Linear Algebra 2:
HW#8

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due Friday, May 16, 2025, at noon (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and it should be either typed or written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Please do **not** send your HW by e-mail. Please write your **name** on top of the first page of your HW.

Unless explicitly stated otherwise, make sure you **show your work** (though you may omit the details of row reduction, as per the Remark below). If you do not show your work, you will receive **no credit** for the exercise/problem in question, even if your final answer is correct.

Remark: Feel free to use a calculator such as the one from

`https://www.dcode.fr/matrix-row-echelon`

for any row reduction that you need to perform. However, keep in mind that on the exam, you might be asked to solve problems like this **without** a calculator, and so you should in principle be able to perform row reduction of this sort by hand.[a] Moreover, make sure you correctly tell the calculator what kinds of numbers you are working with,[b] because otherwise, the calculator may give you a wrong answer.

[a] If you do not yet feel confident in your ability to row reduce, then row reduce by hand first, and then check your answer with a calculator.

[b] For real numbers, you should use their "Echelon Form Matrix Reduction Calculator." For $\mathbb{Z}_2$, use "RREF with Modulo Calculator" with Base/Modulus = 2. For $\mathbb{Z}_3$, use "RREF with Modulo Calculator" with Base/Modulus = 3.

Exercise 1 (10 points)**.** *Suppose you are given a matrix $A \in \mathbb{C}^{14\times 14}$ for which your friend the calculator computes the characteristic polynomial*

$$p_A(\lambda) \quad = \quad (\lambda-2)^4(\lambda+3)^4(\lambda-4)^6,$$

and further helpfully computes the following:

- $\mathit{rank}\big((A-2I_{14})^0\big) = 14$;
- $\mathit{rank}\big((A-2I_{14})^1\big) = 12$;
- $\mathit{rank}\big((A-2I_{14})^2\big) = 11$;
- $\mathit{rank}\big((A-2I_{14})^3\big) = 10$;
- $\mathit{rank}\big((A-2I_{14})^4\big) = 10$;

- $\mathit{rank}\big((A+3I_{14})^0\big) = 14$;
- $\mathit{rank}\big((A+3I_{14})^1\big) = 12$;
- $\mathit{rank}\big((A+3I_{14})^2\big) = 10$;
- $\mathit{rank}\big((A+3I_{14})^3\big) = 10$;

- $\mathit{rank}\big((A-4I_{14})^0\big) = 14$;
- $\mathit{rank}\big((A-4I_{14})^1\big) = 11$;
- $\mathit{rank}\big((A-4I_{14})^2\big) = 10$;
- $\mathit{rank}\big((A-4I_{14})^3\big) = 9$;
- $\mathit{rank}\big((A-4I_{14})^4\big) = 8$;
- $\mathit{rank}\big((A-4I_{14})^5\big) = 8$.

Compute the Jordan normal form of A.

Problem 1 (15 points)**.** *Consider all possible matrices in $\mathbb{C}^{14\times 14}$ whose eigenvalues are (exactly) the following:*

- $\lambda_1 = 3$ *with algebraic multipicity* 5 *and geometric multiplicity* 4*;*
- $\lambda_2 = 5$ *with algebraic multiplicity* 4 *and geometric multiplicity* 1*;*
- $\lambda_3 = 6$ *with algebraic multiplicity* 5 *and geometric multiplicity* 2*.*

Are all such matrices similar? Make sure you fully justify your answer.

> ***Hint:*** *Jordan normal form.*

Definition/Remark. *Let $\mathbb{F}$ be a field. For a positive integer n, let us define the following $n\times n$ matrix with entries understood to be in $\mathbb{F}$.*

$$P_n \quad := \quad \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 1 \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}_{n\times n}$$

*(P_n is **not** the standard notation for this matrix. There probably is no standard notation for it, but we need some way to refer to it.) It is easy to see that RREF($\left[\begin{array}{c|c} P_n & I_n \end{array}\right]$) $= \left[\begin{array}{c|c} I_n & P_n \end{array}\right]$, and so P_n is invertible and is its own inverse, i.e. $P_n^{-1} = P_n$. Moreover, it is not hard to see that for any matrix $A \in \mathbb{F}^{n\times n}$, we have the following:*

(1) AP_n is the matrix obtained by reflecting the entries of A about the vertical axis through the middle of A;[1]

(2) $P_n^{-1}A = P_nA$ is the matrix obtained by reflecting the entries of A about the horizontal axis through the middle of A;[2]

(3) $P_n^{-1}AP_n = P_nAP_n$ is the matrix obtained by rotating the entries of A by $180°$ about the center of the matrix.[3]

(You don't need to give a formal proof, but do try to convince yourself that this really is true!)

Problem 2 (25 points). *Let $\mathbb{F}$ be an algebraically closed field.*[4] *Using the Jordan normal form (plus the Definition/Remark above), prove that every square matrix with entries in $\mathbb{F}$ is similar to its own transpose.*

> ***Hint:*** *First, prove the result for Jordan blocks (use the Definition/Remark above). Then, prove it for Jordan matrices. Finally, prove it for arbitrary square matrices with entries in $\mathbb{F}$. Where exactly do you need the fact that $\mathbb{F}$ is algebraically closed.*

[1] For example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{=P_3} = \begin{bmatrix} 3 & 2 & 1 \\ 6 & 5 & 4 \\ 9 & 8 & 7 \end{bmatrix}.$$

[2] For example:

$$\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{=P_3^{-1}=P_3} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}.$$

[3] For example:

$$\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{=P_3^{-1}=P_3} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{=P_3} = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}.$$

[4] Actually, the assumption that the field $\mathbb{F}$ is algebraically closed is not strictly necessary. However, the proof would become harder without it.

Problem 3 (25 points). *Orthogonally diagonalize the following symmetric matrix in* $\mathbb{R}^{3\times 3}$*:*

$$A \ := \ \begin{bmatrix} 1 & -2 & 2 \\ -2 & 4 & -4 \\ 2 & -4 & 4 \end{bmatrix}.$$

In other words, compute a diagonal matrix D and an orthogonal matrix Q, both in $\mathbb{R}^{3\times 3}$, such that $D = Q^T A Q$.

> ***Hint:*** *This is very similar to Example 8.7.7 of the Lecture Notes.*

Problem 4 (25 points). *Let $A \in \mathbb{R}^{n\times m}$. Prove that there exists an orthonormal basis $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ of $\mathbb{R}^m$ such that vectors $A\mathbf{v}_1, \dots, A\mathbf{v}_m$ are pairwise orthogonal. (Here, orthogonality and orthonormality are assumed to be with respect to the standard scalar product $\cdot$ and the induced norm $||\cdot||$.)*

> ***Remark:*** *It is possible that some of the $A\mathbf{v}_i$'s are zero.*
>
> ***Hint:*** *Explain why $\mathbb{R}^m$ has an orthonormal eigenbasis associated with $A^T A$, and then use that eigenbasis.*