Linear Algebra 2

Lecture #13

Complex numbers. Scalar (inner) products

Irena Penev

February 26, 2025

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  3. Orthogonality

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- The real and imaginary part of a complex number z are denoted by $Re(z)$ and $Im(z)$, respectively.
  - For example, we have the following:
    - $Re(2+i) = 2$ and $Im(2+i) = 1$;
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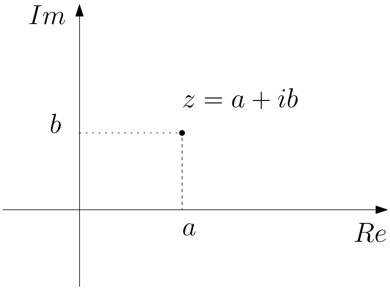
- Note that real numbers are precisely those complex numbers whose imaginary part is zero.
- The set of all complex numbers is denoted by $\mathbb{C}$.

- Complex numbers can be visualized in the "complex plane." This plane has two axes: the *real axis* (denoted by *Re*) and the *imaginary axis* (denoted by *Im*).



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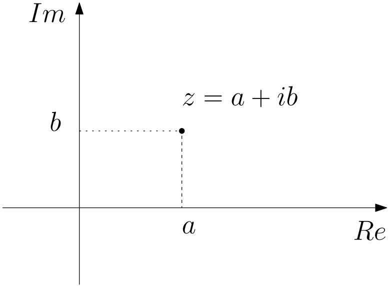

- Note that real numbers are precisely those complex numbers that lie on the real axis.

- We add/subtract complex numbers by adding/subtracting the real and imaginary parts.
  - For example:
    - $(2+3i)+(3-5i) = (2+3)+(3i-5i) = 5-2i$;
    - $(2+3i)-(3-5i) = (2-3)+\big(3i-(-5i)\big) = -1+8i$.

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- To multiply complex numbers, we must keep in mind that $i^2 = -1$.
  - For example:

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\begin{aligned}
(2+3i)(3-5i) &= 2\cdot 3 + 2(-5i) + (3i)3 + (3i)(-5i) \\
&= 6 - 10i + 9i - 15\underbrace{i^2}_{=-1} \\
&= 21 - i.
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- Division: later!

Proposition 0.3.1

All the following hold:

(a) addition and multiplication in $\mathbb{C}$ are commutative, that is, for all $z_1, z_2 \in \mathbb{C}$, we have that $z_1 + z_2 = z_2 + z_1$ and $z_1 z_2 = z_2 z_1$;

(b) addition and multiplication in $\mathbb{C}$ are associative, that is, for all $z_1, z_2, z_3 \in \mathbb{C}$, we have that $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ and $(z_1 z_2) z_3 = z_1 (z_2 z_3)$;

(c) multiplication is distributive over addition in $\mathbb{C}$, that is, for all $z_1, z_2, z_3 \in \mathbb{C}$, we have that $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$.

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  - $z^0 := 1$;
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- So, for all positive integers m, we have the familiar expression

$$z^m = \underbrace{z \dots z}_{m}.$$

Definition

For a complex number $z = a + bi$ (where $a, b \in \mathbb{R}$):

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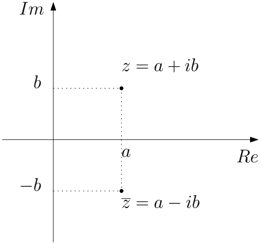

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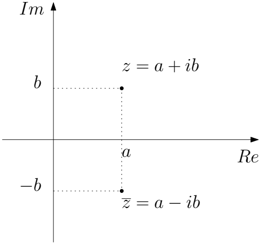

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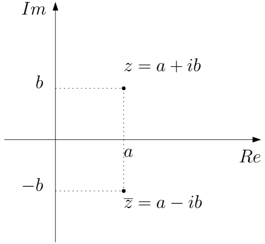

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- Note that $\overline{z} = z$ iff z is in fact a real number, i.e. $Im(z) = 0$.

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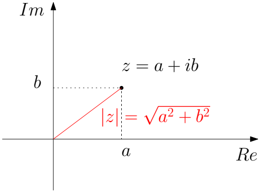

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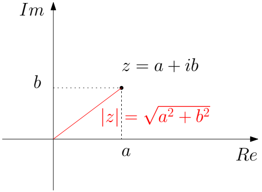

- Note that $|z|$ is a non-negative real number, and moreover, we have that $|z| = 0$ iff $z = 0$.

