mathbf{v}_1$;
  - $\mathbf{u}_2 := \mathbf{v}_2 - \text{proj}_{\mathbf{u}_1}(\mathbf{v}_2)$;
  - $\mathbf{u}_3 := \mathbf{v}_3 - \Big(\text{proj}_{\mathbf{u}_1}(\mathbf{v}_3) + \text{proj}_{\mathbf{u}_2}(\mathbf{v}_3)\Big)$;

    $\vdots$
  - $\mathbf{u}_k := \mathbf{v}_k - \Big(\text{proj}_{\mathbf{u}_1}(\mathbf{v}_k) + \text{proj}_{\mathbf{u}_2}(\mathbf{v}_k) + \dots + \text{proj}_{\mathbf{u}_{k-1}}(\mathbf{v}_k)\Big)$.

Example 6.3.8

Consider the following linearly independent vectors in $\mathbb{R}^4$:

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 4 \\ -4 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -5 \\ 10 \\ 2 \\ 11 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 8 \\ 19 \\ 11 \\ -2 \end{bmatrix}.$$

Set $U := \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$. Using the Gram-Schmidt orthogonalization process (version 1):

(a) compute an orthogonal basis of U (w.r.t. the standard scalar product $\cdot$ in $\mathbb{R}^4$).

(b) compute an orthonormal basis of U (w.r.t. the standard scalar product $\cdot$ in $\mathbb{R}^4$ and the norm $|| \cdot ||$ induced by it).

- **Remark:** To see that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ really are linearly independent, we compute

$$\text{RREF}\Big(\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}\Big) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

and we deduce that $\text{rank}\Big(\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}\Big) = 3$, i.e. $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$ has full column rank. So, by Theorem 3.3.12(a), vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

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- Solution: On the board.

- Let's now prove the correctness of the Gram-Schmidt orthogonalization process! (Or rather: give an outline of it.)

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Proposition 6.3.7

Let V be a real or complex vector space, equipped with a scalar product $\langle \cdot, \cdot \rangle$. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ be an orthogonal set of non-zero vectors in V. Let $\mathbf{v} \in V$, and set $\mathbf{y} := \sum_{i=1}^{k} \text{proj}_{\mathbf{u}_i}(\mathbf{v}) = \sum_{i=1}^{k} \frac{\langle \mathbf{v}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i$ and $\mathbf{z} := \mathbf{v} - \mathbf{y}$. Then all the following hold:

(a) $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{z}\}$ is an orthogonal set of vectors;

(b) $\mathbf{z} = \mathbf{0}$ iff $\mathbf{v} \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$;

(c) $\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v}) = \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{z})$.

Proof.

- Let's now prove the correctness of the Gram-Schmidt orthogonalization process! (Or rather: give an outline of it.)
- We begin with a technical proposition.

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Let V be a real or complex vector space, equipped with a scalar product $\langle \cdot, \cdot \rangle$. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ be an orthogonal set of non-zero vectors in V. Let $\mathbf{v} \in V$, and set $\mathbf{y} := \sum_{i=1}^{k} \text{proj}_{\mathbf{u}_i}(\mathbf{v}) = \sum_{i=1}^{k} \frac{\langle \mathbf{v}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i$ and $\mathbf{z} := \mathbf{v} - \mathbf{y}$. Then all the following hold:

(a) $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{z}\}$ is an orthogonal set of vectors;

(b) $\mathbf{z} = \mathbf{0}$ iff $\mathbf{v} \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$;

(c) $\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v}) = \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{z})$.

Proof. First of all, Proposition 6.3.2 guarantees that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a linearly independent set, and we deduce that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal basis of $\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$.

Proof (continued). Let us first prove (a).

