

Linear Algebra 2: HW#4

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Summer 2025

due Friday, April 4, 2024, at noon (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and it should be either typed or written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Please do **not** send your HW by e-mail. Please write your **name** on top of the first page of your HW.

Unless explicitly stated otherwise, make sure you **show your work** (though you may omit the details of row reduction, as per the Remark below). If you do not show your work, you will receive **no credit** for the exercise/problem in question, even if your final answer is correct.

Remark: Feel free to use a calculator such as the one from

<https://www.dcode.fr/matrix-row-echelon>

for any row reduction that you need to perform. However, keep in mind that on the exam, you might be asked to solve problems like this **without** a calculator, and so you should in principle be able to perform row reduction of this sort by hand.^a Moreover, make sure you correctly tell the calculator what kinds of numbers you are working with,^b because otherwise, the calculator may give you a wrong answer.

^aIf you do not yet feel confident in your ability to row reduce, then row reduce by hand first, and then check your answer with a calculator.

^bFor real numbers, you should use their “Echelon Form Matrix Reduction Calculator.” For \mathbb{Z}_2 , use “RREF with Modulo Calculator” with Base/Modulus = 2. For \mathbb{Z}_3 , use “RREF with Modulo Calculator” with Base/Modulus = 3.

Exercise 1 (15 points). Consider the following matrix and vector, with entries understood to be in \mathbb{R} :

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Find the least-squares solution(s) of the matrix-vector equation $A\mathbf{x} = \mathbf{b}$, and compute the associated least-squares error. Using your answer to the previous question, determine whether the equation $A\mathbf{x} = \mathbf{b}$ is consistent.

Exercise 2 (15 points). Fit a polynomial of degree at most 2 to the data points $(0, 27)$, $(1, 0)$, $(2, 0)$, $(3, 0)$ using least squares.

Problem 1 (20 points). Let $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be an orthonormal basis of \mathbb{R}^n , and set $A = [\mathbf{u}_1 \ \dots \ \mathbf{u}_{n-1}]$. Find the least-squares solution(s) of the matrix-vector equation $A\mathbf{x} = \mathbf{u}_n$. What is the associated least-squares error?

Problem 2 (20 points). Let $Q_1, Q_2 \in \mathbb{R}^{n \times m}$ and $Q \in \mathbb{R}^{m \times m}$. Assume that matrices Q_1 and Q_2 have orthonormal columns (with respect to the standard scalar product \cdot and the induced norm $\|\cdot\|$ in \mathbb{R}^n),¹ and assume furthermore that $Q_1 = Q_2Q$. Prove that the matrix Q is orthogonal.

Hint: Consider the product $Q_1^T Q_1 = (Q_2Q)^T(Q_2Q)$

Problem 3 (30 points). Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix}$$

in \mathbb{R}^4 , and set $C := \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$.

(a) Compute the standard matrix of orthogonal projection onto C .

(b) Compute an orthonormal basis for C and an orthonormal basis for C^\perp .

(c) Let $\mathbf{x} = [1 \ 1 \ 1 \ 1]^T$. Compute vectors $\mathbf{y} \in C$ and $\mathbf{z} \in C^\perp$ such that $\mathbf{x} = \mathbf{y} + \mathbf{z}$.

¹**Remark:** Despite Theorem 6.8.1 from the Lecture Notes, we cannot conclude that matrices Q_1 and Q_2 are orthogonal. This is because Q_1 and Q_2 are not necessarily square matrices. By definition, only square matrices can be orthogonal.