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- Note that Proposition 0,3,2, in particular, establishes that multiplying a complex number z by its conjugate produces a real number; that real number is zero iff $z = 0$.
    - This is important for division!

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- Let us take a look at an example.

Example 0.3.3

Compute the following quotients:

(a) $\frac{7-6i}{3+2i}$; (b) $\frac{1}{2-i}$; (c) $\frac{2-3i}{5}$; (d) $\frac{4-2i}{2-i}$.

Solution.

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Solution. (a) We multiply both the numerator and the denominator by $\overline{3+2i} = 3-2i$, and we obtain

$$\frac{7-6i}{3+2i} = \frac{(7-6i)(3-2i)}{(3+2i)(3-2i)} = \frac{9-32i}{9+4} = \frac{9}{13} - \frac{32}{13}i.$$

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(b) We multiply both the numerator and the denominator by $\overline{2-i} = 2+i$, and we obtain

$$\frac{1}{2-i} = \frac{2+i}{(2-i)(2+i)} = \frac{2+i}{4+1} = \frac{2}{5} + \frac{1}{5}i.$$

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$$\frac{2-3i}{5} \quad = \quad \frac{2}{5} - \frac{3}{5}i.$$

(d) We could multiply both the numerator and the denominator by $\overline{2-i} = 2+i$. However, in this particular case, it is easier to compute as follows:

$$\frac{4-2i}{2-i} \quad = \quad \frac{2(2-i)}{2-i} \quad \overset{(*)}{=} \quad 2,$$

where (*) was obtained by canceling out the common factor $2-i$ in the numerator and the denominator. □

Proposition 0.3.4

For all $z_1, z_2 \in \mathbb{C}$, the following hold:

(a) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$;

(b) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$;

(c) $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$;

(d) if $z_2 \neq 0$, then $\overline{z_1 / z_2} = \overline{z_1} / \overline{z_2}$.

Moreover, for all $z \in \mathbb{C}$ and non-negative integers m, we have that

(e) $\overline{z^m} = (\overline{z})^m$.

Proposition 0.3.5

For all $z_1, z_2 \in \mathbb{C}$, the following hold:

(a) $|z_1 z_2| = |z_1||z_2|$;

(b) if $z_2 \neq 0$, then $|z_1/z_2| = |z_1|/|z_2|$.

Moreover, for all $z \in \mathbb{C}$, the following hold:

(c) $|-z| = |z|$;

(d) for all non-negative integers m, we have $|z^m| = |z|^m$.

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- In the particular case of $p(x) = x^2 - 2x + 2$, the roots could have been found via the familiar quadratic equation.
- There exist formulas for finding the complex roots of all third and fourth degree polynomials with complex coefficients, but no such formula exists for polynomials of degree five or more (although in some special cases, we may be able to use various tricks to find the roots of these higher-degree polynomials).

- Nevertheless, we do have the following **existence** result.
  - A *constant* polynomial is a polynomial of the form $p(x) = c$, where c is a fixed constant/number.

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  - So, the Fundamental Theorem of Algebra, in particular, implies that every non-constant polynomial with real coefficients has a complex root (which may or may not be a real number).
  - For instance, the polynomial $p(x) = x^2 + 1$ is a non-constant polynomial with real (in fact, rational) coefficients, but it has no real roots. It does, of course, have two complex roots, namely i and $-i$.

The Fundamental Theorem of Algebra

Any non-constant polynomial with complex coefficients has a complex root.

- We omit the proof of the Fundamental Theorem of Algebra.
- There are no known elementary proofs of this theorem: all the known proofs of the Fundamental Theorem of Algebra rely on advanced mathematics, such as complex analysis or topology.

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$$p(x) = a(x - \alpha_1)^{n_1} \ldots (x - \alpha_\ell)^{n_\ell},$$

  and moreover, this factorization into linear factors is unique up a permutation of the α_i's and the corresponding n_i's.

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  and moreover, this factorization into linear factors is unique up a permutation of the α_i's and the corresponding n_i's.
    - Here, a is the leading coefficient of $p(x)$, i.e. the coefficient in front of x^n. Complex numbers $\alpha_1, \ldots, \alpha_\ell$ are the roots of $p(x)$ with *multiplicities* $n_1, \ldots, n_\ell$, respectively.

- The Fundamental Theorem of Algebra implies that any polynomial $p(x)$ with complex coefficients and of degree $n \geq 1$ can be factored into n linear factors.
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  - If we think of each α_i as being a root "n_i times" (due to its multiplicity), then we see that the n-th degree polynomial $p(x)$ has exactly n complex roots.
  - This is often summarized as follows: "every n-th degree polynomial (with $n \geq 1$) with complex coefficients has exactly n complex roots, when multiplicities are taken into account."