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$$\begin{aligned}
\langle \mathbf{z}, \mathbf{u}_j \rangle &= \langle \mathbf{v} - \sum_{i=1}^{k} \tfrac{\langle \mathbf{v}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i, \mathbf{u}_j \rangle \\
&\overset{(*)}{=} \langle \mathbf{v}, \mathbf{u}_j \rangle - \sum_{i=1}^{k} \tfrac{\langle \mathbf{v}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \langle \mathbf{u}_i, \mathbf{u}_j \rangle \\
&\overset{(**)}{=} \langle \mathbf{v}, \mathbf{u}_j \rangle - \tfrac{\langle \mathbf{v}, \mathbf{u}_j \rangle}{\langle \mathbf{u}_j, \mathbf{u}_j \rangle} \langle \mathbf{u}_j, \mathbf{u}_j \rangle \\
&= \langle \mathbf{v}, \mathbf{u}_j \rangle - \langle \mathbf{v}, \mathbf{u}_j \rangle \quad = \quad 0,
\end{aligned}$$

where (*) follows from r.2 and r.3 (in the real case) or from c.2 and c.3 (in the complex case), and (**) follows from the fact that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal set.

Proof (continued). Let us first prove (a). By hypothesis, vectors $\mathbf{u}_1, \dots, \mathbf{u}_k$ are pairwise orthogonal. On the other hand, for each $j \in \{1, \dots, k\}$, we have the following:

$$\begin{aligned}
\langle \mathbf{z}, \mathbf{u}_j \rangle &= \left\langle \mathbf{v} - \sum_{i=1}^{k} \frac{\langle \mathbf{v}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i, \mathbf{u}_j \right\rangle \\
&\overset{(*)}{=} \langle \mathbf{v}, \mathbf{u}_j \rangle - \sum_{i=1}^{k} \frac{\langle \mathbf{v}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \langle \mathbf{u}_i, \mathbf{u}_j \rangle \\
&\overset{(**)}{=} \langle \mathbf{v}, \mathbf{u}_j \rangle - \frac{\langle \mathbf{v}, \mathbf{u}_j \rangle}{\langle \mathbf{u}_j, \mathbf{u}_j \rangle} \langle \mathbf{u}_j, \mathbf{u}_j \rangle \\
&= \langle \mathbf{v}, \mathbf{u}_j \rangle - \langle \mathbf{v}, \mathbf{u}_j \rangle \;\; = \;\; 0,
\end{aligned}$$

where (*) follows from r.2 and r.3 (in the real case) or from c.2 and c.3 (in the complex case), and (**) follows from the fact that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal set. Thus, $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{z}\}$ is an orthogonal set of vectors. This proves (a).

Proof (continued). Next, we prove (b).

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Proof (continued). Next, we prove (b). Clearly, $\mathbf{z} = \mathbf{0}$ iff $\mathbf{v} = \sum_{i=1}^{k} \frac{\langle \mathbf{v}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i$.

So, we need to show that $\mathbf{v} = \sum_{i=1}^{k} \frac{\langle \mathbf{v}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i$ iff $\mathbf{v} \in \mathrm{Span}(\mathbf{u}_1, \ldots, \mathbf{u}_k)$.

Proof (continued). Next, we prove (b). Clearly, $\mathbf{z} = \mathbf{0}$ iff $\mathbf{v} = \sum_{i=1}^{k} \frac{\langle \mathbf{v}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i$.

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If $\mathbf{v} = \sum_{i=1}^{k} \frac{\langle \mathbf{v}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i$, then $\mathbf{v}$ is a linear combination of the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$, and consequently, $\mathbf{v} \in \text{Span}(\mathbf{u}_1, \ldots, \mathbf{u}_k)$.

Proof (continued). Next, we prove (b). Clearly, $\mathbf{z} = \mathbf{0}$ iff $\mathbf{v} = \sum_{i=1}^{k} \frac{\langle \mathbf{v}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i$.

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If $\mathbf{v} = \sum_{i=1}^{k} \frac{\langle \mathbf{v}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i$, then $\mathbf{v}$ is a linear combination of the vectors $\mathbf{u}_1, \dots, \mathbf{u}_k$, and consequently, $\mathbf{v} \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$.

On the other hand, if $\mathbf{v} \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$, then Theorem 6.3.5 guarantees $\mathbf{v} = \sum_{i=1}^{k} \frac{\langle \mathbf{v}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i$ (because $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal basis of $\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$). This proves (b).

