Linear Algebra 2:
HW#6

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due Friday, May 2, 2025, at noon (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and it should be either typed or written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Please do **not** send your HW by e-mail. Please write your **name** on top of the first page of your HW.

Unless explicitly stated otherwise, make sure you **show your work** (though you may omit the details of row reduction, as per the Remark below). If you do not show your work, you will receive **no credit** for the exercise/problem in question, even if your final answer is correct.

Remark: Feel free to use a calculator such as the one from

`https://www.dcode.fr/matrix-row-echelon`

for any row reduction that you need to perform. However, keep in mind that on the exam, you might be asked to solve problems like this **without** a calculator, and so you should in principle be able to perform row reduction of this sort by hand.[a] Moreover, make sure you correctly tell the calculator what kinds of numbers you are working with,[b] because otherwise, the calculator may give you a wrong answer.

[a]If you do not yet feel confident in your ability to row reduce, then row reduce by hand first, and then check your answer with a calculator.

[b]For real numbers, you should use their "Echelon Form Matrix Reduction Calculator." For $\mathbb{Z}_2$, use "RREF with Modulo Calculator" with Base/Modulus = 2. For $\mathbb{Z}_3$, use "RREF with Modulo Calculator" with Base/Modulus = 3.

Exercise 1 (25 points). *Consider the following matrix in* $\mathbb{C}^{4\times 4}$*:*

$$A \quad := \quad \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}.$$

Compute the characteristic polynomial and the spectrum of A*. For each eigenvalue* λ *of* A*, determine the algebraic and geometric multiplicity of* λ*, and compute a basis of the eigenspace* $E_\lambda(A)$*.*

Problem 1 (25 points). *Consider the linear function* $f : \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ *given by*

$$f\big(a_2x^2 + a_1x + a_0\big) \quad = \quad (-9a_1 - 8a_2)x^2 + (7a_1 + 6a_2)x + (a_0 + 3a_1 + 2a_2)$$

for all $a_0, a_1, a_2 \in \mathbb{C}$*. (You may assume that* f *is indeed linear, i.e. you do not need to prove that it is linear.) Compute the characteristic polynomial and the spectrum of* f*. For each eigenvalue* λ *of* f*, determine the algebraic and geometric multiplicity of* λ*, and compute a basis of the eigenspace* $E_\lambda(f)$*.*

> ***Hint:*** *Imitate the solution of Example 8.2.16 from the Lecture Notes.*

Problem 2 (25 points). *Consider the linear function* $f : \mathbb{C}^{2\times 2} \to \mathbb{C}^{2\times 2}$ *given by*

$$f\Big(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\Big) \quad = \quad \begin{bmatrix} a+d & b+c \\ b+c & a+d \end{bmatrix}$$

for all $a, b, c, d \in \mathbb{C}$*. (You may assume that* f *is indeed linear, i.e. you do not need to prove that it is linear.) Compute the characteristic polynomial and the spectrum of* f*. For each eigenvalue* λ *of* f*, determine the algebraic and geometric multiplicity of* λ*, and compute a basis of the eigenspace* $E_\lambda(f)$*.*

> ***Hint:*** *Once again, this is similar to Example 8.2.16 from the Lecture Notes, except that it involves matrices rather than polynomials.*

Problem 3 (25 points). *Let* a, b, c *be positive real numbers. Compute the volume of the solid enclosed by the ellipsoid*

$$\Big\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mid x_1, x_2, x_3 \in \mathbb{R},\ \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1 \Big\}.$$

> ***Hint:*** *Imitate the solution of Example 7.10.6 from the Lecture Notes.*