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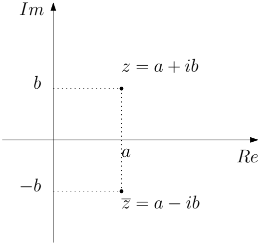
- In fact, not only is no such formula known, but using Galois theory, one can show that no such formula can exist for polynomials of degree at least five.
  - Once again, we may be able to use various tricks to compute the roots of some special high-degree polynomials. However, none of these tricks will work in the general case.

- Recall that, geometrically, the complex conjugate of a complex number *z* is obtained by reflecting *z* about the *Re* axis in the complex plane.



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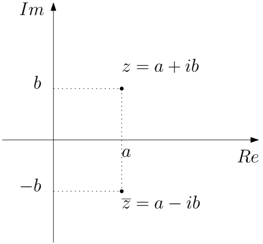

Theorem 0.3.6

Let $p(x)$ be any polynomial with **real** coefficients, and let $z \in \mathbb{C}$. Then z is a root of $p(x)$ iff $\overline{z}$ is a root of $p(x)$.

- First a remark, then a proof.

Theorem 0.3.6

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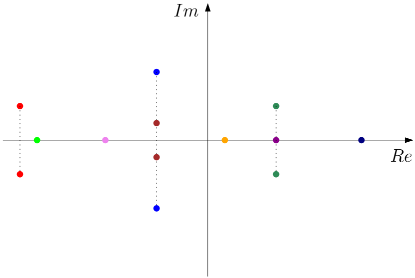
- **Remark:** Note that Theorem 0.3.6 implies that the complex roots of a non-constant polynomial are symmetric about the *Re* axis in the complex plane.
  - Some (or perhaps all) of those roots may lie on the *Re* axis, i.e. they may be real numbers.



Theorem 0.3.6

Let $p(x)$ be any polynomial with **real** coefficients, and let $z \in \mathbb{C}$. Then z is a root of $p(x)$ iff $\overline{z}$ is a root of $p(x)$.

Proof. Set $p(x) = a_n x^n + \cdots + a_1 x + a_0$, where $a_0, a_1, \ldots, a_n \in \mathbb{R}$. Then we have the following sequence of equivalences:

$$
\begin{aligned}
p(z) = 0 \quad &\Longleftrightarrow \quad \overline{p(z)} = \overline{0} \\
&\Longleftrightarrow \quad \overline{a_n z^n + \cdots + a_1 z + a_0} = \overline{0} \\
&\overset{(*)}{\Longleftrightarrow} \quad \overline{a_n}(\overline{z})^n + \cdots + \overline{a_1}(\overline{z}) + \overline{a_0} = \overline{0} \\
&\overset{(**)}{\Longleftrightarrow} \quad a_n(\overline{z})^n + \cdots + a_1\overline{z} + a_0 = 0 \\
&\Longleftrightarrow \quad p(\overline{z}) = 0,
\end{aligned}
$$

where (*) follows from Proposition 0.3.4, and (**) follows from the fact that $a_0, a_1, \ldots, a_n$ and 0 are real numbers. $\square$

2. The scalar product

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    - A scalar product is a way of multiplying two vectors and obtaining a scalar.
    - A norm is a way of measuring the distance of a vector from the origin, or alternatively, measuring the length of a vector.
- As a trade-off for imposing this additional structure, we restrict ourselves to vector spaces over only two fields: $\mathbb{R}$ and $\mathbb{C}$.
    - The theory that we develop over the next few weeks (corresponding to chapter 6 of the Lecture Notes) would not work for vector spaces over general fields $\mathbb{F}$.

Definition

A *scalar product* (also called *inner product*) in a **real** vector space V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ that satisfies the following four axioms:

r.1. for all $\mathbf{x} \in V$, $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, and equality holds iff $\mathbf{x} = \mathbf{0}$;

r.2. for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$;

r.3. for all $\mathbf{x}, \mathbf{y} \in V$ and $\alpha \in \mathbb{R}$, $\langle \alpha\mathbf{x}, \mathbf{y} \rangle = \alpha\langle \mathbf{x}, \mathbf{y} \rangle$;

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- The name "scalar product" comes from the fact that we multiply two vectors and obtain a scalar as a result.