Proof (continued).

Proof (continued). Finally, we prove (c). Fix any vector $\mathbf{x} \in V$. WTS $\mathbf{x} \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v})$ iff $\mathbf{x} \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{z})$. We prove both directions (they are very similar).

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Suppose first that $\mathbf{x} \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v})$.

Proof (continued). Finally, we prove (c). Fix any vector $\mathbf{x} \in V$. WTS $\mathbf{x} \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v})$ iff $\mathbf{x} \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{z})$. We prove both directions (they are very similar).

Suppose first that $\mathbf{x} \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v})$. Then there exist scalars $\alpha_1, \dots, \alpha_k, \beta$ s.t. $\mathbf{x} = \alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k + \beta \mathbf{v}$.

Proof (continued). Finally, we prove (c). Fix any vector $\mathbf{x} \in V$. WTS $\mathbf{x} \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v})$ iff $\mathbf{x} \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{z})$. We prove both directions (they are very similar).

Suppose first that $\mathbf{x} \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v})$. Then there exist scalars $\alpha_1, \dots, \alpha_k, \beta$ s.t. $\mathbf{x} = \alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k + \beta \mathbf{v}$. But now

$$\begin{aligned}
\mathbf{x} &= \alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k + \beta \mathbf{v} \\
&= \Big(\sum_{i=1}^{k} \alpha_i \mathbf{u}_i \Big) + \beta(\mathbf{y} + \mathbf{z}) \\
&= \Big(\sum_{i=1}^{k} \alpha_i \mathbf{u}_i \Big) + \beta\Big(\big(\sum_{i=1}^{k} \tfrac{\langle \mathbf{v}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i \big) + \mathbf{z} \Big) \\
&= \Big(\sum_{i=1}^{k} \big(\alpha_i + \beta \tfrac{\langle \mathbf{v}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle}\big) \mathbf{u}_i \Big) + \beta \mathbf{z},
\end{aligned}$$

and we deduce that $\mathbf{x} \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{z})$.

Proof (continued). Suppose, conversely, that $\mathbf{x} \in \text{Span}(\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{z})$.

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Proof (continued). Suppose, conversely, that $\mathbf{x} \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{z})$. Then there exist scalars $\alpha_1, \dots, \alpha_k, \beta$ s.t. $\mathbf{x} = \alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k + \beta \mathbf{z}$. But now

$$\begin{aligned}
\mathbf{x} &= \alpha_1 \mathbf{u}_1 + \dots + \alpha_k \mathbf{u}_k + \beta \mathbf{z} \\
&= \Big(\sum_{i=1}^{k} \alpha_i \mathbf{u}_i \Big) + \beta(\mathbf{v} - \mathbf{y}) \\
&= \Big(\sum_{i=1}^{k} \alpha_i \mathbf{u}_i \Big) + \beta\Big(\mathbf{v} - \big(\sum_{i=1}^{k} \tfrac{\langle \mathbf{v}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i \big)\Big) \\
&= \Big(\sum_{i=1}^{k} \big(\alpha_i - \beta \tfrac{\langle \mathbf{v}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle}\big) \mathbf{u}_i \Big) + \beta \mathbf{v},
\end{aligned}$$

and we deduce that $\mathbf{x} \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v})$. This proves (c). □

Proposition 6.3.7

Let V be a real or complex vector space, equipped with a scalar product $\langle \cdot, \cdot \rangle$. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ be an orthogonal set of non-zero vectors in V. Let $\mathbf{v} \in V$, and set $\mathbf{y} := \sum_{i=1}^{k} \text{proj}_{\mathbf{u}_i}(\mathbf{v}) = \sum_{i=1}^{k} \frac{\langle \mathbf{v}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i$ and $\mathbf{z} := \mathbf{v} - \mathbf{y}$. Then all the following hold:

(a) $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{z}\}$ is an orthogonal set of vectors;

(b) $\mathbf{z} = \mathbf{0}$ iff $\mathbf{v} \in \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$;

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a. $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{z}\}$ is an orthogonal set of vectors;
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c. $\text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v}) = \text{Span}(\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{z})$.

