Linear Algebra 2:
HW#9

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due Sunday, June 1, 2025, at noon (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and it should be either typed or written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Please do **not** send your HW by e-mail. Please write your **name** on top of the first page of your HW.

Unless explicitly stated otherwise, make sure you **show your work** (though you may omit the details of row reduction, as per the Remark below). If you do not show your work, you will receive **no credit** for the exercise/problem in question, even if your final answer is correct.

Remark: Feel free to use a calculator such as the one from

`https://www.dcode.fr/matrix-row-echelon`

for any row reduction that you need to perform. However, keep in mind that on the exam, you might be asked to solve problems like this **without** a calculator, and so you should in principle be able to perform row reduction of this sort by hand.[a] Moreover, make sure you correctly tell the calculator what kinds of numbers you are working with,[b] because otherwise, the calculator may give you a wrong answer.

[a]If you do not yet feel confident in your ability to row reduce, then row reduce by hand first, and then check your answer with a calculator.

[b]For real numbers, you should use their "Echelon Form Matrix Reduction Calculator." For $\mathbb{Z}_2$, use "RREF with Modulo Calculator" with Base/Modulus = 2. For $\mathbb{Z}_3$, use "RREF with Modulo Calculator" with Base/Modulus = 3.

Exercise 1 (10 points). *Consider the quadratic form q on $\mathbb{R}^3$ given by the formula*

$$q(\mathbf{x}) \quad = \quad x_1^2 - 2x_1x_2 + 3x_1x_3 - 4x_2^2 + 5x_2x_3 - x_3^2$$

for all $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ in $\mathbb{R}^3$. Compute the (symmetric) matrix of the quadratic form q with respect to the standard basis $\mathcal{E}_3$ of $\mathbb{R}^3$.

> ***Hint:*** *See Example 9.3.2 of the Lecture Notes, plus the Remark right above it.*

Exercise 2 (15 points). *Consider the bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^3$ given by*

$$\langle \mathbf{x}, \mathbf{y} \rangle \quad = \quad 2x_1y_1 + 2x_1y_2 - x_1y_3 + 2x_2y_1 + 3x_2y_2 + 2x_2y_3 - x_3y_1 + 2x_3y_2 + 4x_3y_3$$

for all $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ and $\mathbf{y} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^T$ in $\mathbb{R}^3$.

(a) Compute the matrix of the bilinear form $\langle \cdot, \cdot \rangle$ with respect to the standard basis $\mathcal{E}_3$ of $\mathbb{R}^3$.

(b) Is the bilinear form $\langle \cdot, \cdot \rangle$ symmetric?

(c) Is the bilinear form $\langle \cdot, \cdot \rangle$ a scalar product in $\mathbb{R}^3$?

Problem 1 (25 points). *Consider the quadratic form q on $\mathbb{R}^4$ given by the formula*

$$q(\mathbf{x}) \quad = \quad -8x_1^2 + 14x_1x_2 + 8x_1x_3 + 2x_1x_4 - 3x_2^2 - 2x_2x_3 + 2x_2x_4 + 2x_3x_4 + x_4^2$$

for all $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T$ in $\mathbb{R}^4$. Compute the signature (n_+, n_-, n_0) of q, a polar basis $\mathcal{B}$ of $\mathbb{R}^4$ associated with q, and the matrix D of q with respect to $\mathcal{B}$.

> ***Remark:*** *The relevant section for this problem is section 9.4 of the Lecture Notes. A "polar basis" is defined at the beginning of subsection 9.4.2 of the Lecture Notes.*

Problem 2 (25 points). *Let $A \in R^{n \times n}$ be a positive definite matrix. Prove that there exists a positive definite matrix $B \in R^{n \times n}$ such that $A = B^2$.*

Problem 3 (25 points). *Let A and B be symmetric matrices in $R^{n \times n}$, and assume that every eigenvalue of A is strictly greater than every eigenvalue of B. Prove that the matrix $A - B$ is positive definite.*

> ***Hint:*** *Start with an orthonormal eigenbasis of $\mathbb{R}^n$ associated with A, and an orthonormal eigenbasis of $\mathbb{R}^n$ associated with B. (You must explain why such eigenbases exist, and note that the two eigenbases need not be the same!) Can you find a lower bound for the expression $\mathbf{x}^T A\mathbf{x}$ in terms of $\mathbf{x}$ and the eigenvalues of A (where $\mathbf{x}$ is an arbitrary vector in $\mathbb{R}^n$)? And an upper bound for $\mathbf{x}^T B\mathbf{x}$ in terms of $\mathbf{x}$ and the eigenvalues of B? Now what?*