Linear Algebra 2:
HW#5

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due Friday, April 18, 2025, at noon (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and it should be either typed or written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Please do **not** send your HW by e-mail. Please write your **name** on top of the first page of your HW.

Unless explicitly stated otherwise, make sure you **show your work** (though you may omit the details of row reduction, as per the Remark below). If you do not show your work, you will receive **no credit** for the exercise/problem in question, even if your final answer is correct.

Remark: Feel free to use a calculator such as the one from

`https://www.dcode.fr/matrix-row-echelon`

for any row reduction that you need to perform. However, keep in mind that on the exam, you might be asked to solve problems like this **without** a calculator, and so you should in principle be able to perform row reduction of this sort by hand.[a] Moreover, make sure you correctly tell the calculator what kinds of numbers you are working with,[b] because otherwise, the calculator may give you a wrong answer.

[a] If you do not yet feel confident in your ability to row reduce, then row reduce by hand first, and then check your answer with a calculator.

[b] For real numbers, you should use their "Echelon Form Matrix Reduction Calculator." For $\mathbb{Z}_2$, use "RREF with Modulo Calculator" with Base/Modulus = 2. For $\mathbb{Z}_3$, use "RREF with Modulo Calculator" with Base/Modulus = 3.

Exercise 1 (5 points)**.** *Consider the following permutation in S_7:*

$$\pi \quad = \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 3 & 2 & 4 & 1 & 5 \end{pmatrix}.$$

Compute the matrix P_π, and compute $\det(P_\pi)$.

> ***Hint:*** *$\det(P_\pi)$ can be computed in various ways, but it is easiest to use Proposition 7.1.1 from the Lecture Notes.*

Exercise 2 (5 points)**.** *Consider the following permutation matrix (with entries understood to be in some field $\mathbb{F}$):*

$$A \quad = \quad \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Compute the permutation $\pi \in S_6$ such that $A = P_\pi$, and compute $\det(A)$.

> ***Hint:*** *Same hint as for the previous exercise.*

Exercise 3 (15 points)**.** *Compute the determinant of the following matrix (with entries understood to be in $\mathbb{R}$):*

$$A \quad = \quad \begin{bmatrix} -2 & 1 & 1 & 0 & 2 \\ 0 & 1 & -2 & 0 & 0 \\ -3 & 1 & 2 & 0 & -1 \\ 1 & 3 & -1 & 2 & 3 \\ 2 & -1 & 1 & 0 & 0 \end{bmatrix}.$$

You may use any method (or combination of methods) that you like, but make sure you show your work.[1]

Exercise 4 (10 points)**.** *Consider the following matrix and vector, with entries understood to be in $\mathbb{Z}_3$:*

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

*Either use **Cramer's rule** to solve the matrix-vector equation $A\mathbf{x} = \mathbf{b}$, or explain why this equation cannot be solved using Cramer's rule. (If the equation cannot be solved using Cramer's rule, then you do not need to solve it.)*

[1] You should compute this by hand (and show your work!), but you can, and probably should, check your answer with a calculator.

Exercise 5 (15 points)**.** *Consider the matrix*

$$A = \begin{bmatrix} 2 & 1 & -1 \\ -2 & 0 & 2 \\ 0 & -1 & 1 \end{bmatrix}$$

with entries understood to be in $\mathbb{R}$. Compute:

1. *det(A);*
2. *the cofactor matrix of A;*
3. *the adjugate matrix of A.*

Is A invertible? If so, compute its inverse.

Problem 1 (25 points)**.** *Let $\mathbb{F}$ be a field, let $n \geq 2$ be an integer, and let $\mathbf{a}_2, \ldots, \mathbf{a}_n$ be linearly independent vectors in $\mathbb{F}^n$. Define the function $f : \mathbb{F}^n \to \mathbb{F}$ by setting*

$$f(\mathbf{x}) = \begin{vmatrix} \mathbf{x} & \mathbf{a}_2 & \ldots & \mathbf{a}_n \end{vmatrix} \qquad \text{for all } \mathbf{x} \in \mathbb{F}^n.$$

By Proposition 7.2.1 of the Lecture Notes, f is linear. Find a basis of Ker(f), and compute the dimension of Ker(f) and Im(f). Make sure you fully justify your answer.

Problem 2 (25 points)**.** *Let A be an invertible matrix in $\mathbb{R}^{n \times n}$, and assume that the entries of A are all integers. Prove that the entries of A^{-1} are all integers if and only if det$(A) = \pm 1$.*

> ***Hint:*** *Use the adjugate matrix.*