- Using Proposition 6.3.7, we can now prove the correctness of the Gram-Schmidt orthogonalization process (version 1).

Gram-Schmidt orthogonalization process (version 1)

Let V be a real or complex vector space, equipped with a scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $|| \cdot ||$, and let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be linearly independent vectors in V. For all $\ell \in \{1, \dots, k\}$, set

$$\mathbf{u}_\ell \quad := \quad \mathbf{v}_\ell - \sum_{i=1}^{\ell-1} \mathrm{proj}_{\mathbf{u}_i}(\mathbf{v}_\ell) \quad = \quad \mathbf{v}_\ell - \sum_{i=1}^{\ell-1} \frac{\langle \mathbf{v}_\ell, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i.$$

Then $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal basis of $\mathrm{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$, and $\left\{ \frac{\mathbf{u}_1}{||\mathbf{u}_1||}, \dots, \frac{\mathbf{u}_k}{||\mathbf{u}_k||} \right\}$ is an orthonormal basis of $\mathrm{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$.

Proof (outline).

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$$\mathbf{u}_\ell \quad := \quad \mathbf{v}_\ell - \sum_{i=1}^{\ell-1} \text{proj}_{\mathbf{u}_i}(\mathbf{v}_\ell) \quad = \quad \mathbf{v}_\ell - \sum_{i=1}^{\ell-1} \frac{\langle \mathbf{v}_\ell, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \mathbf{u}_i.$$

Then $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal basis of $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$, and $\left\{ \frac{\mathbf{u}_1}{||\mathbf{u}_1||}, \dots, \frac{\mathbf{u}_k}{||\mathbf{u}_k||} \right\}$ is an orthonormal basis of $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$.

Proof (outline). If $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal basis of $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$, then Proposition 6.3.3(b) guarantees that $\left\{ \frac{\mathbf{u}_1}{||\mathbf{u}_1||}, \dots, \frac{\mathbf{u}_k}{||\mathbf{u}_k||} \right\}$ is an orthonormal basis of $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$. So, we just need to show that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal basis of $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$.

Proof (outline, continued).

Proof (outline, continued). How do we prove that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal basis of $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$? The idea is to prove (by induction) that for all $\ell \in \{1, \dots, k\}$, we have that $\{\mathbf{u}_1, \dots, \mathbf{u}_\ell\}$ is an orthogonal basis of $U_\ell := \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_\ell)$.

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For $\ell = 1$, we have that $\mathbf{u}_1 = \mathbf{v}_1$, and the result is immediate.

Proof (outline, continued). How do we prove that $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal basis of $\text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$? The idea is to prove (by induction) that for all $\ell \in \{1, \ldots, k\}$, we have that $\{\mathbf{u}_1, \ldots, \mathbf{u}_\ell\}$ is an orthogonal basis of $U_\ell := \text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_\ell)$.

For $\ell = 1$, we have that $\mathbf{u}_1 = \mathbf{v}_1$, and the result is immediate.

Now fix $\ell \in \{1, \ldots, k-1\}$, and assume that $\{\mathbf{u}_1, \ldots, \mathbf{u}_\ell\}$ is an orthogonal basis of $U_\ell := \text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_\ell)$. Then letting $\mathbf{v} := \mathbf{v}_{\ell+1}$ and $\mathbf{z} := \mathbf{u}_{\ell+1}$, we apply Proposition 6.3.7. $\square$

Gram-Schmidt orthogonalization process (version 2)

Let V be a real or complex vector space, equipped with a scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $|| \cdot ||$, and let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be linearly independent vectors in V. For all $\ell \in \{1, \dots, k\}$, set

$$\mathbf{u}_\ell = \mathbf{v}_\ell - \sum_{i=1}^{\ell-1} \text{proj}_{\mathbf{z}_i}(\mathbf{v}_\ell) = \mathbf{v}_\ell - \sum_{i=1}^{\ell-1} \langle \mathbf{v}_\ell, \mathbf{z}_i \rangle \, \mathbf{z}_i;$$

$$\mathbf{z}_\ell = \frac{\mathbf{u}_\ell}{||\mathbf{u}_\ell||}.$$

Then $\{\mathbf{z}_1, \dots, \mathbf{z}_k\}$ is an orthonormal basis of $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$.