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- Axioms r.2 and r.3 guarantee that the scalar product in a real vector space V is linear in the first variable (when we keep the second variable fixed).
- But in fact, axioms r.2, r.3, and r.4 guarantee that it is linear in the second variable as well (when we keep the first variable fixed).
  - More precisely, we have the following (next slide):

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Proof of r.2'.

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Proof of r.2'. For all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, we have the following:

$$\langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle \overset{\text{r.4}}{=} \langle \mathbf{y} + \mathbf{z}, \mathbf{x} \rangle \overset{\text{r.2}}{=} \langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{z}, \mathbf{x} \rangle \overset{\text{r.4}}{=} \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle.$$

$\square$

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Definition

The *standard scalar product* of vectors $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$ and $\mathbf{y} = \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix}^T$ in $\mathbb{R}^n$ is given by

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- For example, for vectors $\begin{bmatrix} 1 & -2 & 5 \end{bmatrix}^T$ and $\begin{bmatrix} -3 & 2 & 1 \end{bmatrix}^T$ in $\mathbb{R}^3$, we compute:

$$\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \quad = \quad 1 \cdot (-3) + (-2) \cdot 2 + 5 \cdot 1 \quad = \quad -2.$$

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- We still need to check that $\cdot$ really is a scalar product, i.e. that it satisfies axioms r.1-r.4.
    - Later!

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$$\mathbf{x}^T\mathbf{y} \quad = \quad \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \quad = \quad \begin{bmatrix} \sum_{i=1}^{n} x_i y_i \end{bmatrix} \quad = \quad \begin{bmatrix} \mathbf{x} \cdot \mathbf{y} \end{bmatrix}.$$

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- So, if we identify 1×1 matrices with scalars, then we simply get that

$$\mathbf{x} \cdot \mathbf{y} \quad = \quad \mathbf{x}^T\mathbf{y} \qquad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

Proposition 6.1.1

The standard scalar product in $\mathbb{R}^n$ is a scalar product.

Proof.

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Proof. We need to check that the standard scalar product $\cdot$ in $\mathbb{R}^n$ satisfies the four axioms from the definition of a scalar product in a real vector space.

r.1. For a vector $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$ in $\mathbb{R}^n$, we have that

$$\mathbf{x} \cdot \mathbf{x} \;=\; \sum_{i=1}^{n} x_i^2 \;\overset{(*)}{\geq}\; 0,$$

and (*) is an equality iff $x_1 = \dots = x_n = 0$, i.e. iff $\mathbf{x} = \mathbf{0}$.

Proposition 6.1.1

The standard scalar product in $\mathbb{R}^n$ is a scalar product.

Proof (continued). r.2. For vectors $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$, $\mathbf{y} = \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix}^T$, and $\mathbf{z} = \begin{bmatrix} z_1 & \dots & z_n \end{bmatrix}^T$ in $\mathbb{R}^n$, we have that

$$\begin{aligned} (\mathbf{x}+\mathbf{y})\cdot\mathbf{z} &= \sum_{i=1}^{n}(x_i+y_i)z_i \\ &= \Big(\sum_{i=1}^{n} x_i z_i\Big) + \Big(\sum_{i=1}^{n} y_i z_i\Big) \\ &= \mathbf{x}\cdot\mathbf{z} + \mathbf{y}\cdot\mathbf{z}. \end{aligned}$$

Proposition 6.1.1

The standard scalar product in $\mathbb{R}^n$ is a scalar product.

Proof (continued). r.3. For vectors $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$ and $\mathbf{y} = \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix}^T$ in $\mathbb{R}^n$ and a scalar $\alpha \in \mathbb{R}$, we have that

$$(\alpha \mathbf{x}) \cdot \mathbf{y} \;=\; \sum_{i=1}^{n} (\alpha x_i) y_i \;=\; \alpha \sum_{i=1}^{n} x_i y_i \;=\; \alpha (\mathbf{x} \cdot \mathbf{y}).$$

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Proof (continued). r.3. For vectors $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$ and $\mathbf{y} = \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix}^T$ in $\mathbb{R}^n$ and a scalar $\alpha \in \mathbb{R}$, we have that

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Proof (continued). r.3. For vectors $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$ and $\mathbf{y} = \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix}^T$ in $\mathbb{R}^n$ and a scalar $\alpha \in \mathbb{R}$, we have that

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$$\mathbf{x} \cdot \mathbf{y} \;=\; \sum_{i=1}^{n} x_i y_i \;=\; \sum_{i=1}^{n} y_i x_i \;=\; \mathbf{y} \cdot \mathbf{x}.$$

This proves that the standard scalar product in $\mathbb{R}^n$ really is a scalar product. $\square$

Definition

The *standard scalar product* of vectors $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$ and $\mathbf{y} = \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix}^T$ in $\mathbb{R}^n$ is given by

$$\mathbf{x} \cdot \mathbf{y} \quad := \quad \sum_{i=1}^{n} x_i y_i .$$

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The standard scalar product in $\mathbb{R}^n$ is a scalar product.

