Linear Algebra 2: HW#2

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due Friday, March 14, 2025, at noon (Prague time)

Submit your HW through the **Postal Owl** as a **PDF attachment**. Make sure your submission is **printable**: it should be A4 or letter size, and it should be either typed or written in dark ink/pencil (blue, black...) on a light (white, beige...) background. Other formats will not be accepted. Please do **not** send your HW by e-mail. Please write your **name** on top of the first page of your HW.

Unless explicitly stated otherwise, make sure you **show your work** (though you may omit the details of row reduction, as per the Remark below). If you do not show your work, you will receive **no credit** for the exercise/problem in question, even if your final answer is correct.

Remark: Feel free to use a calculator such as the one from

`https://www.dcode.fr/matrix-row-echelon`

for any row reduction that you need to perform. However, keep in mind that on the exam, you might be asked to solve problems like this **without** a calculator, and so you should in principle be able to perform row reduction of this sort by hand.[a] Moreover, make sure you correctly tell the calculator what kinds of numbers you are working with,[b] because otherwise, the calculator may give you a wrong answer.

[a] If you do not yet feel confident in your ability to row reduce, then row reduce by hand first, and then check your answer with a calculator.

[b] For real numbers, you should use their "Echelon Form Matrix Reduction Calculator." For $\mathbb{Z}_2$, use "RREF with Modulo Calculator" with Base/Modulus = 2. For $\mathbb{Z}_3$, use "RREF with Modulo Calculator" with Base/Modulus = 3.

Exercise 1 (10 points). *Compute the angle θ between the vectors*

$$\mathbf{x} = \begin{bmatrix} -1 \\ 2 \\ 4 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{y} = \begin{bmatrix} -4 \\ 0 \\ 4 \\ -2 \end{bmatrix}$$

*in $\mathbb{R}^4$. (Your answer may possibly involve an inverse trigonometric function. In any case, you should give an exact answer rather than an approximate one. Do **not** use a calculator.) Is the angle θ acute, right, or obtuse?*

Problem 1 (20 points). *Let V be a real or complex vector space, equipped with a scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $|| \cdot ||$. Prove that*

$$||\mathbf{x} - \mathbf{y}||^2 + ||\mathbf{x} + \mathbf{y}||^2 \quad = \quad 2||\mathbf{x}||^2 + 2||\mathbf{y}||^2$$

for all $\mathbf{x}, \mathbf{y} \in V$.

Problem 2 (30 points). *The equality from Problem 1 holds for norms induced by scalar products, but it need not hold for other norms.*

*(a) Prove that the equality from Problem 1 does **not** hold for the Manhattan norm $|| \cdot ||_1$ on $\mathbb{R}^2$, that is, find some two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ such that*

$$||\mathbf{x} - \mathbf{y}||_1^2 + ||\mathbf{x} + \mathbf{y}||_1^2 \quad \neq \quad 2||\mathbf{x}||_1^2 + 2||\mathbf{y}||_1^2.$$

*(b) Prove that the equality from Problem 1 does **not** hold for the Chebyshev distance $|| \cdot ||_\infty$ on $\mathbb{R}^2$, that is, find some two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ such that*

$$||\mathbf{x} - \mathbf{y}||_\infty^2 + ||\mathbf{x} + \mathbf{y}||_\infty^2 \quad \neq \quad 2||\mathbf{x}||_\infty^2 + 2||\mathbf{y}||_\infty^2.$$

Problem 3 (40 points). *Let U and V be real vector spaces, equipped with scalar products $\langle \cdot, \cdot \rangle_U$ and $\langle \cdot, \cdot \rangle_V$, respectively. Let $f : U \to V$ be a function that satisfies the property that*

$$\langle f(\mathbf{u}), f(\mathbf{v}) \rangle_V \quad = \quad \langle \mathbf{u}, \mathbf{v} \rangle_U$$

for all $\mathbf{u}, \mathbf{v} \in U$. Prove that f is linear and one-to-one.

Hint: *For linearity, consider*

- $\Big\langle f(\mathbf{u}+\mathbf{v}) - f(\mathbf{u}) - f(\mathbf{v}), f(\mathbf{u}+\mathbf{v}) - f(\mathbf{u}) - f(\mathbf{v}) \Big\rangle_V$
- $\Big\langle f(\alpha\mathbf{u}) - \alpha f(\mathbf{u}), f(\alpha\mathbf{u}) - \alpha f(\mathbf{u}) \Big\rangle_V$

where $\mathbf{u}, \mathbf{v} \in U$ and $\alpha \in \mathbb{R}$.