- The Gram-Schmidt orthogonalization process (version 2) recursively constructs two sequences of vectors, namely, $\mathbf{u}_1, \ldots, \mathbf{u}_k$ and $\mathbf{z}_1, \ldots, \mathbf{z}_k$, as follows:
  - $\mathbf{u}_1 = \mathbf{v}_1$;
  - $\mathbf{z}_1 = \frac{\mathbf{u}_1}{||\mathbf{u}_1||}$;
  - $\mathbf{u}_2 = \mathbf{v}_2 - \text{proj}_{\mathbf{z}_1}(\mathbf{v}_2)$;
  - $\mathbf{z}_2 = \frac{\mathbf{u}_2}{||\mathbf{u}_2||}$;
  - $\mathbf{u}_3 = \mathbf{v}_3 - \Big(\text{proj}_{\mathbf{z}_1}(\mathbf{v}_3) + \text{proj}_{\mathbf{z}_2}(\mathbf{v}_3)\Big)$;
  - $\mathbf{z}_3 = \frac{\mathbf{u}_3}{||\mathbf{u}_3||}$;

    $\vdots$
  - $\mathbf{u}_k = \mathbf{v}_k - \Big(\text{proj}_{\mathbf{z}_1}(\mathbf{v}_k) + \text{proj}_{\mathbf{z}_2}(\mathbf{v}_k) + \cdots + \text{proj}_{\mathbf{z}_{k-1}}(\mathbf{v}_k)\Big)$;
  - $\mathbf{z}_k = \frac{\mathbf{u}_k}{||\mathbf{u}_k||}$.

- The Gram-Schmidt orthogonalization process (version 2) recursively constructs two sequences of vectors, namely, $\mathbf{u}_1, \ldots, \mathbf{u}_k$ and $\mathbf{z}_1, \ldots, \mathbf{z}_k$, as follows:
  - $\mathbf{u}_1 = \mathbf{v}_1$;
  - $\mathbf{z}_1 = \frac{\mathbf{u}_1}{||\mathbf{u}_1||}$;
  - $\mathbf{u}_2 = \mathbf{v}_2 - \text{proj}_{\mathbf{z}_1}(\mathbf{v}_2)$;
  - $\mathbf{z}_2 = \frac{\mathbf{u}_2}{||\mathbf{u}_2||}$;
  - $\mathbf{u}_3 = \mathbf{v}_3 - \Big(\text{proj}_{\mathbf{z}_1}(\mathbf{v}_3) + \text{proj}_{\mathbf{z}_2}(\mathbf{v}_3)\Big)$;
  - $\mathbf{z}_3 = \frac{\mathbf{u}_3}{||\mathbf{u}_3||}$;

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  - $\mathbf{u}_k = \mathbf{v}_k - \Big(\text{proj}_{\mathbf{z}_1}(\mathbf{v}_k) + \text{proj}_{\mathbf{z}_2}(\mathbf{v}_k) + \cdots + \text{proj}_{\mathbf{z}_{k-1}}(\mathbf{v}_k)\Big)$;
  - $\mathbf{z}_k = \frac{\mathbf{u}_k}{||\mathbf{u}_k||}$.
- So, at each step, we obtain a vector $\mathbf{u}_\ell$ that is orthogonal to the previously constructed vectors $\mathbf{z}_1, \ldots, \mathbf{z}_{\ell-1}$, and then we normalize $\mathbf{u}_\ell$ to obtain the unit vector $\mathbf{z}_\ell$ that points in the same direction as $\mathbf{u}_\ell$.