- A similar type of scalar product can be defined for matrices.

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- A similar type of scalar product can be defined for matrices.
- Indeed, for matrices $A = \begin{bmatrix} a_{i,j} \end{bmatrix}_{n \times m}$ and $B = \begin{bmatrix} b_{i,j} \end{bmatrix}_{n \times m}$ in $\mathbb{R}^{n \times m}$, we can define

$$\langle A, B \rangle \quad = \quad \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} b_{ij} .$$

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The *standard scalar product* of vectors $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$ and $\mathbf{y} = \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix}^T$ in $\mathbb{R}^n$ is given by

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- Indeed, for matrices $A = \begin{bmatrix} a_{i,j} \end{bmatrix}_{n\times m}$ and $B = \begin{bmatrix} b_{i,j} \end{bmatrix}_{n\times m}$ in $\mathbb{R}^{n\times m}$, we can define

$$\langle A, B \rangle \quad = \quad \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} b_{ij} .$$

- It is easy to verify that this really is a scalar product in $\mathbb{R}^{n\times m}$ (the proof is similar to that of Proposition 6.1.1).

Definition

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- **Remark:** The standard scalar product is only one of many possible scalar products in $\mathbb{R}^n$.

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Proposition 6.1.1

The standard scalar product in $\mathbb{R}^n$ is a scalar product.

- **Remark:** The standard scalar product is only one of many possible scalar products in $\mathbb{R}^n$.
  - A full characterization of all possible scalar products in $\mathbb{R}^n$ will be given in a later lecture (in a couple of months).

- If you know calculus, here is an example with integrals:

- If you know calculus, here is an example with integrals:

Proposition 6.1.2

Let $a, b \in \mathbb{R}$ be such that $a < b$, and let $\mathcal{C}_{[a,b]}$ be the (real) vector space of all continuous functions from the closed interval $[a, b]$ to $\mathbb{R}$. Then the function $\langle \cdot, \cdot \rangle : \mathcal{C}_{[a,b]} \times \mathcal{C}_{[a,b]} \to \mathbb{R}$ defined by

$$\langle f, g \rangle \quad := \quad \int_a^b f(x)g(x)dx$$

for all $f, g \in \mathcal{C}_{[a,b]}$ is a scalar product.

- Proof: Lecture Notes (optional).

Definition

A *scalar product* (also called *inner product*) in a **complex** vector space V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ that satisfies the following four axioms:

c.1. for all $\mathbf{x} \in V$, $\langle \mathbf{x}, \mathbf{x} \rangle$ is a real number, $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, and equality holds iff $\mathbf{x} = \mathbf{0}$;

c.2. for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$;

c.3. for all $\mathbf{x}, \mathbf{y} \in V$ and $\alpha \in \mathbb{C}$, $\langle \alpha \mathbf{x}, \mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle$;

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- Axioms c.2 and c.3 guarantee that the scalar product in a complex vector space V is linear in the first variable (when we keep the second variable fixed).

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- Axioms c.2 and c.3 guarantee that the scalar product in a complex vector space V is linear in the first variable (when we keep the second variable fixed).
- Unlike in the real case, it is **not** linear in the second variable (when we keep the first variable fixed).
    - We do, however, have the following (next slide):

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c.2. for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$;

c.3. for all $\mathbf{x}, \mathbf{y} \in V$ and $\alpha \in \mathbb{C}$, $\langle \alpha\mathbf{x}, \mathbf{y} \rangle = \alpha\langle \mathbf{x}, \mathbf{y} \rangle$;

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c.2'. for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, $\langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle$;

c.3'. for all $\mathbf{x}, \mathbf{y} \in V$ and $\alpha \in \mathbb{C}$, $\langle \mathbf{x}, \alpha\mathbf{y} \rangle = \overline{\alpha}\langle \mathbf{x}, \mathbf{y} \rangle$.

