Linear Algebra 2:
HW#3

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due Friday, March 28, 2025, at noon (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and it should be either typed or written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Please do **not** send your HW by e-mail. Please write your **name** on top of the first page of your HW.

Unless explicitly stated otherwise, make sure you **show your work** (though you may omit the details of row reduction, as per the Remark below). If you do not show your work, you will receive **no credit** for the exercise/problem in question, even if your final answer is correct.

Remark: Feel free to use a calculator such as the one from

`https://www.dcode.fr/matrix-row-echelon`

for any row reduction that you need to perform. However, keep in mind that on the exam, you might be asked to solve problems like this **without** a calculator, and so you should in principle be able to perform row reduction of this sort by hand.[a] Moreover, make sure you correctly tell the calculator what kinds of numbers you are working with,[b] because otherwise, the calculator may give you a wrong answer.

[a]If you do not yet feel confident in your ability to row reduce, then row reduce by hand first, and then check your answer with a calculator.

[b]For real numbers, you should use their "Echelon Form Matrix Reduction Calculator." For $\mathbb{Z}_2$, use "RREF with Modulo Calculator" with Base/Modulus = 2. For $\mathbb{Z}_3$, use "RREF with Modulo Calculator" with Base/Modulus = 3.

Exercise 1 (20 points). *In this exercise, we consider* $\mathbb{R}^5$ *to be equipped with the standard scalar product* $\cdot$ *and the induced norm* $||\cdot||$. *Consider the following linearly independent vectors in* $\mathbb{R}^5$*:*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 4 \\ -4 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ -4 \\ -3 \\ 7 \\ 1 \end{bmatrix}.$$

(You may assume that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ *really are linearly independent.) Using the Gram-Schmidt orthogonalization process (either version), compute an orthonormal basis of* $U := \mathit{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

Exercise 2 (20 points). *In this exercise, we assume that* $\mathbb{R}^4$ *is equipped with the standard scalar product* $\cdot$ *and the induced norm* $||\cdot||$. *Consider the following matrix in* $\mathbb{R}^{4\times 5}$*:*

$$A = \begin{bmatrix} 3 & -5 & 1 & 1 & 4 \\ 1 & 1 & 3 & 1 & 2 \\ -1 & 5 & 3 & -2 & -3 \\ 3 & -7 & -1 & 8 & 11 \end{bmatrix}.$$

Compute an orthonormal basis of $\mathit{Col}(A)$.

Exercise 3 (30 points). *In this exercise, we assume that* $\mathbb{R}^3$ *is equipped with the standard scalar product* $\cdot$ *and the induced norm* $||\cdot||$. *Consider the following vectors in* $\mathbb{R}^3$*:*

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \quad \mathbf{a}_4 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Set $U := \mathit{Span}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4)$.

(a) Compute an orthonormal basis of U *and an orthonormal basis of* $U^\perp$.

> ***Hint:*** *See Proposition 6.4.4 and Example 6.4.5 of the Lecture Notes.*

(b) Let $\mathbf{x} = \begin{bmatrix} 4 & 2 & 4 \end{bmatrix}^T$. *Compute vectors* $\mathbf{y} \in U$ *and* $\mathbf{z} \in U^\perp$ *such that* $\mathbf{x} = \mathbf{y} + \mathbf{z}$.

> ***Hint:*** *See Corollary 6.5.3 of the Lecture Notes.*

Problem 1 (30 points)**.** *Let* $\langle \cdot, \cdot \rangle$ *be a scalar product in* $\mathbb{R}^n$ *(not necessarily the standard one). Let* V *be a real vector space, and let* $f : V \to \mathbb{R}^n$ *be a linear function. We define* $\langle \cdot, \cdot \rangle_f : V \times V \to \mathbb{R}$ *by setting*

$$\langle \mathbf{u}, \mathbf{v} \rangle_f \quad = \quad \Big\langle f(\mathbf{u}), f(\mathbf{v}) \Big\rangle$$

for all $\mathbf{u}, \mathbf{v} \in V$*. Prove that* $\langle \cdot, \cdot \rangle_f$ *is a scalar product in* V *if and only if* f *is one-to-one.*