Gram-Schmidt orthogonalization process (version 2)

Let V be a real or complex vector space, equipped with a scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $|| \cdot ||$, and let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be linearly independent vectors in V. For all $\ell \in \{1, \dots, k\}$, set

$$\mathbf{u}_\ell = \mathbf{v}_\ell - \sum_{i=1}^{\ell-1} \text{proj}_{\mathbf{z}_i}(\mathbf{v}_\ell) = \mathbf{v}_\ell - \sum_{i=1}^{\ell-1} \langle \mathbf{v}_\ell, \mathbf{z}_i \rangle \, \mathbf{z}_i;$$

$$\mathbf{z}_\ell = \frac{\mathbf{u}_\ell}{||\mathbf{u}_\ell||}.$$

Then $\{\mathbf{z}_1, \dots, \mathbf{z}_k\}$ is an orthonormal basis of $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$.

- The proof of correctness is similar to that of version 1.
- A numerical example is given in the Lecture Notes.

Corollary 6.3.11

Let V be a finite-dimensional real or complex vector space, equipped with a scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $|| \cdot ||$. Let U be a subspace of V. Then all the following hold:

(a) U has an orthogonal basis;

(b) any orthogonal basis of U can be extended to an orthogonal basis of V;[a]

(c) U has an orthonormal basis;

(d) any orthonormal basis of U can be extended to an orthonormal basis of V.[b]

[a] This means that for any orthogonal basis $\mathcal{B}$ of U, there exists an orthogonal basis $\mathcal{C}$ of V s.t. $\mathcal{B} \subseteq \mathcal{C}$.

[b] This means that for any orthonormal basis $\mathcal{B}$ of U, there exists an orthonormal basis $\mathcal{C}$ of V s.t. $\mathcal{B} \subseteq \mathcal{C}$.

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Proof. We first prove (a) and (c). Since V is finite-dimensional, Theorem 3.2.21 guarantees that the subspace U of V is also finite-dimensional. Consider any basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ of U. Then the Gram-Schmidt orthogonalization process (version 1) applied to the vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ yields a sequence of vectors $\mathbf{u}_1, \dots, \mathbf{u}_k$ s.t. $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal and $\left\{\frac{\mathbf{u}_1}{||\mathbf{u}_1||}, \dots, \frac{\mathbf{u}_k}{||\mathbf{u}_k||}\right\}$ an orthonormal basis of $U = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$.

Proof. We first prove (a) and (c). Since V is finite-dimensional, Theorem 3.2.21 guarantees that the subspace U of V is also finite-dimensional. Consider any basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ of U. Then the Gram-Schmidt orthogonalization process (version 1) applied to the vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ yields a sequence of vectors $\mathbf{u}_1, \dots, \mathbf{u}_k$ s.t. $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is an orthogonal and $\left\{\frac{\mathbf{u}_1}{||\mathbf{u}_1||}, \dots, \frac{\mathbf{u}_k}{||\mathbf{u}_k||}\right\}$ an orthonormal basis of $U = \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$. This proves (a) and (c).

Proof (continued). For (b), consider any orthogonal basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ of U, and using Theorem 3.2.19, extend it to a basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n\}$ of V.

Proof (continued). For (b), consider any orthogonal basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ of U, and using Theorem 3.2.19, extend it to a basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n\}$ of V.

We apply the Gram-Schmidt orthogonalization process (version 1) to the sequence $\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n$, and we obtain a sequence $\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n$ s.t. $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an orthogonal basis of V.

Proof (continued). For (b), consider any orthogonal basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ of U, and using Theorem 3.2.19, extend it to a basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n\}$ of V.

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However, since $\mathbf{v}_1, \dots, \mathbf{v}_k$ were pairwise orthogonal to begin with, we see from the description of the Gram-Schmidt orthogonalization process that $\mathbf{u}_1 = \mathbf{v}_1, \dots, \mathbf{u}_k = \mathbf{v}_k$.

Proof (continued). For (b), consider any orthogonal basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ of U, and using Theorem 3.2.19, extend it to a basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n\}$ of V.