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Proof.

c.2'. for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, $\langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle$;

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Proof. c.2'. For all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, we have the following:

$$\begin{aligned} \langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle &\overset{c.4}{=} \overline{\langle \mathbf{y} + \mathbf{z}, \mathbf{x} \rangle} \\ &\overset{c.2}{=} \overline{\langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{z}, \mathbf{x} \rangle} \\ &= \overline{\langle \mathbf{y}, \mathbf{x} \rangle} + \overline{\langle \mathbf{z}, \mathbf{x} \rangle} \\ &\overset{c.4}{=} \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle. \end{aligned}$$

c.2'. for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, $\langle \mathbf{x}, \mathbf{y}+\mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle$;

c.3'. for all $\mathbf{x}, \mathbf{y} \in V$ and $\alpha \in \mathbb{C}$, $\langle \mathbf{x}, \alpha\mathbf{y} \rangle = \overline{\alpha}\langle \mathbf{x}, \mathbf{y} \rangle$.

Proof. c.2'. For all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, we have the following:

$$\begin{aligned} \langle \mathbf{x}, \mathbf{y}+\mathbf{z} \rangle &\overset{c.4}{=} \overline{\langle \mathbf{y}+\mathbf{z}, \mathbf{x} \rangle} \\ &\overset{c.2}{=} \overline{\langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{z}, \mathbf{x} \rangle} \\ &= \overline{\langle \mathbf{y}, \mathbf{x} \rangle} + \overline{\langle \mathbf{z}, \mathbf{x} \rangle} \\ &\overset{c.4}{=} \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle. \end{aligned}$$

c.3'. For all $\mathbf{x}, \mathbf{y} \in V$ and $\alpha \in \mathbb{C}$, we have the following:

$$\langle \mathbf{x}, \alpha\mathbf{y} \rangle \overset{c.4}{=} \overline{\langle \alpha\mathbf{y}, \mathbf{x} \rangle} \overset{c.3}{=} \overline{\alpha\langle \mathbf{y}, \mathbf{x} \rangle} = \overline{\alpha}\,\overline{\langle \mathbf{y}, \mathbf{x} \rangle} \overset{c.4}{=} \overline{\alpha}\langle \mathbf{x}, \mathbf{y} \rangle.$$

□

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c.2. for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$;

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Definition

The *standard scalar product* of vectors $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^T$ and $\mathbf{y} = \begin{bmatrix} y_1 & \dots & y_n \end{bmatrix}^T$ in $\mathbb{C}^n$ is given by

$$\mathbf{x} \cdot \mathbf{y} \quad := \quad \sum_{i=1}^{n} x_i \overline{y_i}.$$

- For example, for vectors $\begin{bmatrix} 1-2i & -2+i \end{bmatrix}^T$ and $\begin{bmatrix} 2+i & 1+3i \end{bmatrix}^T$ in $\mathbb{C}^2$, we compute:

$$\begin{aligned} \begin{bmatrix} 1-2i \\ -2+i \end{bmatrix} \cdot \begin{bmatrix} 2+i \\ 1+3i \end{bmatrix} &= (1-2i)\overline{(2+i)} + (-2+i)\overline{(1+3i)} \\ &= (1-2i)(2-i) + (-2+i)(1-3i) \\ &= 1+2i. \end{aligned}$$

Definition

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$$\mathbf{x} \cdot \mathbf{y} \quad := \quad \sum_{i=1}^{n} x_i \overline{y_i}.$$

Proposition 6.1.3

The standard scalar product in $\mathbb{C}^n$ is a scalar product.

- Proof: Lecture Notes (similar to the real case).

3. Orthogonality

3. Orthogonality

Definition

Given a real or complex vector space V, equipped with a scalar product $\langle \cdot, \cdot \rangle$, we say that vectors $\mathbf{x}$ and $\mathbf{y}$ in V are *orthogonal*, and we write $\mathbf{x} \perp \mathbf{y}$, if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

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- When our scalar product is the **standard** scalar product in $\mathbb{R}^n$, this corresponds to the usual geometric interpretation.
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- However, for general scalar products, this is how we **define** orthogonality.

