Linear Algebra 2:
HW#7

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due Friday, May 9, 2025, at noon (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and it should be either typed or written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Please do **not** send your HW by e-mail. Please write your **name** on top of the first page of your HW.

Unless explicitly stated otherwise, make sure you **show your work** (though you may omit the details of row reduction, as per the Remark below). If you do not show your work, you will receive **no credit** for the exercise/problem in question, even if your final answer is correct.

Remark: Feel free to use a calculator such as the one from

`https://www.dcode.fr/matrix-row-echelon`

for any row reduction that you need to perform. However, keep in mind that on the exam, you might be asked to solve problems like this **without** a calculator, and so you should in principle be able to perform row reduction of this sort by hand.[a] Moreover, make sure you correctly tell the calculator what kinds of numbers you are working with,[b] because otherwise, the calculator may give you a wrong answer.

[a] If you do not yet feel confident in your ability to row reduce, then row reduce by hand first, and then check your answer with a calculator.

[b] For real numbers, you should use their "Echelon Form Matrix Reduction Calculator." For $\mathbb{Z}_2$, use "RREF with Modulo Calculator" with Base/Modulus = 2. For $\mathbb{Z}_3$, use "RREF with Modulo Calculator" with Base/Modulus = 3.

Exercise 1 (20 points). *Consider the following matrices in* $\mathbb{C}^{6\times 6}$*:*

$$D = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \qquad P := \begin{bmatrix} 1 & 5 & -2 & 3 & 0 & 8 \\ 0 & 0 & 3 & -2 & 1 & -2 \\ -4 & 2 & 1 & 1 & 1 & 5 \\ 6 & -6 & -6 & 7 & 0 & 1 \\ 9 & 1 & 9 & 1 & 9 & 1 \\ 2 & 3 & 2 & 4 & 4 & 4 \end{bmatrix}.$$

You may assume that P is invertible.[1] *Set $A := PDP^{-1}$ (so, $D = P^{-1}AP$). Compute the characteristic polynomial of A and the spectrum of A. Identify the eigevalues of A, and for each eigenvalue, specify its algebraic and geometric multiplicity. Find a basis of each eigenspace of A.*

> ***Hint:*** *See Proposition 8.5.12 and Example 8.5.13 of the Lecture Notes.*

Problem 1 (20 points). *For what values of complex constants a, b, c is the complex matrix A (below) diagonalizable? Make sure you prove that your answer is correct. (However, you do not have to actually diagonalize A for those values of a, b, c for which A is diagonalizable.)*

$$A \quad = \quad \begin{bmatrix} 3 & a & b \\ 0 & 3 & c \\ 0 & 0 & 2 \end{bmatrix}$$

Problem 2 (20 points). *Consider the following matrix in* $\mathbb{C}^{4\times 4}$*:*

$$A \quad := \quad \begin{bmatrix} 20 & 0 & 9 & 0 \\ 0 & -1 & 0 & 0 \\ -42 & 0 & -19 & 0 \\ -21 & 0 & -9 & -1 \end{bmatrix}.$$

Diagonalize the matrix A, and then compute a formula for A^m, where m is a non-negative integer. Does your formula also work for negative integers m? Why or why not?

Problem 3 (20 points). *Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{n\times n}$ be a square matrix such that $\mathbb{F}^n$ has an eigenbasis associated with A. Prove that $\mathbb{F}^n$ has an eigenbasis associated with A^T.*

> ***Hint:*** *Diagonalization.*

[1] Indeed, the calculator tells us that $\det(P) = 4040 \neq 0$, and so P is invertible.

Problem 4 (20 points). *In what follows, $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the standard basis vectors of $\mathbb{C}^n$. Prove or disprove the following statement:*

> *If $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are eigenvectors of a matrix $A \in \mathbb{C}^{n\times n}$, then A must be diagonal.*

Remark: *First, state clearly whether the statement is true or false. If it is true, then prove it. If it is false, then construct a counterexample (and prove that your counterexample really is a counterexample).*