We apply the Gram-Schmidt orthogonalization process (version 1) to the sequence $\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}, \dots, \mathbf{v}_n$, and we obtain a sequence $\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n$ s.t. $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is an orthogonal basis of V.

However, since $\mathbf{v}_1, \dots, \mathbf{v}_k$ were pairwise orthogonal to begin with, we see from the description of the Gram-Schmidt orthogonalization process that $\mathbf{u}_1 = \mathbf{v}_1, \dots, \mathbf{u}_k = \mathbf{v}_k$.

So, the orthogonal basis $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ of V extends the orthogonal basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ of U. This proves (b).

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Proof (continued). For (d), consider any orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ of U. In particular, the basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ of U is orthogonal, and so by (b), it can be extended to an orthogonal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ of V.

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Then by Proposition 6.3.3(c),

$$\left\{ \frac{\mathbf{u}_1}{||\mathbf{u}_1||}, \dots, \frac{\mathbf{u}_k}{||\mathbf{u}_k||}, \frac{\mathbf{u}_{k+1}}{||\mathbf{u}_{k+1}||}, \dots, \frac{\mathbf{u}_n}{||\mathbf{u}_n||} \right\}$$

is an orthonormal basis of V.

Proof (continued). For (d), consider any orthonormal basis $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ of U. In particular, the basis $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ of U is orthogonal, and so by (b), it can be extended to an orthogonal basis $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ of V.

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But since the basis $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ of U is orthonormal, we know that $||\mathbf{u}_1|| = \dots = ||\mathbf{u}_k|| = 1$,

Proof (continued). For (d), consider any orthonormal basis $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ of U. In particular, the basis $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ of U is orthogonal, and so by (b), it can be extended to an orthogonal basis $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ of V.

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But since the basis $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ of U is orthonormal, we know that $||\mathbf{u}_1|| = \dots = ||\mathbf{u}_k|| = 1$, and it follows that $\frac{\mathbf{u}_1}{||\mathbf{u}_1||} = \mathbf{u}_1, \dots, \frac{\mathbf{u}_k}{||\mathbf{u}_k||} = \mathbf{u}_k$.

Proof (continued). For (d), consider any orthonormal basis $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ of U. In particular, the basis $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ of U is orthogonal, and so by (b), it can be extended to an orthogonal basis $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ of V.

Then by Proposition 6.3.3(c),

$$\left\{ \frac{\mathbf{u}_1}{||\mathbf{u}_1||}, \dots, \frac{\mathbf{u}_k}{||\mathbf{u}_k||}, \frac{\mathbf{u}_{k+1}}{||\mathbf{u}_{k+1}||}, \dots, \frac{\mathbf{u}_n}{||\mathbf{u}_n||} \right\}$$

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But since the basis $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ of U is orthonormal, we know that $||\mathbf{u}_1|| = \dots = ||\mathbf{u}_k|| = 1$, and it follows that $\frac{\mathbf{u}_1}{||\mathbf{u}_1||} = \mathbf{u}_1, \dots, \frac{\mathbf{u}_k}{||\mathbf{u}_k||} = \mathbf{u}_k$.

So, our orthonormal basis $\left\{ \frac{\mathbf{u}_1}{||\mathbf{u}_1||}, \dots, \frac{\mathbf{u}_k}{||\mathbf{u}_k||}, \frac{\mathbf{u}_{k+1}}{||\mathbf{u}_{k+1}||}, \dots, \frac{\mathbf{u}_n}{||\mathbf{u}_n||} \right\}$ of V in fact extends the orthonormal basis $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ of U. This proves (d). $\square$

Corollary 6.3.11

Let V be a finite-dimensional real or complex vector space, equipped with a scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $|| \cdot ||$. Let U be a subspace of V. Then all the following hold:

a. U has an orthogonal basis;
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[b] This means that for any orthonormal basis $\mathcal{B}$ of U, there exists an orthonormal basis $\mathcal{C}$ of V s.t. $\mathcal{B} \subseteq \mathcal{C}$.