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Given a real or complex vector space V, equipped with a scalar product $\langle \cdot, \cdot \rangle$, we say that vectors $\mathbf{x}$ and $\mathbf{y}$ in V are *orthogonal*, and we write $\mathbf{x} \perp \mathbf{y}$, if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

- When our scalar product is the **standard** scalar product in $\mathbb{R}^n$, this corresponds to the usual geometric interpretation.
  - Details: Later!
- However, for general scalar products, this is how we **define** orthogonality.
  - For example, for the scalar product defined on $\mathcal{C}_{[-\pi,\pi]}$ in Proposition 6.1.2 (the one with integrals), we have that

$$\sin x \perp \cos x,$$

since $\left\langle \sin x, \cos x \right\rangle = \int_{-\pi}^{\pi} \sin x \cos x dx = 0.$

Proposition 6.1.4

Let V be a real or complex vector space, equipped with a scalar product $\langle \cdot, \cdot \rangle$. Then all the following hold:

a. for all vectors $\mathbf{x}, \mathbf{y} \in V$, we have that $\mathbf{x} \perp \mathbf{y}$ iff $\mathbf{y} \perp \mathbf{x}$;
b. for all vectors $\mathbf{x}, \mathbf{y} \in V$ and scalars α, β,[a] if $\mathbf{x} \perp \mathbf{y}$ then $(\alpha\mathbf{x}) \perp (\beta\mathbf{y})$;
c. for all vectors $\mathbf{x} \in V$, we have that $\mathbf{x} \perp \mathbf{0}$ and $\mathbf{0} \perp \mathbf{x}$.

[a] Here, α and β are real or complex numbers, depending on whether V is a real or complex vector space.

Proof.

Proposition 6.1.4

Let V be a real or complex vector space, equipped with a scalar product $\langle \cdot, \cdot \rangle$. Then all the following hold:

(a) for all vectors $\mathbf{x}, \mathbf{y} \in V$, we have that $\mathbf{x} \perp \mathbf{y}$ iff $\mathbf{y} \perp \mathbf{x}$;

(b) for all vectors $\mathbf{x}, \mathbf{y} \in V$ and scalars α, β,[a] if $\mathbf{x} \perp \mathbf{y}$ then $(\alpha\mathbf{x}) \perp (\beta\mathbf{y})$;

(c) for all vectors $\mathbf{x} \in V$, we have that $\mathbf{x} \perp \mathbf{0}$ and $\mathbf{0} \perp \mathbf{x}$.

[a] Here, α and β are real or complex numbers, depending on whether V is a real or complex vector space.

Proof. We prove the proposition for the case when V is a complex vector space. The real case is similar but slightly easier (because we do not have to deal with complex conjugates).

Proposition 6.1.4

(a) for all vectors $\mathbf{x}, \mathbf{y} \in V$, we have that $\mathbf{x} \perp \mathbf{y}$ iff $\mathbf{y} \perp \mathbf{x}$

Proof (continued). (a) For vectors $\mathbf{x}, \mathbf{y} \in V$, we have the following sequence of equivalences:

$$\begin{aligned}
\mathbf{x} \perp \mathbf{y} \quad &\iff \quad \langle \mathbf{x}, \mathbf{y} \rangle = 0 && \text{by definition} \\
&\iff \quad \overline{\langle \mathbf{y}, \mathbf{x} \rangle} = 0 && \text{by c.4} \\
&\iff \quad \langle \mathbf{y}, \mathbf{x} \rangle = 0 && \\
&\iff \quad \mathbf{y} \perp \mathbf{x} && \text{by definition.}
\end{aligned}$$

Proposition 6.1.4

- (b) for all vectors $\mathbf{x}, \mathbf{y} \in V$ and scalars α, β, if $\mathbf{x} \perp \mathbf{y}$ then $(\alpha\mathbf{x}) \perp (\beta\mathbf{y})$

Proof (continued). (b) Fix vectors $\mathbf{x}, \mathbf{y} \in V$ and scalars $\alpha, \beta \in \mathbb{C}$, and assume that $\mathbf{x} \perp \mathbf{y}$. Then we compute:

$$\begin{aligned}
\langle \alpha\mathbf{x}, \beta\mathbf{y} \rangle &= \alpha\langle \mathbf{x}, \beta\mathbf{y} \rangle && \text{by c.3} \\
&= \alpha\overline{\beta}\langle \mathbf{x}, \mathbf{y} \rangle && \text{by c.3'} \\
&= \alpha\overline{\beta}0 && \text{beause } \mathbf{x} \perp \mathbf{y} \\
&= 0.
\end{aligned}$$

So, $(\alpha\mathbf{x}) \perp (\beta\mathbf{y})$.

Proposition 6.1.4

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(a) for all vectors $\mathbf{x}, \mathbf{y} \in V$, we have that $\mathbf{x} \perp \mathbf{y}$ iff $\mathbf{y} \perp \mathbf{x}$;

(b) for all vectors $\mathbf{x}, \mathbf{y} \in V$ and scalars α, β,[a] if $\mathbf{x} \perp \mathbf{y}$ then $(\alpha \mathbf{x}) \perp (\beta \mathbf{y})$;

(c) for all vectors $\mathbf{x} \in V$, we have that $\mathbf{x} \perp \mathbf{0}$ and $\mathbf{0} \perp \mathbf{x}$.

---

[a] Here, α and β are real or complex numbers, depending on whether V is a real or complex vector space.

Proof (continued). (c) Fix any vector $\mathbf{x} \in V$. We then have that

$$\langle \mathbf{0}, \mathbf{x} \rangle \;=\; \langle 0\mathbf{0}, \mathbf{x} \rangle \;\stackrel{\text{c.3}}{=}\; 0\langle \mathbf{0}, \mathbf{x} \rangle \;=\; 0,$$

and so $\mathbf{0} \perp \mathbf{x}$. The fact that $\mathbf{x} \perp \mathbf{0}$ now follows from (a). $\square$

Definition

Given a real or complex vector space V, equipped with a scalar product $\langle \cdot, \cdot \rangle$, we say that vectors $\mathbf{x}$ and $\mathbf{y}$ in V are *orthogonal*, and we write $\mathbf{x} \perp \mathbf{y}$, if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

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- Suppose that V is a real or complex vector space, equipped with a scalar product $\langle \cdot, \cdot \rangle$.
  - For a vector $\mathbf{v} \in V$ and a set of vectors $A \subseteq V$, we say that $\mathbf{v}$ is *orthogonal* to A, and we write $\mathbf{v} \perp A$, provided that $\mathbf{v}$ is orthogonal to all vectors in A.
    - By definition, this means that:

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- Suppose that V is a real or complex vector space, equipped with a scalar product $\langle \cdot, \cdot \rangle$.
    - For a vector $\mathbf{v} \in V$ and a set of vectors $A \subseteq V$, we say that $\mathbf{v}$ is *orthogonal* to A, and we write $\mathbf{v} \perp A$, provided that $\mathbf{v}$ is orthogonal to all vectors in A.
        - By definition, this means that:

$$\langle \mathbf{v}, \mathbf{a} \rangle = 0 \;\; \forall \mathbf{a} \in A.$$

    - For sets of vectors $A, B \subseteq V$, we say that A is *orthogonal* to B, and we write $A \perp B$, if every vector in A is orthogonal to every vector in B.

Proposition 6.1.5

Let V be a real or complex vector space, equipped with a scalar product $\langle \cdot, \cdot \rangle$. Let $\mathbf{a}_1, \dots, \mathbf{a}_p, \mathbf{b}_1, \dots, \mathbf{b}_q \in V$, and assume that $\{\mathbf{a}_1, \dots, \mathbf{a}_p\} \perp \{\mathbf{b}_1, \dots, \mathbf{b}_q\}$. Then $\text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_p) \perp \text{Span}(\mathbf{b}_1, \dots, \mathbf{b}_q)$.

Proof.

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Proof. Fix $\mathbf{a} \in \text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_p)$ and $\mathbf{b} \in \text{Span}(\mathbf{b}_1, \dots, \mathbf{b}_q)$. Then there exist scalars $\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q$ s.t.

$$\mathbf{a} = \alpha_1 \mathbf{a}_1 + \dots + \alpha_p \mathbf{a}_p \qquad \text{and} \qquad \mathbf{b} = \beta_1 \mathbf{b}_1 + \dots + \beta_q \mathbf{b}_q.$$

We now compute (next slide):

Proof (continued).

$$\begin{aligned}
\langle \mathbf{a}, \mathbf{b} \rangle &= \left\langle \sum_{i=1}^{p} \alpha_i \mathbf{a}_i, \sum_{j=1}^{q} \beta_j \mathbf{b}_j \right\rangle \\
&= \sum_{i=1}^{p} \left\langle \alpha_i \mathbf{a}_i, \sum_{j=1}^{q} \beta_j \mathbf{b}_j \right\rangle \qquad \text{by r.2 or c.2} \\
&= \sum_{i=1}^{p} \sum_{j=1}^{q} \underbrace{\langle \alpha_i \mathbf{a}_i, \beta_j \mathbf{b}_j \rangle}_{\overset{(*)}{=} 0} \qquad \text{by r.2' or c.2'} \\
&= 0,
\end{aligned}$$

where (*) follows from Proposition 6.1.4(b) and from the fact that $\{\mathbf{a}_1, \dots, \mathbf{a}_p\} \perp \{\mathbf{b}_1, \dots, \mathbf{b}_q\}$. This proves that $\mathbf{a} \perp \mathbf{b}$, and the result follows. □

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